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TRIVIALLY AND LOCALLY TRIVIALLY $\mathcal{C}$ MAPS

DAVID BUHAGIAR, TAKUO MIWA AND BORIS A. PASYNKOV

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Abstract. In this work the authors give a possible systematic way of extending definitions from the category $\mathcal{TOP}$ to the category $\mathcal{TOP}_Y$, the object of research in Fibrewise General Topology. This is done by introducing the notions of *trivially $\mathcal{C}$-map* and *locally trivially $\mathcal{C}$-map*, where $\mathcal{C}$ is some class of topological spaces closed under homeomorphisms. Two classes of spaces are considered as the collection $\mathcal{C}$, the class of all metrizable spaces and the class of all linearly ordered topological spaces (i.e., $LOTS$). In particular, this method gives another possible way in defining a metrizable map, thus introducing the notion of TM-map (trivially metrizable) and its local version called LTM-map (locally trivially metrizable).

1. Introduction

Fibrewise General Topology (FGT) is a branch of General Topology which concerns itself with the study of the category $\mathcal{TOP}_Y$, the objects of which are continuous maps into a fixed topological space Y, and for the objects $f : X \to Y$ and $g : Z \to Y$, a morphism from f into g is a continuous map $\lambda : X \to Z$ with the property $f = g \circ \lambda$. This field of research can be justified by the fact that the two concepts of *topological space* and *continuous map* are equally important and one can look at a space as a map from this space onto a singleton space and in this manner identify these two concepts. Thus, the category $\mathcal{TOP}$ of topological spaces as objects, and continuous maps as morphisms, is a particular case of $\mathcal{TOP}_Y$, where Y is a singleton space.

Research in FGT is mainly aimed at extending the main notions and results concerning topological spaces to that of continuous maps. The carried out research showed a strong analogy in the behavior of spaces and maps and it was possible to extend the main notions and results of spaces to that of maps. In most cases there is some choice in defining properties on maps and one usually prefers the simplest and the one that gives the most complete generalization of the corresponding results in the category $\mathcal{TOP}$. It would be beneficial to have a more systematic way of extending definitions and results from the category $\mathcal{TOP}$ to the category $\mathcal{TOP}_Y$ and

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some hope is provided by the link between Fibrewise Topology and Topos Theory [7, 8, 9, 10]. Unfortunately, as was noted in [6], this approach has several drawbacks. In defining *compact maps* [12, Proposition 2.2 (V.P.Norin)], *paracompact maps* [1], *metacompact maps*, *subparacompact maps*, *submetacompact maps* [2] and *metrizable type maps* [3], one can see a systematic method in defining notions in the category $\mathcal{TOP}_Y$ corresponding to definitions which involve coverings or bases of topological spaces. This construction gave satisfactory definitions which can be seen from the results obtained for such maps [1, 2, 3, 12]. One can also add that the definitions of paracompact maps, metacompact maps, subparacompact maps and submetacompact maps strengthened the result that paracompactness, metacompactness, subparacompactness and submetacompactness are all inverse invariant of perfect maps. Namely, it was proved that the inverse image of a paracompact T_2 (resp. subparacompact, metacompact, submetacompact) space by a paracompact T_2 (resp. subparacompact, metacompact, submetacompact) map is paracompact T_2 (resp. subparacompact, metacompact, submetacompact) [1, 2].

In this work we give a possible systematic way of extending definitions from the category $\mathcal{TOP}$ to the category $\mathcal{TOP}_Y$. This is done by introducing the notions of *trivially $\mathcal{C}$-map* and *locally trivially $\mathcal{C}$-map*, where $\mathcal{C}$ is some class of topological spaces closed under homeomorphisms.

We consider two classes of spaces as the collection $\mathcal{C}$, the class of all metrizable spaces and the class of all linearly ordered topological spaces (i.e., $LOTS$). The authors have already introduced one possible way in defining a metrizable map [3], these maps are called MT-maps (metrizable type maps). The above mentioned method gives another possible way in defining a metrizable map, thus introducing the notion of TM-map (trivially metrizable) and its local version called LTM-map (locally trivially metrizable). Examples are given to clarify the definitions and results.

For undefined term related with FGT one can consult [1, 2, 3, 12]

2. Trivially and locally trivially $\mathcal{C}$ maps

Throughout this section the T_1 separation axiom is not assumed in the definition of *regular, completely regular, normal and collectionwise normal* space. Thus, for example, a T_3-space is a T_1 regular space.

Let $f : X \to Y$ be a continuous map and let τ be a fixed topology on the space Y. Also, let $\mathcal{C}$ be some class of topological spaces closed under homeomorphisms.

Definition 2.1. The map f is said to be *trivially $\mathcal{C}$ ($\equiv T\mathcal{C}$)* if it is parallel to a space $C \in \mathcal{C}$, i.e. there exists a space $C \in \mathcal{C}$ and an embedding $e : X \to Y \times C$ such that $f = pr_Y \circ e$, where $pr_Y : Y \times C \to Y$ is the projection of the product onto the factor Y. Thus e is an embedding of f into pr_Y.

Definition 2.2. The map f is said to be *locally trivially $\mathcal{C}$* ($\equiv LT\mathcal{C}$) if for any $y \in Y$, there exists a neighborhood O_y of y such that the restriction $f|_{f^{-1}O_y} : f^{-1}O_y \to O_y$ is a $T\mathcal{C}$-map and so, there exists a space $C_y \in \mathcal{C}$ and an embedding $e_y : f^{-1}O_y \to O_y \times C_y$ such that $f|_{f^{-1}O_y} = pr_{O_y} \circ e_y$.

Remark 2.1. One can note that in the definition of $LT\mathcal{C}$-map, the space $C_y \in \mathcal{C}$ can be different for every $f|_{f^{-1}O_y} : f^{-1}O_y \to O_y$.

3. Trivially metrizable maps

Let $f : X \to Y$ be a continuous map and let τ be a fixed topology on the space Y. Let $\mathcal{M}$ be the class of metrizable spaces.

Definition 3.1. The map f is said to be *trivially metrizable ($\equiv$ TM)* if it is a $T\mathcal{M}$-map, i.e. there exists a metrizable space M and an embedding $e : X \to Y \times M$ such that $f = pr_Y \circ e$. Thus e is an embedding of f into pr_Y. The map f is said to be *trivially metric ($\equiv$ TMC)* if M is a metric space and the embedding e is fixed.

Theorem 3.1. *A map $f : X \to Y$ is TM if and only if f is a T_0-map and there exists a pseudometric ρ on X such that $\tau(\rho)$, the topology on X generated by ρ, is a base for the map f.*

Proof. Let $f : X \to Y$ be a TM-map. Since f is parallel to a metrizable space then it is a T_0-map. Without loss of generality, one can assume that $X \subset Y \times M$ for some metrizable space M. We define a pseudometric ρ on X in the following way. For elements $x = (y, m)$ and $x' = (y', m')$ of X, let $\rho(x, x') = d(m, m')$, where d is a metric compatible with the topology of M. We now show that ρ is the required pseudometric. Below, by B_ρ and B_d we denote the neighborhood balls with respect to ρ and d respectively.

Let $U(x)$ be an arbitrary neighborhood of $x = (y, m)$. There exists some $O_y \in \tau$ and $\epsilon > 0$ such that $x \in (O_y \times B_d(m, \epsilon)) \cap X \subset U(x)$. Thus $x \in (pr_Y^{-1} O_y \cap pr_M^{-1} B_d(m, \epsilon)) \cap X = pr_Y^{-1} O_y \cap pr_M^{-1} B_\rho(m, \epsilon) \subset U(x)$.

Conversely, say f is a T_0-map and there exists a pseudometric ρ on X such that $\tau(\rho)$ is a base for the map f. We construct the following equivalence relation $\sim$ on X, $x_1 \sim x_2$ if $\rho(x_1, x_2) = 0$. Let X/ρ be the factor space and let $d(\bar{x}, \bar{x}') = \rho(x, x')$. It is not difficult to see that d is well defined and that it is a metric on X/ρ. Consider the map $i : X \to Y \times X/\rho$ defined by $i(x) = (f(x), \bar{x})$.

Since f is a T_0-map we have that i is 1–1 map. Let $U(x)$ be an arbitrary neighborhood of an element $x \in X$ and say $f(x) = y$. There exists $O_y \in \tau$ and $\epsilon > 0$ such that $x \in f^{-1}O_y \cap B_\rho(x, \epsilon) \subset U(x)$. Consequently $x \in i^{-1}(O_y \times B_d(\bar{x}, \epsilon)) \subset U(x)$. On the other hand, if $i(x) \in O_{f(x)} \times B_d(\bar{x}, \epsilon)$ then $i(x) \in i(f^{-1}O_{f(x)} \cap B_\rho(x, \epsilon)) \subset O_{f(x)} \times B_d(\bar{x}, \epsilon)$. □

Example 3.2. Let $X = S$ be the Sorgenfrey line and let $Y = \mathbb{R}$ be the real line with the right topology, that is the topology generated on $\mathbb{R}$ be the neighborhood system $\{R(x) : x \in \mathbb{R}\}$, where $R(x) = \{y \in \mathbb{R} : y \geqslant x\}$. Then the map $f = \mathrm{id}_\mathbb{R} : X \to Y$ is an TM-map. One can take the pseudometric ρ to be the standard metric on $\mathbb{R}$.

Although we cannot cite any reference to the following result, one can consider it as *mathematical folklore.*

Theorem 3.2. *The following are equivalent for a topological space X:*

(1) *X is pseudometrizable;*

(2) *X is regular and has a σ-locally finite base;*
(3) *X is regular and has a σ-discrete base;*
(4) *X is collectionwise normal and has a development;*
(5) *X has a normal development.*

We thus have the following result.

Theorem 3.3. *The following are equivalent for a continuous map $f : (X, \Omega) \to (Y, \tau)$:*

(1) *f is a TM-map;*
(2) *f is a T_0-map and there exists a regular topology $\Omega' \subset \Omega$ on X which has a σ-locally finite base and is a base for f;*
(3) *f is a T_0-map and there exists a regular topology $\Omega' \subset \Omega$ on X which has a σ-discrete base and is a base for f;*
(4) *f is a T_0-map and there exists a collectionwise normal topology $\Omega' \subset \Omega$ on X which has a development and is a base for f;*
(5) *f is a T_0-map and there exists a topology $\Omega' \subset \Omega$ on X which has a normal development and is a base for f.*

As is the case for metrizable spaces, one has the following result (for a proof, one can follow the proof of [Engelking, Theorem 4.4.15] for the case of metrizable spaces).

Proposition 3.4. *Pseudometrizability is an invariant of perfect maps.*

Following the above proposition, one would expect the following result to hold.

- Let $f : X \to Y$ be a TM-map and $g : Z \to Y$ a continuous map. Then if $\lambda : f \to g$ is a perfect morphism of f onto g, g is also a TM-map.

Unfortunately this is not the case since the morphism λ may not be a continuous map of the corresponding pseudometrizable space as the following example shows.

Example 3.3. Let Y be the space of all ordinal numbers $\leqslant \omega_1$, i.e. the compact $LOTS$ $[0, \omega_1]$, and let X be the product of Y with the two-point discrete space $D = \{0, 1\}$. Let f be the projection of X onto the factor Y. Now let Z be the quotient space $X/\{(\omega_1, 0), (\omega_1, 1)\}$ and let ϕ be the natural projection of X onto Z. Finally, let $g : Z \to Y$ be such a map satisfying $f = g \circ \phi$. Then, the map f is TM but g is not, though ϕ is perfect, because any continuous map of Y to a metric space is finally constant.

From the definition of TM-map we have the following result.

Theorem 3.5. *If $f : X \to Y$ is a TM-map and $Z \subset Y$ is metrizable then so is $f^{-1}Z$.*

The following result for products of TM-maps holds.

Theorem 3.6. *Let the maps $p_i : P_i \to Y$, $i < \omega$, be TM-maps. Then the projection $p : P = \prod\{P_i \ rel \ p_i : i < \omega\} \to Y$ is also a TM-map, where P is the fan product of P_i relative to the maps p_i.*

Proof. For each $i < \omega$ let ρ_i be a pseudometric on P_i whose topology $\tau(\rho_i)$ forms a base for p_i. Consider the pseudometric

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i}\rho_i(x_i, y_i)$$

on the Tychonoff product $X = \prod_{i=1}^{\infty} P_i$. We show that the topology $\tau(\rho)$ induced on P is a base for the map p.

Let $x = \{x_i\}$ be any point in P and let $U(x)$ be any neighborhood of x in P. Let $y = p_i(x_i) = p_j(x_j)$ for any $i, j < \omega$ and consider any open set $\hat{U}(x)$ in X such that $\hat{U}(x) \cap P = U(x)$. There exists $V_1, \dots, V_k$, open in $P_1, \dots, P_k$ respectively, such that $x \in \bigcap pr_i^{-1} V_i \subset \hat{U}(x)$. By definition, there exist a neighborhood O of y in Y and $\epsilon > 0$ such that $p_i^{-1}O \cap B_{\rho_i}(x_i, \epsilon) \subset V_i$ for every $i = 1, \dots, k$. Thus $p^{-1}O \cap (\bigcap_{i=1}^{k} pr_i^{-1}(B_{\rho_i}(x_i, \epsilon))) \subset U(x)$. Finally, since $\rho_i(x_i, y_i) < \epsilon$ whenever $\rho(x,y) < \frac{\epsilon}{2^i}$, we get $x \in p^{-1}O \cap B_\rho(x, \frac{\epsilon}{2^k}) \subset U(x)$. □

4. Locally trivially metrizable maps

We now define a local version of a TM-map. As in the above sections, let $f : X \to Y$ be a continuous map and let τ be a fixed topology on the space Y.

Definition 4.1. The map f is said to be *locally trivially metrizable* ($\equiv$ *LTM*) if it is an $LT\mathcal{M}$-map, i.e. for any $y \in Y$, there exists a neighborhood O_y of y such that the restriction $f|_{f^{-1}O_y} : f^{-1}O_y \to O_y$ is a TM-map.

Similar to Theorem 3.1 we have the following result.

Theorem 4.1. *A map $f : X \to Y$ is LTM if and only if f is a T_0-map and there exists an open cover $\mathcal{O} = \{O_\alpha : \alpha \in \mathcal{A}\}$ of Y and a collection of pseudometrics $\Xi = \{\rho_\alpha : \alpha \in \mathcal{A}\}$, where ρ_α is a pseudometric on $f^{-1}O_\alpha$, such that $\tau(\rho_\alpha)$ is a base for the map $f|_{f^{-1}O_\alpha}$ for every $\alpha \in \mathcal{A}$.*

Example 4.2. We now give an example of an LTM-map which is not a TM-map. Let X be the subset of the plane consisting of all points (x, y) with either x irrational and $y \geqslant 0$, or with $x = r_n$ (where $r_1, r_2, \dots$ is an enumeration of the rationals) and $0 < y \leqslant \frac{1}{n}$. Let $A = \{(x, y) \in X : x \text{ rational }\}$ and $B = \{(x, y) \in X : y = 0\}$. We will now construct the following topologies on the set X.

- A basic neighborhood of (x, y) in X is a vertical open interval about (x, y) if x is irrational, and is an ordinary plane neighborhood of (x, y) in X if x is rational. This defines a completely regular topology Ω_T on X (see [4]).
- Let Ω_B be the topology on the set X whose open sets are $X, \emptyset, G = X \setminus A, H = X \setminus B$ and $G \cap H$.

Now we consider the map $f = \mathrm{id}_X : (X, \Omega_T) \to (X, \Omega_B)$. This map is an LTM-map but not a TM-map.

Assume that f is a TM-map and so there exists a regular topology $\Omega' \subset \Omega_T$ on X which has a σ-discrete base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n < \omega\}$, where each $\mathcal{B}_n$ is discrete with respect to Ω', and is a base for the map f. Let $\hat{\mathcal{B}}_n = \mathcal{B}_n \wedge H$, $\mathcal{B}'_m = \mathcal{B}_m \wedge G$ and

$\hat{\mathcal{B}'}_k = \mathcal{B}_k \wedge (G \cap H)$. Finally, let $\widetilde{\mathcal{B}} = \bigcup\{\hat{\mathcal{B}}_n, \mathcal{B}'_m, \hat{\mathcal{B}'}_k : n, m, k < \omega\}$. Then $\widetilde{\mathcal{B}}$ is a σ-discrete base for some topology $\widetilde{\Omega} \subset \Omega_T$.

Take any point $(x, y) \in X$ with rational x and let W be a basic Ω_T neighborhood of (x, y). By definition, there exists $U \in \mathcal{B}$ such that $(x, y) \in U \cap H \subset W$ and so there exists some n and $U' \in \hat{\mathcal{B}}_n$ such that $(x, y) \in U' \subset W$.

Now let $(x, y) \in X$ with irrational x and let W be a basic Ω_T neighborhood of (x, y). There can be two cases, either $y = 0$ or $y > 0$. Analogous to the above, but using $\mathcal{B}'_m$ or $\hat{\mathcal{B}'}_k$, there exists a $U' \in \widetilde{\mathcal{B}}$ such that $(x, y) \in U' \subset W$.

This proves that the space $(X, \widetilde{\Omega})$ is homeomorphic to the space (X, Ω_T) and so is a T_3-space. Thus $(X, \widetilde{\Omega})$, an so (X, Ω_T), is a metrizable space which contradicts the fact that (X, Ω_T) is not a normal space (see [4]).

The proof that f is an LTM-map follows from the fact that the subspaces G and H are metrizable.

Of course one can also come up with local variants of the characterizations given in Theorem 3.3 for TM-maps.

As in Theorem 3.5 we have the following result.

Theorem 4.2. *If $f : X \to Y$ is an LTM-map and $Z \subset Y$ is metrizable then so is $f^{-1}Z$.*

Proof. Without loss of generality we prove the result for $Z = Y$. By the hypothesis, there exists a collection $\mathcal{O} = \{O_y : y \in Y\}$ such that $f^{-1}O_y$ is metrizable for every y. Since Y is paracompact, there exists a closed locally finite refinement $\mathcal{F}$ of $\mathcal{O}$. Finally, the proof follows from the fact that a space which is the union of a locally finite collection of closed metrizable subspaces is metrizable (see for example [5]). □

As mentioned in the introduction, the authors have already introduced one possible way in defining a metrizable map, these maps are called MT-maps (metrizable type maps). The notion of MT-space was also introduced as the MT-map preimage of metrizable spaces. For details concerning MT-maps and MT-spaces one can consult [3]. Any non metrizable MT-space (for example $X = I \times I$) gives an MT-map which is not an LTM-map (and so neither a TM-map). In fact the example $X = I \times I$ with the lexicographic order topology gives a compact MT-map (a CMT-map) which is not LTM (and so neither TM). Conversely, if $f_1 = id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \{0\} \subset \mathbb{R}$, then $f_1 \times f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \{0\}$ is a TM-map (and so also LTM-map) which is not an MT-map (since it is not closed). On the other hand, any closed LTM-map (and so, any closed TM-map) is an MT-map.

5. GO-maps and LOTM

Let there be given a set X and a relation among its elements written as $x \leqslant y$. Consider the following four conditions (of reflexivity, antisymmetry, transitivity and connectedness):

(1) for all $x \in X$, $x \leqslant x$;

(2) if $x \leqslant y$ and $y \leqslant x$, then $x = y$;
(3) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$;
(4) for each pair x, y, either $x \leqslant y$ or $y \leqslant x$.

It is well known that if conditions (1)–(3) are satisfied, we say that the relation $x \leqslant y$ is an *ordering* of X (or that the set X is *ordered*); the relation $x \leqslant y$ is a *quasi-ordering* if it satisfies conditions (1) and (3) only; it is a *linear ordering* if it satisfies conditions (1)–(4). We now come to the following definition.

Definition 5.1. The relation $x \leqslant y$ is a *pseudo-ordering* of X if it satisfies conditions (1), (3) and (4).

Remark 5.1. One can note that in this case one can have two distinct elements $x, y \in X$ satisfying $x < y$ and $y < x$.

For $a, b \in X$ satisfying $a < b$ and $b \not< a$ let

$$\begin{aligned}]a, b[&= \{x \in X : a < x < b, x \not< a, b \not< x\}; \\ ]a, \rightarrow [&= \{x \in X : a < x, x \not< a\}; \\ ] \leftarrow, b[&= \{x \in X : x < b, b \not< x\}; \\ [a, \rightarrow [&= \{x \in X : a \leqslant x\}; \\ ] \leftarrow, b] &= \{x \in X : x \leqslant b\}. \end{aligned}$$

Definition 5.2. Let X be a set equipped with a pseudo-order $\leqslant$. The set X with the topology generated by the subsets $]a, b[, X$ and $\emptyset$ as a base is called a *pseudo LOTS* ($\equiv$ *PLOTS*). We denote this topology by $\lambda(\leqslant)$.

We now consider arbitrary subsets R and L of X and for each $x \in R$ we consider the set $[x, \rightarrow [$, while for each $x \in L$ we consider the set $] \leftarrow, x]$. Let $\tau(R, L)$ be the topology on X generated by $\lambda(\leqslant)$, $\{[x, \rightarrow [: x \in R\}$ and $\{] \leftarrow, x] : x \in L\}$. A space X with such a topology is called a *pseudo generalized ordered space* ($\equiv$ *PGO-space*). It can be easily seen that every *PGO*-space can be embedded into a *PLOTS* as a closed subspace (or as a dense subspace). Also, every subspace of a *PLOTS* is a *PGO*-space.

Although *PLOTS* and *PGO*-spaces are of interest on their own, our main interest in this paper is their application to fibrewise topology. Let $\mathcal{LOTS}$ be the class of *linearly ordered topological spaces* ($\equiv$ *LOTS*).

Definition 5.3. The map $f : X \to Y$ is said to be a *trivially GO- ($\equiv$ TGO-) map* if it is a $T\mathcal{LOTS}$-map, i.e. there exists a *LOTS* Z and an embedding $e : X \to Y \times Z$ such that $f = pr_Y \circ e$. Thus e is an embedding of f into pr_Y. The map f is said to be *trivially LOTM ($\equiv$ TLOTM)* if f is a *TGO*-map and each fibre is a *LOTS*.

Theorem 5.1. *A map $f : X \to Y$ is a TGO-map if and only if f is a T_0-map and there exists a pseudo-ordering $\leqslant$ of X such that $\tau(R, L)$, is a base for the map f for some subsets $R, L \subset X$.*

Proof. Let $f : X \to Y$ be a *TGO*-map. Since f is parallel to a *LOTS* Z then it is a T_0-map. Without loss of generality, one can assume that $X \subset Y \times Z$. We define a pseudo-ordering $\leqslant$ of X in the following way. For elements $x = (y, z)$ and $x' = (y', z')$ of X, let $x \leqslant x'$ if $z \preccurlyeq z'$, where $\preccurlyeq$ is a linear ordering of Z generating its topology. We now show that $\leqslant$ is the required pseudo-order.

Let $U(x)$ be an arbitrary neighborhood of $x = (y, z)$. There exists some $O_y \in \tau$ and $a, b \in Z$ such that $x \in (O_y \times]a, b[_{\preccurlyeq}) \cap X \subset U(x)$. Thus $x \in (pr_Y^{-1} O_y \cap pr_Z^{-1}(]a, b[_{\preccurlyeq})) \cap X \subset U(x)$. We now define the subsets L and R of X included in the definition of the topology $\tau(R, L)$. By definition, $x = (y, z) \in L$, i.e. $] \leftarrow, x]_{\leqslant} \in \tau(R, L)$, if there exists $a \in Z$ satisfying:

(i) there does not exist any $x' \in X$ with $x' = (y', a)$; and

(ii) $] \leftarrow, x]_{\leqslant} = (Y \times] \leftarrow, a[_{\preccurlyeq}) \cap X$.

Similarly, one defines the subset R. One can now easily see that $\tau(R, L)$ is the required topology.

Conversely, say f is a T_0-map and there exists a pseudo-ordering $\leqslant$ of X such that $\tau(R, L)$ is a base for the map f, where $\tau(R, L)$ is a PGO-topology relative to the pseudo-linear topology $\lambda(\leqslant)$. We construct the following equivalence relation $\sim$ on X, $x_1 \sim x_2$ if $x_1 \leqslant x_2$ and $x_2 \leqslant x_1$. Let $X/\!\sim$ be the quotient space and let $\bar{x} \preccurlyeq \bar{x}'$ if $x \leqslant x'$. We further take the Dedekind completion $(X/\!\sim)^*$, which is a $LOTS$, and denote its order by $\preccurlyeq'$. Consider the map $i : X \to Y \times (X/\!\sim)^*$ defined by $i(x) = (f(x), \bar{x})$.

Since f is a T_0-map we have that i is 1–1 map. Let $U(x)$ be an arbitrary neighborhood of an element $x \in X$ and say $f(x) = y$. Assume that there exists $O_y \in \tau$ and $a, b \in X$, satisfying $a < b$ and $b \not\leqslant a$, such that $x \in f^{-1}O_y \cap]a, b[_{\leqslant} \subset U(x)$. Consequently, $x \in i^{-1}(O_y \times]\bar{a}, \bar{b}[_{\preccurlyeq'}) \subset U(x)$. Other cases can be treated similarly. On the other hand, if $i(x) \in O_{f(x)} \times]\bar{a}, \bar{b}[_{\preccurlyeq'}$ and $]\bar{a}, \bar{b}[_{\preccurlyeq'} \cap X/\!\sim = [\bar{c}, \bar{b}[_{\preccurlyeq}$ (other cases can be treated similarly), then $i(x) \in i(f^{-1}O_{f(x)} \cap [c, b[_{\leqslant}) \subset O_{f(x)} \times]\bar{a}, \bar{b}[_{\preccurlyeq'}$, for any $c \in \bar{c}$ and any $b \in \bar{b}$. $\square$

Definition 5.4. The map f is said to be a *locally trivially GO- ($\equiv$ LTGO-) map* if it is an $LT\mathcal{LOTS}$-map, i.e. for any $y \in Y$, there exists a neighborhood O_y of y such that the restriction $f|_{f^{-1}O_y} : f^{-1}O_y \to O_y$ is a TGO-map. Similarly one can define an $LTLOTM$.

Remark 5.2. If X is a GO-space, then the projection $pr_X : X \times X \to X$ is a TGO-map onto a GO-space while the total space $X \times X$ is not generally a GO-space. Similarly, if X is a $LOTS$, then one obtains a $TLOTM$ onto a $LOTS$ where the total space is not generally a $LOTS$. Naturally, any constant map $p : X \to \{p\}$, where X is a non-$LOTS$, GO-space, gives us a TGO-map which is not a $TLOTM$.

Example 5.5. Let X be the 1-dimensional sphere $\mathbb{S}^1$ with its usual topology (as a subset of $\mathbb{R}^2$). Consider the points $p_1 = (0, 1), p_{-1} = (0, -1) \in \mathbb{S}^1$ and let Y be the set $\mathbb{S}^1$ with the topology having the following open sets as a base $\{\mathbb{S}^1, \emptyset, \mathbb{S}^1 \setminus \{p_1\}, \mathbb{S}^1 \setminus \{p_{-1}\}\}$. Then the identity map $f \equiv \mathrm{id} : X \to Y$ is a $LTLOTM$ but is not a $TLOTM$ (neither a TGO-map).

Indeed, consider an embedding $e : \mathbb{S}^1 \to Y \times L$, where L is a $LOTS$, such that $f = pr_Y \circ e$. Then $e(\mathbb{S}^1) \subset Y \times L$. It is not difficult to prove that for every two distinct elements $s_1, s_2 \in \mathbb{S}^1$ we have that $e(s_1) = (y_1, l_1), e(s_2) = (y_2, l_2)$ with

$l_1 \neq l_2$. That is, the composition $pr_L \circ e$, where $pr_L : Y \times L \to L$ is the projection, is a 1-1 map. One can also note that $pr_L(e(\mathbb{S}^1))$ is connected and compact and therefore, is a $LOTS$ with no gaps nor jumps (see for example [11]). In particular, $pr_L(e(\mathbb{S}^1))$ has a maximal point which we denote by l_x. Let $s_x = (u, v)$ be the unique element of $\mathbb{S}^1$ satisfying $pr_L(e(s_x)) = l_x$, and let $A = \{z \in \mathbb{S}^1 : d(s_x, z) < \epsilon\}$ for sufficiently small $\epsilon > 0$, where d is the usual distance in $\mathbb{S}^1$. Then A is a connected subset of $\mathbb{S}^1$ and therefore, $pr_L(e(A))$ is also connected and contains l_x. Thus, $pr_L(e(A))$ is a convex open neighborhood of l_x. Take any two points a and b in A on opposite sides of s_x satisfying $d(a, s_x) < d(s_x, b) < \epsilon$, without loss of generality one can assume that $pr_L(e(a)) < pr_L(e(b))$. Now let $B = \{z \in \mathbb{S}^1 : d(s_x, z) < d(s_x, b)\}$, then B is a connected open neighborhood of s_x containing a but not b, and therefore, $pr_L(e(B))$ is an open connected neighborhood of l_x containing $pr_L(e(a))$ but not $pr_L(e(b))$, which is a contradiction.

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Mathematics Department, Faculty of Science, University of Malta, Msida MSD.06, Malta

E-mail address: `david.buhagiar@um.edu.mt`

Department of Mathematics, Shimane University, Matsue 690-8504, Japan

E-mail address: `miwa@riko.shimane-u.ac.jp`

Department of General Topology, Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia

E-mail address: `pasynkov@mech.math.msu.su`