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# On Right Self-injective, Right Non-singular Semigroups

Kunitaka SHOJI

*Department of Mathematics, Faculty of Science, Shimane University*
*Matsue, 690 JAPAN*

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It is shown that each right self-injective, right nonsingular semigroup is isomorphic to a direct product of right self-injective, right non-singular semigroups of types (I), (II), (III), (IV). The structures of those semigroups of four types are studied. In particular, it is shown that every semigroup of type (I) is a semilattice of groups. It is proved that every right self-injective, right non-singular regular semigroup is strongly left reversible. This gives another proof that every semigroup of type (I) is absolutely flat and, consequently, a semigroup amalgamation base.

## Introduction

Let $S$ be a semigroup with zero 0 and $M_S$ a nonempty set with an operation of $S$ on the right. Then $M_S$ is called a *right S-set with zero* $0_M$ (simply called a *right S-set*) if $(ms)t = m(st)$ and $0_M s = m0 = 0_M$ for all $m \in M_S$ and all $s, t \in S$. Dually, a *left S-set* is defined. Let $\phi$ be a mapping of $A_S$ into $B_S$, where $A_S$, $B_S$ are right $S$-sets. Then $\phi$ is called an *S-map* if $\phi(as) = \phi(a)s$ for all $a \in A_S$ and $s \in S$. A right $S$-set $M_S$ is called *injective* if for any injective $S$-map $\xi: A_S \to B_S$ and any $S$-map $\eta: A_S \to M_S$, there exists an $S$-map $\theta: B_S \to M_S$ with $\theta\xi = \eta$. A semigroup $S$ is called *right self-injective* if the right $S$-set $S_S$ is injective. Dually, a *left self-injective* semigroup is defined. A both left and right self-injective semigroup is simply called *self-injective.*

Let $S$ be a semigroup with zero and $I$, $J$ right ideals of $S$ with $I \subset J$. Then we say that $I$ is *intersection large* in $J$ if $I \cap K \neq 0$ for all nenzero right ideal $K$ of $S$ with $K \subset J$. In particular, if $I$ is intersection large in $S$, the $I$ is simply called an *intersection large right ideal.* A right ideal $R$ of $S$ is called *dense* if for any triple of $a$, $b$ and $c \in S$ with $a \neq b$, there exists $z \in S$ such that $cz \in R$ and $az \neq bz$. A semigroup $S$ with zero is called *right non-singular* if every intersection large right ideal of $S$ is dense. A *left non-singular* semigroup is dually defined. A both left and right non-singular semigroup is simply called *non-singular.* In [12] the author has studied the structure of self-injective non-singular semigroups. According to Hinkle [8], all right self-injective, right nonsingular semigroups are obtained as maximal right quotient semigroups of right non-singular semigroups. As far as the author knows, the structure of these semigroups has not been known except in special cases (see [7]). In Section 1, we give a decomposition of a right self-injective, right non-singular semigroup into semigroups of types (I), (II), (III) and (IV). In Section 2, we investigate the structures of semigroups of four types. In the last section, we show that every right self-injective, right non-singular semigroup is strongly left reversible. Consequently, we obtain another proof that every semigroup of type (I) is a semigroup amalgamation base. Finally, we prove that a semigroup $S$ of type (II) with the right socle $\Sigma$ being intersection large is never strongly right reversible. Terminology and notations are referred to Clifford and Preston [5], unless otherwise stated.

## §1. Decomposition theorem

Throughout this paper, let $S$ denote a semigroup. An element $x \in S$ is called nilpotent if $x^n = 0$ for some positive integer $n$. An ideal $I$ of $S$ is called *nilpotent* if $I^n = 0$ for some positive integer $n$. The purpose of this section is to prove the following:

THEOREM 1 (Decomposition Theorem). *Every right self-injective, right non-singular semigroup is isomorphic to the direct product of right self-injective, right non-singular semigroups $S_1$, $S_2$, $S_3$, $S_4$ of the following types* (III) *and* (IV):

(Type I) $S_1$ *is a regular semigroup containing no nonzero nilpotent elements.*

(Type II) $S_2$ *is a regular semigroup, each of which nonzero ideal contains nonzero nilpotent elemets.*

(Type III) $S_3$ *contains no nonzero nilpotent elements and each nonzero ideal of $S_3$ is not regular as a semigroup.*

(Type IV) *Each nonzero ideal of $S_4$ contains nonzero nilpotent elements and is not regular as a semigroup.*

The proof of Theorem 1 follows from the following results. In Lemmas 1 through to 6, we assume that $S$ is a right self-injective, right non-singular semigroup.

LEMMA 1. (from [1, Theorem 10] and [8, Proposition 3.3]).

(1) *For each right ideal $I$ of $S$, there is an injective right ideal $K$ of $S$ such that $I$ is intersection large in $K$.*

(2) *Let $M$, $N$ be right ideals of $S$ such that $M$ is intersection large in $N$. If $M$ is injective, then $M = N$.*

LEMMA 2. *Let $J$ be a right ideal of $S$ and let $J^C = \{x \in T \mid xS \cap J = 0\}$. Then $J^C$ is an injective right ideal of $S$ such that $J \cap J^C = 0$ and $J \cup J^C$ is intersection large. Also, $J$ is intersection large in $(J^C)^C$. In this case $J^C$ is called the complement of $J$.*

PROOF. By lemma 1, there is an injective right ideal $K$ of $S$ such that $J^C$ is intersection large in $K$. Obviously $K \cap J = 0$. Hence $K = J^C$ and $J^C$ is injective. Let $t \in S$ with $t \neq 0$. If $ts \cap J = 0$, then $ts \subset J^C$. Otherwise we have $ts \cap J \neq 0$. Thus $J \cup J^C$ is intersection large. We shall show next that $J$ is intersection large in $(J^C)^C$. Let $u \in (J^C)^C$ with $u \neq 0$. Then $uS \cap (J \cup J^C) \neq 0$ and hence $us \cap J \neq 0$. The lemma is proved.

COROLLARY 1. *Let $I$, $J$ be injective right ideals of $S$. Then $I \cap J$ is injective.*

PROOF. From Lemmas 1 and 2, we have $I = (I^C)^C$, $J = (J^C)^C$. Clearly, $I \cap J \subset (I^C \cup J^C)^C$. On the other hand, $(I^C \cup J^C)^C \subset (I^C)^C = I$ and $(I^C \cup J^C)^C \subset (J^C)^C = J$. Hence $I \cap J = (I^C \cup J^C)^C$ and hence $I \cap J$ is injective by Lemma 2.

LEMMA 3. *Let $I$ be a right ideal of $S$. Then the following are equivalent.*

(1) *$I$ is injective.*

(2) *$I$ is generated by an idempotent.*

(3) *There is a unique idempotent $e \in S$ such that $es = I$ and $eI^C = 0$. In this case, $e$ is called a projection of $I$.*

PROOF. (1)$\Rightarrow$(3): Take an $S$-homomorphism $\xi: I \cup I^C \to I$ such that $\xi(I^C) = 0$ and $\xi(a) = a$ for all $a \in I$. Since $I$ is injective, $\xi$ extends to an $S$-homomorphism $\bar{\xi}: S \to I$. Put $\bar{\xi}(1) = e \in I$. Then $eI^C = 0$, $e^2 = e$ and $eS = I$. Since $I \cup I^C$ is intersection large in $S$ and $S$ is right non-singular, there are no such idempotents in $S$ but $e$.

(3)$\Rightarrow$(2): Obvious.

(2)$\Rightarrow$(1): This was proved in [12, Lemma 3].

LEMMA 4. *Let $e$ be a central idempotent and $f$ the projection of $(eS)^C$. Then $f$ is a central idempotent and $S$ is isomorphic to the direct product of two semigroups $eSe$, $fSf$.*

PROOF. Firstly we have $fS = SfS$, since $SfS \cap eS = 0$. Hence $sf = fsf$ for all $s \in S$. We shall show that $fsf = fs$ for all $s \in S$. Let $s \in S$. Then $fsa = fsfa$ for $a \in eS \cup fS$. Since $eS \cup fS$ is intersection large in $S$ and $S$ is right non-singular, we get $fs = fsf$. Therefore $sf = fs$ for all $s \in S$, that is $f$ is central. Now define a semigroup-homomorphism $v: S \to eSe \times fSf$ by $v(s) = (es, fs)$ for all $s \in S$. Since $eS \cup fS$ is dense, $v$ is one-to-one. Let $x \in eSe$, $y \in fSf$. Then there is an $S$-homomorphism $\xi: eS \cup fS \to S$ such that $\xi(e) = x$, $\xi(f) = y$. Since $S_S$ is injective, there exists $w \in S$ such that $\xi(a) = wa$ for all $a \in eS \cup fS$, Then $ew = x$, $fw = y$. This implies that $v$ is onto. Hence $S \simeq eSe \times fSf$. The lemma is proved,

LEMMA 5. *Let $I$ be an ideal of $S$ containing no nonzero nilpotent ideals of $S$. Then there exist central idempotents $e, f \in S$ such that $S \simeq eSe \times fSf$ and $I$ is an* *intersection large right ideal of eSe.*

PROOF. By Lemma 1, there exists an injective right ideal $K$ of $S$ in which $I$ is intersection large, By virtue of Lemma 4, we let $e, f$ be the projections of $K$, $K^C$, respectively. Set $V=\{x\in S|ex\in I\}$. Then we shall show that $V$ is an intersection large right ideal of $S$. Let $t\in S$ with $t\neq 0$. If $et=0$, then $t\in V$. Otherwise we get $etS\cap I\neq 0$. Then $tS\cap V\neq 0$. Whence $V$ is interesection large in $S$. So, $V$ is dense and $(fSe)V\subset fSI=fI=0$, so that $fSe=0$. Also since $(eSfS\cap I)^2=0$, we have $S(eSfS\cap I)=0$, by assumption. Moreover $eSfS=0$ since $I$ is intersection large in $eS$, Therefore, $fSe=eSf=0$. Let $s\in S$. Then $ese=se$ since $eS\cup fS$ is dense and $(ese)a=(se)a$ for all $a\in eS\cup fS$. Whence $ese=se$ for all $s\in S$. On the other hand, for any nonzero right ideal $A$ of $S$ with $A\subset SeS$, $A\cap eS=A\cap(eS\cup fS)\neq 0$. Thus $eS$ is intersection large in $SeS$, while $eS$ is injective, by Lemma 3. Consequently, $SeS=eS$, and so $se=ese$ for all $s\in S$. Therefore, $e$ is central. The result follows from Lemma 4.

The following is essentially due to [8, Theorem 4.2].

LEMMA 6. *Let $I$ be an ideal of $S$ such that $I$ is intersection large in $S$ as a right ideal. If $I$ is a regular semigroup, then so is $S$.*

PROOF. Let $x\in S$. Consider the set $\pi(x)$ of all ordered pair $(\phi, K)$ such that $K$ is a right ideal of $S$ contained in $xS$ and $\phi$ is an $S$-homomorphism of $K$ into $S$ satisfying $x\phi(u)=u$ for all $u\in K$. Define an order relation $\geq$ on $\pi(x)$ by $(\phi, K)\geq(\eta, J)$ if and only if $K\supset J$ and $\eta=\phi|_J$ (the restriction of $\phi$ to $J$). Then $\pi(x)$ is an inductive ordered set respectively to $\geq$. By Zorn's lemma, there exists a maximal element $(\phi, K)$ in $\pi(x)$. We shall show that $K$ is intersection large in $xS$. Suppose that $0\neq a\in xS$ with $aS\cap K=0$. Then $ay\neq 0$ for some $y\in I$, since $I$ is dense. Also since $I$ is regular, there exists $z\in I$ with $(ay)z(ay)=ay$. Put $t=ryz$ where $a=xr$. Now define an $S$-homomorphism $\xi\colon ayS\cup K\to S$ such that $\xi|_K=\phi$ and $\xi(v)=tv$ for all $v\in ayS$. Then $x(tay)=x(ryz(ay))=(ay)z(ay)=ay$ and so $x(\xi(w))=w$ for all $w\in K\cup ayS$, a contradiction. Thus $K$ is intersection large in $xS$. Since $S$ is right self-injective, there exists $b\in S$ such that $bu=\phi(u)$ for all $u\in K$. We shall next that $xbx=x$. Suppose that $x\neq xbx$. Set $V=\{c\in S|xc\in K\}$. As shown in the proof of Lemma 5, we can show that $V$ is an intersection large right ideal of $S$. Then $V$ is dense. On the other haned, $(xbx)s=xs$ for all $s\in V$, a contradiction. It must be that $xbx=x$. Therefore $S$ is regular, as required.

LEMMA 7. *Let $U$, $V$ be semigroups. Then the direct product $U\times V$ of $U$, $V$ is right self-injective and right non-singular if and only if so are both $U$ and $V$.*

PROOF. Necessity: By [13, Theorem 9], $U\times V$ is right self-njective. Also it is clear that $U\times V$ is right non-singular.

Sufficiency: It suffices to how that both $U$ and $V$ are right self-injective. Set $T = U \times V$, $e = (1, 0) \in T$. Since $e$ is central in $T$, it is easily shown that $eT$ is a right self-injective semigroup if and only if $eT$ is injective as a right $T$-set. On the other hand, by Lemma 3, $eT$ is an injective right $T$-set. Hence $eT$ is a right self-injective semigroup. Since $U \simeq eT$, we have $U$ is a right self-injective semigroup. Similarly one can show that $V$ is right self-injective. The lemma is proved.

The proof of Theorem 1 follows immediately from the following results.

PROPOSITION 1. *Let $S$ be a right self-injective, right non-singular semigroup. Then there exist right self-injective, right non-singular semigroups $U$, $V$ such that $U$ is a regular semigroup, each nonzero ideal of $V$ is not a regular semigroup and $U \times V \simeq S$.*

PROOF. Let $I$ be the union of all ideals of $S$ which are regular semigroups. The $I$ is an ideal of $S$ and a regular semigroup. Whence $I$ contains no nonzero nilpotent ideals of $S$. So by using Lemmas 5 and 6, we can obtain the reguired semigroups $U$, $V$.

PROPOSITION 2. *Let $S$ be the same as in Proposition 1. The there exist right self-injective, right non-singular semigroups $Y$, $Z$ such that $Y$ contains no nonzero nilpotent element, each nonzero ideal of $Z$ contains nonzero nilpotent elements and $Y \times Z \simeq S$.*

PROOF. Let $J$ be the union of all ideals of $S$ which contains nonzero nilpotent elements. Then $J$ is an ideal of $S$ and contains no nonzero nilpotent elements. By Lemma 5, there exist semigroups $Y$, $Z$ such that $Y$ contains $J$ as an intersection large ideal, each nonzero ideal of $Z$ contains nonzero nilpotent elements and $Y \times Z \simeq S$. To prove the proposition, it suffices to show that $Y$ contains no nonzero nilpotent elements, Let $x \in Y$ with $x^2 = 0$. The $xJx = 0$ since $(xJx)^2 = 0$. Further, $xJ = 0$. Since $J$ is a dense right ideal of $Y$, we obtain $x = 0$, proving the proposition.

## §2. Semigroups of types (I), (II), (III), (IV)

By a *semigroup of type* (I) [resp. (II), (III), (IV)], we mean right self-injective, right non-singular semigroup of type (I) [resp. (II), (III), (IV)]. The purpose of this section is to clarify the structures of semigroups of these types. We begin with semigroups of type (I).

LEMMA 8. *Let $S$ be a right self-injective, right non-singular regular semigroup. Then the following are equivalent.*

(1) *$S$ is self-injective and non-singular.*

(2) *$S$ has nonzero nilpotent elements.*

(3) *For any idempotents $e, f \in S$, $ef = 0$ implies $fe = 0$.*

PROOF. (1) $\Rightarrow$ (2): By [12, Theorem 3], $S$ is a semilattice of groups. So the result holds.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Let $e$, $f$ be any idempotents of $S$ with $e \mathscr{L} f$ (where $\mathscr{L}$ denotes the Green's L-relation on $S$). Suppose that $eS \not\subset fS$. By Lemma 1, $eS \cap fS$ is not intersection large in $eS$ since $eS \cap fS$ is injective, by Corollary 1. Then there exists a nonzero $x \in eS$ such that $xS \cap fS = 0$. Since $S$ is regular, $xS$ is generated by an idempotent and hence, by Lemma 3, $xS$ is injective. By Lemma 3, $xS$ contains a projection $h$. Then $hf = 0$ and so, by assumption, $fh = 0$. Consequently, $eh = efh = 0$, so that $he = 0$. Thus $h = heh = 0$, a contradiction. Hence $eS \subset fS$. Dually, we get $fS \subset eS$, Whence $e \mathscr{R} f$ (where $\mathscr{R}$ denotes the Green's R-relation on $S$) and so, $e = f$. Thus each $\mathscr{L}$-class of $S$ contains a unique idempotent. Next we let $a$, $b$ be any idempotents of $S$ with $a \mathscr{R} b$. Set $I = \{s \in S \mid as = bs\}$. Then we shall show that $I$ is an intersection large right ideal of $S$. So suppose that there is a nonzero $c \in S$ with $cS \cap I = 0$. Since $aS \subset I$, we have $cS \cap aS = 0$. Since $S$ is regular, by Lemma 3(3), $cS$ contains a projection $k$. Then $ka = kb = 0$. So, by assumption, $ak = bk$. Hence $k \in I$, a contradiction. Thus $I$ is intersection large and hence $I$ is dense. Then we obtain $a = b$. Therefore each $\mathscr{R}$-class of $S$ contains a unique idempotent. By [5, Theorem 1.17], $S$ is an inverse semigroup. In this case, $S$ is anti-isomorphic to $S$ itself. So we conclude that $S$ is self-injective and non-singular. The lemma is proved.

From [12, Theorem 3] and Lemma 7, we have

THEOREM 2. *Every semigroup of type* (I) *is a self-injective, non-singular semigroup which is a semilattice of groups.*

Thus the structure of semigroups of type (I) has been clarified by [12] and [14].

In the remaining part of this section (except Theorem 4), we assume that $S$ satisfies the following (*):

(*) The right socle $\Sigma = \Sigma_r(S)$ of $S$ is an intersection large in $S$.

REMARK 1. Without the assumption, the structures of semigroups of type (II), (III), (IV) seem difficult to be handled.

Let $S$ be right self-injective, right non-singular semigroup satisfying (*). Since $\Sigma$ is dense, it follows easily from [5, II, Theorem 6.19] that $\Sigma$ is a 0-direct union ideals $SR_i$, where $R_i$'s are non-nilpotent 0-minimal right ideals of $S$. By applying [5, Theorem 6.5 and Lemma 5.2], it follows that $S$ is isomorphic to the direct product of semigroups $Hom_S(SR_i, SR_i)$ consisting of all $S$-endomorphisms of $SR_i$, where we use the convertion: $f \cdot g(s) = f(g(s))(s \in SR_i, f, g \in Hom_{SR_i}(SR_i, SR_i))$. In this case, $Hom_S(SR_i, SR_i) = Hom_{SR_i}(SR_i, SR_i)$, that is, the semigroup of all left translations of $SR_i$, and $SR_i$ is a right non-sigular semigroup, equivalently, a right reductive semigroup.

In the case of semigroups of type (II), the semigroups above $SR_i$ are right reductive completely 0-simple semigroups (see [13]).

From [7, Theorem 7.16] and [8, Theorem 5.6], it follows that

THEOREM 3. *Every semigroup of type* (II) *satisfying* (*) *is isomorphic to a direct product of square column monomial matrix semigroups over groups with zero, and vice-versa.*

REMARK 2. Let $S$ be a square column monomial matrix semigroup over a group with zero and $T$ its subsemigroup consisting of all matrices with at most one nonzero entry. Then $T$ is a completely 0-simple inverse semigroup (so-called, a Brandt semigroup) and $S$ is isomorphic to $\Lambda(T)$. More generally, one can see that the maximal right quotient semigroup of a non-singular inverse semigroup which is not a semilattice of groups is a semigroup of type (II).

As for semigroups of type (III), we shall show:

A semigroup $S$ is called *indecomposable* if it is not isomorphic to a direct product of two non-trivial semigroups.

THEOREM 4. *An indecomposable semigroup $S$ of type* (III) *is the semigroup obtained from a right cancellative semigroup without idempotents by adjoining a zero. Specially, $S$ is an infinite semigroup.*

PROOF. Suppose that there exists $0 \neq x \in S$ such that $xS$ is not intersection large in $S$. Then $(xS)^c \neq 0$ and $((xS)^c)^c \neq 0$. Let $e, f$ be projections of $(xS)^c$, $((xS)^c)^c$, respectively. Then $ef = 0$ and hence $(fSe)^2 = 0$. Since $S$ contains no zero nilpotent elements, we get $fSe = 0$. Similarly we obtain $SeS \cap fS = 0$, so that $SeS \cap xS = 0$. By Lemma 2, $SeS \subset eS$. From the proof of Lemma 5, there exists a central idempotent $h \in S$ such that $eS$ is intersection large in $hS$. By Lemma 1 and Lemma 3, we have $e = h$. By lemma 4, $eS$ is a direct summand of $S$ as a subsemigroup. Since $S$ is indecomposable, we know that $e$ equals 1 or 0, a contradiction. So we conclude that every nonzero right ideal of $S$ is intersection large in $S$. By [8, Theorem 4.3], $S\text{-}\{0\}$ is a right cancellative subsemigroup. All remaining parts of the therem are easily proved.

THEOREM 5. *Every semigroup of type* (III) *satisfying* (*) *is isomorphic to a direct product of the semigroups of left translations of semigroups obtained from right cancellative, right simple semigroups without idempotents by adjoining a zero element, and vice-versa.*

PROOF. Let $S$ be a semigroup of type (III) and $R$ a 0-minimal right ideal. We shall show first that the ideal $SR$ is a 0-minimal right ideal. By [5, II, Theorem 6.23], $SR$ is a 0-minimal ideal of $S$. By Lemma 5, there exists a central idempotent $e \in S$ such that $I$ is intersection large in $eS$ and $eS$ is a (semigroup) direct summand of $S$. Since clearly $eS$ is indecomposable, as shown in the proof of Theorem 4, it follows that every nonzero right ideal of $eS$ is intersection large in $eS$. Hence $SR = R$. By Theorem 4, it follows that $SR - \{0\}$ is a right cancellative subsemigroup without idempotents. The proof of the theorem is complete.

REMARK 3. Baer-Levi semigroup is an important example of right cancellative, right simple semigroups without idempotents. Thus one can construct semigroups of type (III) satisfying (*) from Bear-Levi semigroups with zero adjoined (see [10]).

Finally, we shall study the structure of semigroups of type (IV) satisfying (*).

Let $S$ be a semigroup of type (IV) satisfying (*). As is shown in the argument before Theorem 2, $S$ is isomorphic to the direct product of semigroups $\Lambda(SR_i)$, where $R_i$'s are 0-minimal right ideals of $S$. So we assume that the right socle of $S$ is of form $SR$, where $R$ is a non-nilpotent 0-minimal right ideal. Then by [5, II, Theorem 6.19] and our observation mentioned above, we can show that the semigroup $SR$ satisfies any one of the following two conditions:

($IV_1$) $SR$ is a right reductive, non-regular 0-simple semigroup which is a union of at least two non-nilpotent 0-minimal right ideals.

($IV_2$) $SR$ is a right reductive, semigroup which is a union non-nilpotent 0-minimal right ideals and nilpotent 0-minimal right ideals and satisfies that (1) $R_1R_2 = R_1$, $R_2R_1 = R_2$ for any non-nilpotent 0-minimal right ideals $R_1$, $R_2$ and (2) $NR = N$, $RN = 0$ for any non-nilpotent 0-minimal right ideal $R$ and any nilpotent 0-minimal right ideal $N$.

Summarizing up the above, we obtain

THEOREM 6. *Every semigroup of type* (IV) *satisfying* (*) *is isomorphic to a direct product of the semigroups of all left translations of semigroups satisfying any one of the conditions* ($IV_1$) *or* ($IV_2$).

REMARK 4. (1) The semigroup $\Lambda(S)$ of a semigroup $S$ satisfying ($IV_1$) is of type (IV) if it is not regular. Saito and Hori [11] gave method of constructing semigroups satisfying ($IV_1$). (Also, semigroups of ($IV_1$) are obtained from factor semigroups of Croisot-Teisser semigroups.) one can show that the semigroup $\Lambda(S)$ of a Saito-Hori semigroup $S$ is of type (IV).

(2) If a semigroup $S$ satisfies ($IV_2$) and an additional condition (**) that all $S$-homomorphisms from nilpotent 0-minimal right ideals to non-nilpotent 0-minimal right ideals are the zero map, then it is easy to see that $\Lambda(S)$ is a semigroup of type (IV). A method of constructing a semigroup which is a 0-union of nilpotent 0-minimal right ideals and non-nilpotent 0-minimal right ideals from a 0-simple

semigroup with a non-nilpotent 0-minimal right ideal is described in [5, II, P.11-12]. This enables us to construct semigroups satisfying ($IV_2$) and (∗∗).

Following the last statement of Remark 4, we give a simple example of a semigroup of type (IV).

EXAMPLE. Let $G$ be a group and $H$ its subgroup such that there does not exist any normal subgroup between $H$ and $G$. Let $N = G/H$ denote the set of left cosets of $G$ mod $H$. Set $S = G \cup N \cup \{0\}$. Define multiplication on $S$ as follows: (1) $x \circ y = xy$ if $x, y \in G$, or $x \in N$, $y \in G$ (since $N$ is a right $G$-set), (2) $x \circ y = 0$ if $y \in N$, (3) $x \circ 0 = 0 \circ x = 0$. Then $S$ is a right reductive semigroup such that $S$ is a 0-union of a non-nilpotent right 0-minimal ideal $G^0$ and a nilpotent right 0-minimal ideal $N$, and all $S$-homomorphism of $N$ to $G$ are the zero map. Also it is easy to see that $\Lambda(S) = \{e\} \cup S$, $e^2 = e$, $en = ne = n$ for all $n \in N$, $N^2 = 0$ and $eg = ge = e$ for all $g \in G$. Then $\Lambda(S)$ is a semigroup of type (IV).

## §3. Strong reversibility

Howie [9] and Hall [6] proved that every inverse semigroup is an amalgamation base in the class of semigroups (hereafter, called a *semigroup amalgamation base*). Consequently, we know that every semigroup of type (I) is a semigroup amalgamation base. In the same paper [6], Hall also showed that every semigroup which is a semigroup amalgamation base has the representation extension property (*REP*) and its dual $(REP)^{op}$. The author [14] showed that every right self-injective semigroup has $(REP)^{op}$. On the other hand, Bulman-Fleming and McDowell [2] proved that every absolutely flat semigroup (that is, every left or right $S$-set is $S$-flat) is a semigroup amalgamation base (see also [3]). Bulman-Fleming and McDowell [4] introduced the stronger property "strong reversibility" than the property "Absolute flatness".

According to [4], a monoid $S$ is called *strongly left* [*right*] *reversible* if for any $x, y \in S$, there exists $z \in S$ such that $zx = x$ and $zy \in xS \cap yS$ [respectively, $xz = x$ and $yz \in Sx \cap Sy$], and *strongly reversible* if it is both strongly left and right reversible.

THEOREM 7. *Let $S$ be a right self-injective, right non-singular regular semigroup. Then $S$ is strongly left reversible.*

PROOF. Let $x, y \in S$. By lemma 3, there exist projections $e, f$ of $xS$, $xS \cap yS$, respectively. Then we shall show that $ey = fy$. So suppose that $ey \neq fy$. Clearly, there exists a right ideal $A$ of $S$ such that $A \cap fyS = 0$ and $A \cup fyS$ is intersection large in $yS$, so that the set $I = \{s \in S \mid ys \in A \cup fyS\}$ is an intersection large right ideal of $S$. Since $S$ is right non-singular, $I$ is dense and so, there eixsts $t \in S$ such that $yt \in I$ and $eyt \neq fyt$. However if $yt \in fyS$, then $eyt = efyt = fyt$, or if $yt \in A$, then $eyt = 0 = fyt$ (since $A \cap xS = 0$ and $e$, $f$ are projections of $xS, xS \cup yS$), a contradiction. Hence $ey = fy$. Now it is easily seen that $ex = x$ and $ey \in xS \cap yS$. Therefore $S$ is strongly left reversible. The therorem is proved.

Now it follows immediately from Theorem 2 and Theorem 7 that

THEOREM 8. *Every semigroup of type* (I) *is strongly reversible.*

From [4, Fleischer's Theorem], [3, Proposition 1.2] and [6, Theorems 3 and 4], we have

COROLLARY 2. *Every semigroup of type* (I) *is absolutely flat, and hence it is a semigroup amalgamation base.*

Further we have

THEOREM 9. Let $S$ be a *semigroup of* (II) *satisfying* (*). *Then*

(1) *$S$ is always strongly left reversible.*

(2) *$S$ is never strongly right reversible.*

PROOF. The part (1) of the theorem is from Theorem 7. The part (2) will follow from Theorem 3 and the next lemma.

LEMMA 9. *Let $S$ be a right reductive completely* 0*-simple semigroup. Then $\Lambda(S)$ is strongly right reversible if and only if $S$ is a group with zero.*

PROOF. Sufficiency: It is obvious.

Necessity: Assume that $\Lambda(S)$ is strongly right reversible. Then it suffices to show that $S$ is itself a 0-minimal right ideal. Suppose that $S$ contains two right 0-minimal right ideals $R_1$, $R_2$. Then there exist $f, g \in \Lambda(S)$ such that $f|_{R_1} = \iota_{R_1}$, $f(R) = 0$ for all 0-minimal right ideal $R$ other than $R_1$, and $g(R_1) = R_2$, $g(R_2) = R_2$ and $g(R') = 0$ for all 0-minimal right ideal $R'$ other than $R_1$, $R_2$. Then there does not exist $h \in \Lambda(S)$ such that $fh = f$ and $gh \in \Lambda(S)f \cap \Lambda(S)g$. For if there exists such $h \in \Lambda(S)$, then $gh(R_2) = 0$ (since $gh \in \Lambda(S)f$), and so, $gh(R_1) = ghg(R_1)$ (since $gh \in \Lambda(S)g) = gh(R_2) = 0$. Evidently, $h(R_1) \cap R_1 = 0$. Hence $fh(R_1) = 0$. This contradicts $fh = f$. The lemma is proved, and the proof of Theorem 9 is complete.

In a subsequent paper [16], we will discuss when semigroups of type (II) are semigroup amalgamation bases.

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