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THE CLASSIFICATION OF SIMPLE IRREDUCIBLE PSEUDO-HERMITIAN SYMMETRIC SPACES: FROM A VIEWPOINT OF ELLIPTIC ORBITS

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ABSTRACT. In this paper, we call a special elliptic element an *Spr*-element, we define an equivalence relation on the set of *Spr*-elements of a real form of a complex simple Lie algebra, and we classify *Spr*-elements of each real form of all complex simple Lie algebras under the equivalence relation. Besides, we demonstrate that the classification of *Spr*-elements under the equivalence relation corresponds to the classification of simple irreducible pseudo-Hermitian symmetric Lie algebras under Berger's equivalence relation. In terms of the correspondence, we achieve the classification of simple irreducible pseudo-Hermitian symmetric Lie algebras without Berger's classification.

1. INTRODUCTION

Let $\mathfrak{g}$ be a real simple Lie algebra, let σ be an involutive automorphism of $\mathfrak{g}$, and let $\mathfrak{h}$ be the +1-eigenspace of σ in $\mathfrak{g}$. Then, the pair $(\mathfrak{g}, \sigma)$ or $(\mathfrak{g}, \mathfrak{h})$ is said to be a simple *symmetric pair* or *symmetric Lie algebra*. This pair corresponds to an infinitesimal version of a simple (affine) symmetric space, and these pairs have been already classified by Berger [Be] in 1957. Notice that he achieves the classification under the following equivalence relation (cf. Definition 7.2 in [Be, pp. 96]):

> **Berger's equivalence relation**. Let $(\mathfrak{g}, \sigma_1)$ and $(\mathfrak{g}, \sigma_2)$ be two symmetric Lie algebras. Then $(\mathfrak{g}, \sigma_1)$ is said to be *ext-isomorphic* to $(\mathfrak{g}, \sigma_2)$, if there exists an automorphism ϕ of $\mathfrak{g}$ such that $\sigma_2 = \phi \circ \sigma_1 \circ \phi^{-1}$.

In the same paper [Be], he has introduced the notion of pseudo-Hermitian symmetric space: A symmetric space G/H is called *pseudo-Hermitian*, if it admits an invariant complex structure J and an invariant pseudo-Hermitian metric g (with respect to J). Simple pseudo-Hermitian symmetric Lie algebras are infinitesimal

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versions of simple pseudo-Hermitian symmetric spaces (see Definition 2.1.3 for detail). If $(\mathfrak{g}, \sigma_1)$ is ext-isomorphic to $(\mathfrak{g}, \sigma_2)$, and if $(\mathfrak{g}, \sigma_1)$ is pseudo-Hermitian, then so is $(\mathfrak{g}, \sigma_2)$. Therefore, simple pseudo-Hermitian symmetric Lie algebras have been classified by Berger, under the equivalence relation mentioned above.

In 1971, Shapiro [Sh] has clarified relation between semisimple pseudo-Hermitian symmetric spaces and elliptic (adjoint) orbits, which is as follows: For every almost effective, semisimple pseudo-Hermitian symmetric space L/R, there exists an elliptic element $T \in \mathfrak{l} = \mathrm{Lie}(L)$ such that R is the centralizer $C_L(T)$ of T in L. Hence, it follows that any almost effective semisimple pseudo-Hermitian symmetric space L/R is an elliptic orbit $\mathrm{Ad}(L)T = L/C_L(T)$. Note that every elliptic orbit is not always a pseudo-Hermitian symmetric space—for example, $G_{2(2)}/U(2)$ is an elliptic orbit but it can not be a pseudo-Hermitian symmetric space (see Example 4.1.3). Therefore, an adjoint orbit through a special elliptic element is a pseudo-Hermitian symmetric space. Expressing respect to Shapiro, we want to call the following special elliptic element S an Spr-element: Let $\mathfrak{l}$ be a real semisimple Lie algebra, and let S be a semisimple element of $\mathfrak{l}$. Then, $\mathfrak{l}$ is decomposed as

$$\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(S) \oplus [S, \mathfrak{l}].$$

In the setting, we call the semisimple element $S \in \mathfrak{l}$ an *Spr-element*, if $\mathrm{ad}_{\mathfrak{l}} S|_{[S,\mathfrak{l}]}$ is a complex structure on the vector space $[S, \mathfrak{l}]$. By the definition of Spr-element, the following five items are deduced (see Remark 2.1.2, Lemma 2.1.6, Remark 2.1.7-(1) and Lemma 3.1.1):

- Any Spr-element $S \in \mathfrak{l}$ is a non-zero elliptic element.
- The canonical central element of $\mathfrak{r}$ relative to $(\mathfrak{l}, \mathfrak{r})$ is an Spr-element of $\mathfrak{l}$, for each semisimple pseudo-Hermitian symmetric Lie algebra $(\mathfrak{l}, \mathfrak{r})$.
- An inner automorphism $\rho := \exp \pi \,\mathrm{ad}_{\mathfrak{l}} S$ of $\mathfrak{l}$ is involutive and its $+1$ (resp. -1)-eigenspace accords with $\mathfrak{c}_{\mathfrak{l}}(S)$ (resp. $[S, \mathfrak{l}]$).
- The pair $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ is the pseudo-Hermitian symmetric Lie algebra by the involution ρ, and S is the canonical central element of $\mathfrak{c}_{\mathfrak{l}}(S)$ relative to $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$.
- Let L be a connected semisimple Lie group with Lie algebra $\mathfrak{l}$. For each almost effective pseudo-Hermitian symmetric space L/R, there exists an Spr-element $S \in \mathfrak{l}$ such that L/R coincides with the adjoint orbit $\mathrm{Ad}(L)S = L/C_L(S)$. Conversely, the adjoint orbit $\mathrm{Ad}(L)S' = L/C_L(S')$ through any Spr-element $S' \in \mathfrak{l}$ is a pseudo-Hermitian symmetric space.

Here in the above items, we assume the element S to be an Spr-element of a real semisimple Lie algebra $\mathfrak{l}$, without the statement. These items suggest that the classification of simple pseudo-Hermitian symmetric Lie algebras under Berger's equivalence relation corresponds to a classification of Spr-elements of each real simple Lie algebra under some equivalence relation.

The main purpose of this paper is to deduce Theorem 5.6.10, which is the classification of Spr-elements of each real form $\mathfrak{g}$ of all complex simple Lie algebras under the following equivalence relation:

Our equivalence relation. Let S_1 and S_2 be two Spr-elements of $\mathfrak{g}$. We say that S_1 is *equivalent* to S_2, if there exists an automorphism ϕ of $\mathfrak{g}$ satisfying $\phi(S_1) = S_2$ or $\phi(S_1) = -S_2$.

Let us comment on a correspondence between our equivalence relation and Berger's equivalence one. Fix a real form $\mathfrak{g}$ of a complex simple Lie algebra, denote by $Spr_{\mathfrak{g}}$ the set of Spr-elements of $\mathfrak{g}$, and denote by $\mathrm{Inv}(\mathfrak{g})^{pH}$ the set of involutions σ of $\mathfrak{g}$ such that $(\mathfrak{g}, \sigma)$ is a pseudo-Hermitian symmetric Lie algebra. Let $Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$ and $\mathrm{Inv}(\mathfrak{g})^{pH}/\mathrm{Aut}(\mathfrak{g})$ be the quotient set of $Spr_{\mathfrak{g}}$ by our equivalence relation and of $\mathrm{Inv}(\mathfrak{g})^{pH}$ by Berger's equivalence relation, respectively. Then, the following mapping F_1 is a bijection of $Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$ onto $\mathrm{Inv}(\mathfrak{g})^{pH}/\mathrm{Aut}(\mathfrak{g})$ (see Theorem 3.2.1):

$$\begin{array}{rcll} F_1 : Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) & \longrightarrow & \mathrm{Inv}(\mathfrak{g})^{pH}/\mathrm{Aut}(\mathfrak{g}) & \quad \text{(bijective)} \\ [\,S\,] & \mapsto & [\exp \pi \, \mathrm{ad}_{\mathfrak{g}} S\,]. & \end{array}$$

Consequently, the classification of Spr-elements under our equivalence relation is on a parity with that of simple irreducible pseudo-Hermitian symmetric Lie algebras under Berger's equivalence relation. Hence, two Theorems 3.2.1 and 5.6.10 enable us to achieve the classification of simple irreducible pseudo-Hermitian symmetric Lie algebras without Berger's classification (see Corollary 5.6.11). Here, a simple pseudo-Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{r})$ is irreducible (resp. reducible) if and only if $\mathfrak{g}^{\mathbb{C}}$ is simple (resp. $\mathfrak{g}^{\mathbb{C}}$ is not simple) (cf. Shapiro [Sh, pp. 532]).

Remark that Wolf [Wo] has achieved the classification of Hermitian symmetric spaces without É. Cartan's classification of Riemannian symmetric spaces, and his results in [Wo] enable us to complete the classification of not only Hermitian symmetric Lie algebras but also simple reducible pseudo-Hermitian symmetric Lie algebras without Berger's classification. Consequently, this paper and Wolf's paper [Wo] enable us to completely determine all simple pseudo-Hermitian symmetric Lie algebras without Berger's classification.

This paper consists of six sections, and an outline of each section is as follows:

§2 Preliminaries.
  In this section, we recall the notion of elliptic element, the canonical central element, and so forth; and we introduce Murakami's setting utilized in [Mu1] and [Mu3].

§3 Semisimple pseudo-Hermitian symmetric spaces and Spr-elements.
  This section is devoted to investigating relation between semisimple pseudo-Hermitian symmetric Lie algebras (or symmetric spaces) and Spr-elements of real semisimple Lie algebras.

§4 Necessary and sufficient conditions for an elliptic element to be an Spr-element.
  In this section, we provide necessary and sufficient conditions for an elliptic element to be an Spr-element. Consequences obtained in this section will play an important role in Section 5.

§5 The classification of Spr-elements of each simple Lie algebra.

In this section, we achieve the classification of *Spr*-elements of each real form of all complex simple Lie algebras under our equivalence relation (cf. Theorem 5.6.10).

§6 A classification of simple irreducible pseudo-Hermitian symmetric spaces.

Finally in this section, we define an equivalence relation on the set of simple irreducible pseudo-Hermitian symmetric spaces $(G/R, \Sigma, J, \mathrm{g})$, where Σ is an involution of G such that $(G_\Sigma)_0 \subset R \subset G_\Sigma$ (see page 112 for $(G_\Sigma)_0$ and G_Σ). Moreover, we give a correspondence between the equivalence relation on the set of $(G/R, \Sigma, J, \mathrm{g})$ and that on the set of *Spr*-elements of $\mathfrak{g} = \mathrm{Lie}(G)$ (cf. Theorem 6.2.1).

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2. Preliminaries

This section consists of four subsections. In Subsection 2.1, we give definitions used in this paper, and we demonstrate Lemma 2.1.6. In Subsection 2.2, we introduce Murakami's setting utilized in [Mu1] and [Mu3]. Subsection 2.3 is a review of elementary facts about root theory. Lastly in Subsection 2.4, we enumerate notation utilized in this paper.

2.1. Definitions.

In this subsection, we recall the notion of elliptic element, the canonical central element, and so on. Moreover, we study properties of the canonical central elements (see Lemma 2.1.6).

Definition 2.1.1 (Kobayashi [Kt, pp. 5]). Let $\mathfrak{l}$ be a real semisimple Lie algebra. An element $X \in \mathfrak{l}$ is called *semisimple*, if $\mathrm{ad}_{\mathfrak{l}} X$ is a semisimple endomorphism of $\mathfrak{l}$. A semisimple element $T \in \mathfrak{l}$ is said to be *elliptic*, if the eigenvalues of $\mathrm{ad}_{\mathfrak{l}} T$ are all purely imaginary. Let L be a connected Lie group with Lie algebra $\mathfrak{l}$. The adjoint orbit $\mathrm{Ad}(L)T$ through an elliptic element $T \in \mathfrak{l}$ is said to be an *elliptic orbit.*

Remark 2.1.2. It is possible to restate the definition of *Spr*-element as follows (see Section 1 for the definition of *Spr*-element): Let $\mathfrak{l}$ be a real semisimple Lie algebra. A semisimple element $S \in \mathfrak{l}$ is an *Spr*-element if and only if $S \neq 0$ and the eigenvalue of $\mathrm{ad}_{\mathfrak{l}} S$ is $\pm i$ or zero. Hence, an *Spr*-element of $\mathfrak{l}$ is a non-zero elliptic element, and the set of *Spr*-elements of $\mathfrak{l}$ is invariant under the action of $\{\pm 1\}$ and $\mathrm{Aut}(\mathfrak{l})$.

Definition 2.1.3 (Berger [Be, pp. 94]). Let $(\mathfrak{l}, \sigma)$ be a semisimple symmetric Lie algebra, and let $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{q}$ be its canonical decomposition, where $\mathfrak{r} := \{R \in \mathfrak{l} \,|\, \sigma(R) = R\}$ and $\mathfrak{q} := \{Q \in \mathfrak{l} \,|\, \sigma(Q) = -Q\}$. Then, the symmetric Lie algebra $(\mathfrak{l}, \sigma)$ or $(\mathfrak{l}, \mathfrak{r})$

is called *pseudo-Hermitian*, if there exist an $\mathrm{ad}_{\mathfrak{l}}\,\mathfrak{r}$-invariant complex structure I on $\mathfrak{q}$ and an $\mathrm{ad}_{\mathfrak{l}}\,\mathfrak{r}$-invariant pseudo-Hermitian form (with respect to I) on $\mathfrak{q}$. Remark that a symmetric Lie algebra $(\mathfrak{l}, \sigma_2)$ is pseudo-Hermitian in case of being ext-isomorphic to a pseudo-Hermitian symmetric Lie algebra $(\mathfrak{l}, \sigma_1)$.

Definition 2.1.4 (Shapiro [Sh, pp. 533]). Let $(\mathfrak{l}, \mathfrak{r})$ be a semisimple pseudo-Hermitian symmetric Lie algebra, and let $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{q}$ be its canonical decomposition. Then a central element Z of $\mathfrak{r}$ (i.e., an element Z which belongs to the center of $\mathfrak{r}$) is called *the canonical central element* of $\mathfrak{r}$ relative to $(\mathfrak{l}, \mathfrak{r})$, if $\mathrm{ad}_{\mathfrak{l}}\,Z|_{\mathfrak{q}}$ is a complex structure on $\mathfrak{q}$. Remark that the canonical central element is defined under existence of a semisimple pseudo-Hermitian symmetric Lie algebra, so that there is an essential difference between the canonical central elements and *Spr*-elements.

Remark 2.1.5 (Shapiro [Sh, pp. 534]). Shapiro's result assures the following:

(I) For any almost effective semisimple pseudo-Hermitian symmetric space L/R, there exists the canonical central element Z of $\mathfrak{r}$ relative to $(\mathfrak{l}, \mathfrak{r})$ such that (a) $R = C_L(Z)$ and (b) $\mathrm{ad}_{\mathfrak{l}}\,Z$ induces the complex structure J, where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{r} = \mathrm{Lie}(R)$.

(I′) For any effective semisimple pseudo-Hermitian symmetric Lie algebra $(\mathfrak{l}, \mathfrak{r})$, there exists the canonical central element Z of $\mathfrak{r}$ relative to $(\mathfrak{l}, \mathfrak{r})$.

Now, we will investigate properties of the canonical central elements.

Lemma 2.1.6. *Let $(\mathfrak{l}, \sigma)$ be a semisimple pseudo-Hermitian symmetric Lie algebra, let $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{q}$ be its canonical decomposition (where $\mathfrak{r} := \{R \in \mathfrak{l} \,|\, \sigma(R) = R\}$, $\mathfrak{q} := \{Q \in \mathfrak{l} \,|\, \sigma(Q) = -Q\}$), and let Z be the canonical central element of $\mathfrak{r}$ relative to $(\mathfrak{l}, \mathfrak{r})$. Then, the following three items hold*:

(1) *Z is a semisimple element of $\mathfrak{l}$.*
(2) *$\mathfrak{r} = \mathfrak{c}_{\mathfrak{l}}(Z)$ and $\mathfrak{q} = [Z, \mathfrak{l}]$.*
(3) *$\sigma = \exp \pi\, \mathrm{ad}_{\mathfrak{l}}\, Z$, where $\exp \pi\, \mathrm{ad}_{\mathfrak{l}}\, Z$ is the inner automorphism of $\mathfrak{l}$ determined by an element $\pi \cdot Z \in \mathfrak{l}$.*

Therefore by (1) *and* (2), *the canonical central element Z is an Spr-element of $\mathfrak{l}$.*

Proof. (1) The first item has been already demonstrated by Shapiro (ref. pp. 531, line 22 on [Sh]).

(2) Let us prove that the second item holds. It is immediate from (1) that $\mathfrak{l}$ is decomposed as follows:

$$\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(Z) \oplus [Z, \mathfrak{l}] \tag{2.1.1}$$

(see Notation (n6) in Subsection 2.4, for $\mathfrak{c}_{\mathfrak{l}}(Z)$). Since Z belongs to the center of $\mathfrak{r}$, one perceives that

$$\mathfrak{r} \subset \mathfrak{c}_{\mathfrak{l}}(Z). \tag{2.1.2}$$

The restriction of $\mathrm{ad}_{\mathfrak{l}}\,Z$ to $\mathfrak{q}$ is a linear isomorphism of $\mathfrak{q}$ by Definition 2.1.4; and thus

$$\mathfrak{q} \subset [Z, \mathfrak{l}].$$

This, together with (2.1.1), (2.1.2) and $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{q}$, deduces the second item.

(3) We aim to show the last item. Since $\mathrm{ad}_{\mathfrak{l}} Z|_{\mathfrak{q}}$ is a complex structure on $\mathfrak{q}$, one confirms that $(\mathrm{ad}_{\mathfrak{l}} Z)^2 Q = -Q$ for any $Q \in \mathfrak{q}$; and so

$$\begin{aligned}
\exp \pi\, \mathrm{ad}_{\mathfrak{l}} Z(Q) &= \sum_{l \geq 0} \frac{1}{l\,!} (\pi\, \mathrm{ad}_{\mathfrak{l}} Z)^l (Q) \\
&= \sum_{m \geq 0} \frac{1}{2m!} (\pi\, \mathrm{ad}_{\mathfrak{l}} Z)^{2m} (Q) + \sum_{n \geq 0} \frac{1}{(2n+1)!} (\pi\, \mathrm{ad}_{\mathfrak{l}} Z)^{2n+1} (Q) \\
&= \sum_{m \geq 0} (-1)^m \cdot \frac{\pi^{2m}}{2m!} \cdot Q + \sum_{n \geq 0} (-1)^n \cdot \frac{\pi^{2n+1}}{(2n+1)!} \cdot [Z, Q] \\
&= \cos \pi \cdot Q + \sin \pi \cdot [Z, Q] \\
&= -Q.
\end{aligned}$$

On the other hand, it follows from (2.1.2) that

$$\exp \pi\, \mathrm{ad}_{\mathfrak{l}} Z(R) = R$$

for every element $R \in \mathfrak{r}$. These, combined with $\mathfrak{l} = \mathfrak{r} \oplus \mathfrak{q}$, allow us to see that $\exp \pi\, \mathrm{ad}_{\mathfrak{l}} Z$ is an involution of $\mathfrak{l}$, and that $\sigma = \exp \pi\, \mathrm{ad}_{\mathfrak{l}} Z$. So, the last item has been shown. Hence, we have proved Lemma 2.1.6. □

Remark 2.1.7. (1) Remark 2.1.5-(I) and Lemma 2.1.6 imply the following: Let L/R be an almost effective, semisimple pseudo-Hermitian symmetric space defined by an involutive automorphism Σ of L. Then, there exists an Spr-element $S \in \mathfrak{l} = \mathrm{Lie}(L)$ satisfying three conditions

(a) $R = C_L(S)$;

(b) $\mathrm{ad}_{\mathfrak{l}} S$ induces the complex structure J—that is, J is an invariant complex structure J_s on $L/C_L(S)$ given by

$$(J_s)_o(d\pi(X)) := d\pi(\mathrm{ad}_{\mathfrak{l}} S(X)) \quad \text{for } X \in [S, \mathfrak{l}] = T_o(L/C_L(S)),$$

where π denotes the projection of L onto $L/C_L(S)$ and $T_o(L/C_L(S))$ denotes the tangent space of $L/C_L(S)$ at the origin o (see Kobayashi and Nomizu [Ks-No, pp. 216–217] for J_s);

(c) $\Sigma = \mathrm{A}_{\exp \pi S}$, where $\mathrm{A}_{\exp \pi S}$ is the inner automorphism of L determined by an element $\exp \pi S \in L$.

(2) Remark 2.1.5-(I′) and Lemma 2.1.6 imply that for any effective semisimple pseudo-Hermitian symmetric Lie algebra $(\mathfrak{l}, \sigma)$, there exists an Spr-element $S \in \mathfrak{l}$ satisfying $\sigma = \exp \pi\, \mathrm{ad}_{\mathfrak{l}} S$.

2.2. **Murakami's setting.** Our consideration to the group $\mathrm{Aut}(\mathfrak{l})$ of automorphisms of a real semisimple Lie algebra $\mathfrak{l}$, root theory for a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{l}$, and so forth will depend on the results of Murakami [Mu1] and [Mu3]. For the reason, we are going to introduce Murakami's setting utilized in [Mu1] and [Mu3].

Let $\mathfrak{l}_u$ be a compact real form of a complex semisimple Lie algebra $\tilde{\mathfrak{l}}$, and let θ be an involutive automorphism of $\mathfrak{l}_u$. Then, $\mathfrak{l}_u$ is decomposed as the direct sum of the two eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ of θ in $\mathfrak{l}_u$;

$$\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}. \tag{2.2.1}$$

Here $\mathfrak{k}$ and $\mathfrak{p}$ are defined by

$$\begin{cases} \mathfrak{k} := \{K \in \mathfrak{l}_u \,|\, \theta(K) = K\}, \\ \mathfrak{p} := \{P \in \mathfrak{l}_u \,|\, \theta(P) = -P\}. \end{cases} \tag{2.2.2}$$

In the setting, we give a real form $\mathfrak{l}$ of $\tilde{\mathfrak{l}}$ by setting

$$\mathfrak{l} := \mathfrak{k} \oplus i\mathfrak{p}. \tag{2.2.3}$$

Remark that $\mathfrak{l}$ is a real semisimple Lie algebra, $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ is a Cartan decomposition, $\mathfrak{k}$ $(= \mathfrak{l} \cap \mathfrak{l}_u)$ is a maximal compact subalgebra of $\mathfrak{l}$, and $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$ is the compact dual of $\mathfrak{l}$. Furthermore, remark that each real semisimple Lie algebra can be, up to isomorphic, given by the above fashion (ref. Theorem 2.1 in Wallach [Wa, pp. 5]).

Remark 2.2.1. Henceforth in this paper, we suppose that each real semisimple Lie algebra $\mathfrak{l}$ is given by the above fashion (2.2.1), (2.2.2) and (2.2.3).

Let $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ be a real form of a complex semisimple Lie algebra $\tilde{\mathfrak{l}}$, and let $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$ be the compact dual of $\mathfrak{l}$. Following Murakami's setting [Mu1, pp. 105], we identify the group $\mathrm{Aut}(\mathfrak{l})$ of automorphisms of $\mathfrak{l}$ with $\{\phi \in \mathrm{Aut}(\tilde{\mathfrak{l}}) \,|\, \phi(\mathfrak{l}) \subset \mathfrak{l}\}$, and identify $\mathrm{Aut}(\mathfrak{l}_u)$ with $\{\phi \in \mathrm{Aut}(\tilde{\mathfrak{l}}) \,|\, \phi(\mathfrak{l}_u) \subset \mathfrak{l}_u\}$. This identification allows us to assume that θ is an involution of not only $\mathfrak{l}_u$ but $\mathfrak{l}$ as well. Then, it is a Cartan involution of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$.

Remark 2.2.2. In this paper, we suppose that

$$\mathrm{Aut}(\mathfrak{l}) = \{\phi \in \mathrm{Aut}(\tilde{\mathfrak{l}}) \,|\, \phi(\mathfrak{l}) \subset \mathfrak{l}\} \text{ and } \mathrm{Aut}(\mathfrak{l}_u) = \{\phi \in \mathrm{Aut}(\tilde{\mathfrak{l}}) \,|\, \phi(\mathfrak{l}_u) \subset \mathfrak{l}_u\}.$$

We refer to the result of Murakami, and finish this subsection.

Proposition 2.2.3 (Murakami [Mu1, pp. 106]). *In the setting on Subsection* 2.2*; for an automorphism ϕ of $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$, the following three conditions* (i)*,* (ii) *and* (iii) *are mutually equivalent*:

$$\text{(i) } \phi \circ \theta = \theta \circ \phi, \quad \text{(ii) } \phi \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u), \quad \text{(iii) } \phi(\mathfrak{k}) \subset \mathfrak{k}.$$

Here, $\mathfrak{l}$ is related to $\mathfrak{l}_u$ as in the formulae (2.2.1)*,* (2.2.2) *and* (2.2.3) *by means of θ.*

2.3. **Elementary facts about root theory.** This subsection is a review of elementary facts about root theory, and consists of two paragraphs. In Paragraph 2.3.1, we state relation between root theory for a complex semisimple Lie algebra $\tilde{\mathfrak{l}}$ and that for its compact real form $\mathfrak{l}_u$. In Paragraph 2.3.2, we review root theory for a maximal compact subalgebra $\mathfrak{k}$ $(= \mathfrak{l} \cap \mathfrak{l}_u)$ of $\mathfrak{l}$. In addition, we recall the result of Murakami [Mu1] (see Proposition 2.3.4).

2.3.1. *Root theory for* $\tilde{\mathfrak{l}}$ *and for* $\mathfrak{l}_u$. Let $\tilde{\mathfrak{l}}$ be a complex semisimple Lie algebra, let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\tilde{\mathfrak{l}}$, and let $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$ denote the set of non-zero roots of $\tilde{\mathfrak{l}}$ with respect to $\tilde{\mathfrak{h}}$. Then, there exists a Weyl basis $\{X_\alpha \,|\, \alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})\}$ of $\tilde{\mathfrak{l}}$ such that for all $\alpha, \beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$

$$\begin{aligned}
&[X_\alpha, X_{-\alpha}] = H_\alpha, \quad [H, X_\alpha] = \alpha(H)\cdot X_\alpha \text{ for } H \in \tilde{\mathfrak{h}};\\
&[X_\alpha, X_\beta] = 0 \quad \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}});\\
&[X_\alpha, X_\beta] = N_{\alpha,\beta}\cdot X_{\alpha+\beta} \quad \text{if } \alpha+\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}),
\end{aligned}$$

where the real constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ (cf. Helgason [He, Theorem 5.5, pp. 176]). Here for $\alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$, one defines the element $H_\alpha \in \tilde{\mathfrak{h}}$ by $B_{\tilde{\mathfrak{l}}}(H, H_\alpha) = \alpha(H)$ for all $H \in \tilde{\mathfrak{h}}$. By using this Weyl basis, we give a compact real form $\mathfrak{l}_u$ of $\tilde{\mathfrak{l}}$ as follows:

$$\mathfrak{l}_u = i\tilde{\mathfrak{h}}_{\mathbb{R}} \quad \oplus \bigoplus_{\alpha\in\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})} \mathrm{span}_{\mathbb{R}}\{X_\alpha - X_{-\alpha}\} \oplus \mathrm{span}_{\mathbb{R}}\{i(X_\alpha + X_{-\alpha})\} \tag{2.3.1}$$

(ref. the proof of Theorem 6.3 in Helgason [He, pp. 181]), where $\tilde{\mathfrak{h}}_{\mathbb{R}}$ is a real vector subspace of $\tilde{\mathfrak{h}}$ determined by

$$\begin{aligned}
\tilde{\mathfrak{h}}_{\mathbb{R}} &:= \mathrm{span}_{\mathbb{R}}\{H_\alpha \,|\, \alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})\}\\
&\quad \big(= \{H \in \tilde{\mathfrak{h}} \,|\, \alpha(H) \in \mathbb{R} \text{ for all } \alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})\}\big).
\end{aligned} \tag{2.3.2}$$

Remark 2.3.1. (i) $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ is a maximal abelian subalgebra of $\mathfrak{l}_u$. (ii) Decomposition (2.3.1) is the root-space decomposition of the compact real form $\mathfrak{l}_u$ of $\tilde{\mathfrak{l}}$ with respect to $i\tilde{\mathfrak{h}}_{\mathbb{R}}$. In this case, the root system for $(\mathfrak{l}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ coincides with that for $(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$ multiplied by $-i$, namely

$$\triangle(\mathfrak{l}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}}) = \big\{ -i\alpha \,\big|\, \alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})\big\}$$

(cf. Toda and Mimura [To-Mi, Chapter 5]). (iii) In this paper, we fix a linear order in $\triangle(\mathfrak{l}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ as follows: $-i\alpha \in \triangle(\mathfrak{l}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is a positive root if so is $\alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$.

Theorem 5.1 in Helgason [He, pp. 421] and its proof enable us to demonstrate the following:

Proposition 2.3.2 (Helgason [He, pp. 421–423]). *In the setting on Paragraph* 2.3.1; *let* ϕ' *be a real, linear isomorphism of* $i\tilde{\mathfrak{h}}_{\mathbb{R}}$. *Suppose that the transposed mapping of* $\phi'_{\mathbb{C}}$ *satisfies*

$${}^t\phi'_{\mathbb{C}}\big(\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})\big) = \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}),$$

where $\phi'_{\mathbb{C}}$ *denotes the complex linear extension of* ϕ' *to* $\tilde{\mathfrak{h}}$. *Then, there exists an automorphism* ϕ *of* $\tilde{\mathfrak{l}}$ *which satisfies three conditions*

$$\begin{aligned}
&\text{(i)}\ \ \phi(\mathfrak{l}_u) \subset \mathfrak{l}_u, \qquad \text{(ii)}\ \ \phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi',\\
&\text{(iii)}\ \ \phi(X_{\pm\alpha_b}) = X_{\pm{}^t\phi^{-1}(\alpha_b)} \textit{ for all } b \in \{1, \dots, r\}.
\end{aligned}$$

Moreover, ϕ *is involutive if so is* ϕ'. *Here,* $\{\alpha_b\}_{b=1}^r$ *denotes the set of simple roots in* $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$.

2.3.2. *Root theory for* $\mathfrak{k}$ $(= \mathfrak{l} \cap \mathfrak{l}_u)$*, and automorphisms of* $\mathfrak{l}$. Let $\tilde{\mathfrak{l}}$ be a complex semisimple Lie algebra, let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\tilde{\mathfrak{l}}$, let $\mathfrak{l}_u$ be the compact real form of $\tilde{\mathfrak{l}}$ given by a Weyl basis $\{X_\alpha \,|\, \alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})\}$ of $\tilde{\mathfrak{l}}$ and (2.3.1), and let θ be an involutive automorphism of $\tilde{\mathfrak{l}}$ satisfying three conditions

$$\text{(c1) } \theta(\mathfrak{l}_u) \subset \mathfrak{l}_u, \quad \text{(c2) } \theta(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}, \quad \text{(c3) } {}^t\theta(\Pi_{\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})},$$

where $\Pi_{\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})}$ is the set of simple roots in $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$. Denote by $\mathfrak{k}$ the $+1$-eigenspace of θ in $\mathfrak{l}_u$—that is, $\mathfrak{k} = \{X + \theta(X) \,|\, X \in \mathfrak{l}_u\}$. Then, $\mathfrak{k}$ is described as follows (read Murakami [Mu3, pp. 300]):

$$\text{(2.3.3)} \quad \mathfrak{k} = \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \quad \oplus \bigoplus_{\gamma \in \triangle_1(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta) \cup \triangle_3(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta)} \operatorname{span}_{\mathbb{R}}\{X_\gamma - X_{-\gamma} + \theta(X_\gamma - X_{-\gamma})\} \oplus \operatorname{span}_{\mathbb{R}}\{i(X_\gamma + X_{-\gamma}) + i\theta(X_\gamma + X_{-\gamma})\}.$$

Here, $\triangle_1(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta)$ and $\triangle_3(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta)$ are defined by

$$\text{(2.3.4)} \quad \begin{cases} \triangle_1(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta) := \{\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \,|\, {}^t\theta(\beta) = \beta \text{ and } \theta(X_\beta) = X_\beta\}, \\ \triangle_3(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta) := \{\xi \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \,|\, {}^t\theta(\xi) \neq \xi\}. \end{cases}$$

Remark 2.3.3. (i) At page 300 on [Mu3], Murakami only treats the case where θ is of outer type. However, we take both the case where θ is of outer type and inner type into consideration. (ii) $\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$ is a maximal abelian subalgebra of $\mathfrak{k}$, because it follows from ${}^t\theta(\Pi_{\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})}$ that θ leaves fixed a regular element of $\tilde{\mathfrak{l}}$ contained in $\tilde{\mathfrak{h}}$ (ref. Murakami [Mu2, Proposition 1, pp. 106]). (iii) Decomposition (2.3.3) is the root-space decomposition of $\mathfrak{k}$ with respect to $\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$. The set of non-zero roots of $\mathfrak{k}$ with respect to $\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$ is as follows:

$$\begin{aligned} &\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}) \\ &= \left\{ -i\gamma|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} = -\frac{i}{2}(\gamma + {}^t\theta(\gamma)) \,\middle|\, \gamma \in \triangle_1(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta) \cup \triangle_3(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta) \right\} \end{aligned}$$

(read Murakami [Mu3, pp. 300] again; recall our Remark 2.3.1-(ii)). (iv) In this paper, we fix a linear order in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ as follows: $-i\gamma|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is a positive root if γ is a positive root in $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$. Remark that this linear order is the same one used in [Mu3, pp. 300].

Now, let $\mathfrak{p}$ denote the -1-eigenspace of θ in $\mathfrak{l}_u$, and let $\mathfrak{l}$ be the real form of $\tilde{\mathfrak{l}}$ determined by (2.2.3) $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$. Then, Theorem 3 in Murakami [Mu1] and its proof allow us to assert the following:

Proposition 2.3.4 (Murakami [Mu1, pp. 118–121]). *In the setting on Paragraph* 2.3.2*; let* ψ *be an automorphism of* $\tilde{\mathfrak{l}}$ *which stabilizes* $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$*. Suppose that it satisfies the following two conditions*:

(a) $\psi(i\tilde{\mathfrak{h}}_{\mathbb{R}}) \subset i\tilde{\mathfrak{h}}_{\mathbb{R}}$*, and* $\psi \circ \theta = \theta \circ \psi$ *on* $i\tilde{\mathfrak{h}}_{\mathbb{R}}$;

(b) ${}^t\psi\big(\triangle_1(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta)\big) = \triangle_1(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}} : \theta)$.

Then, there exists an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ *such that* $\psi \circ \exp \operatorname{ad}_{\tilde{\mathfrak{l}}} iH \in \operatorname{Aut}(\mathfrak{l}) \cap \operatorname{Aut}(\mathfrak{l}_u)$.

2.4. **Notation.** Throughout this paper, we utilize the following notation:

(n1) Ad: the adjoint representation of a Lie group.
(n2) $\mathrm{Ad}(D)X$: the adjoint orbit of a Lie group D through an element $X \in \mathrm{Lie}(D)$.
(n3) $C_D(X)$: the centralizer of X in a Lie group D, for an element $X \in \mathrm{Lie}(D)$.
(n4) $B_{\mathfrak{a}}$: the Killing form of a Lie algebra $\mathfrak{a}$.
(n5) $\mathrm{ad}_{\mathfrak{a}}$: the adjoint representation of a Lie algebra $\mathfrak{a}$.
(n6) $\mathfrak{c}_{\mathfrak{a}}(X)$: the centralizer of X in a Lie algebra $\mathfrak{a}$, for an element $X \in \mathfrak{a}$.
(n7) $\mathfrak{a}_{\mathfrak{z}}$: the center of a Lie algebra $\mathfrak{a}$.
(n8) $\mathfrak{y}^{\mathbb{C}}$: the complexification of a real Lie algebra $\mathfrak{y}$.
(n9) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$.
(n10) $f|_A$: the restriction of a mapping f to a set A.
(n11) $\delta_{a,b}$: Kronecker's delta.
(n12) i: the imaginary unit, namely $i = \sqrt{-1}$.

If $\tilde{\mathfrak{l}}$ is a complex semisimple Lie algebra and if $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\tilde{\mathfrak{l}}$, then we specially utilize the following notation:

(n13) $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$: the set of non-zero roots of $\tilde{\mathfrak{l}}$ with respect to $\tilde{\mathfrak{h}}$.
(n14) $\triangle^+(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$: the set of positive roots in $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$.
(n15) $\Pi_{\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})}$: the set of simple roots in $\triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$.

Let $\mathfrak{l}$ be a real semisimple Lie algebra with Cartan decomposition (2.2.3) $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$. Then, we utilize the following notation:

(n16) $\mathfrak{t}^n$: an n-dimensional abelian subalgebra of $\mathfrak{l}$ which is contained in $\mathfrak{k}$.

3. Semisimple pseudo-Hermitian symmetric spaces and Spr-elements

This section consists of two subsections. Subsection 3.1 is an investigation into relation between semisimple pseudo-Hermitian symmetric Lie algebras and Spr-elements of $\mathfrak{l}$, for a real semisimple Lie algebra $\mathfrak{l}$ (refer to Section 1 for the definition of Spr-element). In Subsection 3.2, we consider the case where $\mathfrak{g}$ is a real form of a complex simple Lie algebra, and we verify that the mapping $F_1 : Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) \to \mathrm{Inv}(\mathfrak{g})^{pH}/\mathrm{Aut}(\mathfrak{g})$, $[\,S\,] \mapsto [\,\exp \pi\, \mathrm{ad}_{\mathfrak{g}} S\,]$, is bijective (see Theorem 3.2.1).

3.1. **Semisimple case.** Our aim in this subsection is to prove Lemma 3.1.1.

Lemma 3.1.1. *Let $\mathfrak{l}$ be a real semisimple Lie algebra, and let S be an Spr-element of $\mathfrak{l}$. Then, the following three items hold*:

(1) *An inner automorphism $\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{l}} S$ of $\mathfrak{l}$ is involutive and its $+1$ (resp. -1)-eigenspace accords with $\mathfrak{c}_{\mathfrak{l}}(S)$ (resp. $[S, \mathfrak{l}]$).*
(2) *The pair $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ is the pseudo-Hermitian symmetric Lie algebra by the involution ρ, and S is the canonical central element of $\mathfrak{c}_{\mathfrak{l}}(S)$ relative to $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$.*
(3) *Let L be a connected Lie group with Lie algebra $\mathfrak{l}$. The adjoint orbit $\mathrm{Ad}(L)S = L/C_L(S)$ is a pseudo-Hermitian symmetric space defined by an involutive automorphism $\mathrm{A}_{\exp \pi S}$ of L, an invariant complex structure J_s and an invariant pseudo-Hermitian metric $\mathsf{g}_{B_{\mathfrak{l}}}$ (with respect to J_s).*

Here, J_s *is given in Remark* 2.1.7*, and* $\mathrm{g}_{B_{\mathfrak{l}}}$ *is an invariant pseudo-Riemannian metric on* $L/C_L(S)$ *given by* $\mathrm{g}_{B_{\mathfrak{l}}}(X,Y)_o := B_{\mathfrak{l}}(X,Y)$ *for* $X, Y \in [S,\mathfrak{l}] = T_o(L/C_L(S))$ (*refer to Kobayashi and Nomizu* [Ks-No, pp. 200-201] *for* $\mathrm{g}_{B_{\mathfrak{l}}}$).

Proof. (1) Since S is a semisimple element, $\mathfrak{l}$ is decomposed as $\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(S) \oplus [S,\mathfrak{l}]$. Therefore the proof of Lemma 2.1.6 allows us to have the first item, because $\mathrm{ad}_{\mathfrak{l}}\, S|_{[S,\mathfrak{l}]}$ is a complex structure on $[S,\mathfrak{l}]$.

(2) It is clear from (1) that $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ is the symmetric Lie algebra by ρ, and that $\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(S) \oplus [S,\mathfrak{l}]$ is its canonical decomposition. Moreover, one perceives that $I := \mathrm{ad}_{\mathfrak{l}}\, S|_{[S,\mathfrak{l}]}$ is an $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{c}_{\mathfrak{l}}(S))$-invariant complex structure on the vector space $[S,\mathfrak{l}]$, and that $B_{\mathfrak{l}}|_{[S,\mathfrak{l}]\times[S,\mathfrak{l}]}$ is an $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{c}_{\mathfrak{l}}(S))$-invariant pseudo-Hermitian form (with respect to I) on $[S,\mathfrak{l}]$ (recall Notation (n4) and (n10) in Subsection 2.4, for $B_{\mathfrak{l}}|_{[S,\mathfrak{l}]\times[S,\mathfrak{l}]}$). Therefore, we deduce that the symmetric Lie algebra $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ is pseudo-Hermitian and that S is the canonical central element of $\mathfrak{c}_{\mathfrak{l}}(S)$ relative to $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$.

(3) The last item follows from (2) and the result of Shapiro [Sh, Proposition 2.5 and its proof, pp. 533–534]. Consequently, we have shown Lemma 3.1.1. □

Remark 3.1.2. Fix a real simple Lie algebra $\mathfrak{l}$, denote by $Spr_{\mathfrak{l}}$ the set of Spr-elements of $\mathfrak{l}$, and denote by $\mathrm{Inv}(\mathfrak{l})^{pH}$ the set of involutions σ of $\mathfrak{l}$ such that $(\mathfrak{l},\sigma)$ is a pseudo-Hermitian symmetric Lie algebra. Then, the following mapping F' is a surjection of $Spr_{\mathfrak{l}}$ onto $\mathrm{Inv}(\mathfrak{l})^{pH}$:

$$\begin{aligned} F' : Spr_{\mathfrak{l}} &\longrightarrow \mathrm{Inv}(\mathfrak{l})^{pH} \qquad \text{(surjective)} \\ S &\mapsto \exp \pi\, \mathrm{ad}_{\mathfrak{l}}\, S \end{aligned}$$

because of Remark 2.1.7-(2) and Lemma 3.1.1.

3.2. **Simple case.** We have so far argued about a real semisimple Lie algebra $\mathfrak{l}$. In this subsection, we consider a real simple Lie algebra $\mathfrak{g}$ whose complexification is also simple, and we aim to demonstrate Theorem 3.2.1.

Theorem 3.2.1. *Let* $\mathfrak{g}$ *be a real form of a complex simple Lie algebra. Then, the following mapping* F_1 *is a bijection of* $Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$ *onto* $\mathrm{Inv}(\mathfrak{g})^{pH}/\,\mathrm{Aut}(\mathfrak{g})$:

$$\begin{aligned} F_1 : Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) &\longrightarrow \mathrm{Inv}(\mathfrak{g})^{pH}/\,\mathrm{Aut}(\mathfrak{g}) \qquad \text{(bijective)} \\ [\,S\,] &\mapsto [\exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, S\,]. \end{aligned}$$

Here, $Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$ *and* $\mathrm{Inv}(\mathfrak{g})^{pH}/\,\mathrm{Aut}(\mathfrak{g})$ *are the quotient set of* $Spr_{\mathfrak{g}}$ *by our equivalence relation and of* $\mathrm{Inv}(\mathfrak{g})^{pH}$ *by Berger's equivalence relation, respectively* (*cf. Section* 1).

Proof. In the first place, we will confirm that the mapping F_1 is well-defined. Let S_1 and S_2 be two elements of $Spr_{\mathfrak{g}}$, and let ϕ be an automorphism of $\mathfrak{g}$ such that $\phi(S_1) = S_2$ or $\phi(S_1) = -S_2$. Then, it is natural that $\phi \circ \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, S_1 \circ \phi^{-1} = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, \phi(S_1) = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, S_2 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}(-S_2)$ since $\exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, S_2$ is involutive. Hence, F_1 is well-defined.

In the second place, let us verify that the mapping F_1 is injective. Suppose that there exists an automorphism ψ of $\mathfrak{g}$ satisfying $\psi \circ \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, S_1 \circ \psi^{-1} = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, S_2$

$(S_1, S_2 \in Spr_{\mathfrak{g}})$. Then, Lemma 3.1.1-(1) means that $\psi(\mathfrak{c}_{\mathfrak{g}}(S_1)) = \mathfrak{c}_{\mathfrak{g}}(S_2)$, and so

$$\psi(\mathfrak{c}_{\mathfrak{g}}(S_1)_{\mathfrak{z}}) = \mathfrak{c}_{\mathfrak{g}}(S_2)_{\mathfrak{z}}$$

(recall Notation (n7) in Subsection 2.4, for $\mathfrak{c}_{\mathfrak{g}}(S_p)_{\mathfrak{z}}$). Corollary 2.3 in Shapiro [Sh, pp. 532] and our Lemma 3.1.1-(2) imply that $\mathfrak{c}_{\mathfrak{g}}(S_p)_{\mathfrak{z}} = \mathrm{span}_{\mathbb{R}}\{S_p\}$ $(p = 1, 2)$, and thus $\psi(\mathrm{span}_{\mathbb{R}}\{S_1\}) = \mathrm{span}_{\mathbb{R}}\{S_2\}$. Accordingly, there exists a non-zero number $\lambda \in \mathbb{R}$ satisfying

$$\psi(S_1) = \lambda \cdot S_2.$$

We obtain $\lambda = \pm 1$, since both $\mathrm{ad}_{\mathfrak{g}}\, \psi(S_1) = \lambda\, \mathrm{ad}_{\mathfrak{g}}\, S_2$ and $\mathrm{ad}_{\mathfrak{g}}\, S_2$ are complex structures on $[\psi(S_1), \mathfrak{g}] = [S_2, \mathfrak{g}]$. For the reasons, it follows that $\psi(S_1) = \pm S_2$. This deduces that the mapping F_1 is injective. It is immediate from Remark 3.1.2 that the mapping F_1 is surjective. Consequently, we have completed the proof of Theorem 3.2.1. □

4. Necessary and sufficient conditions for an elliptic element to be an Spr-element

This section is organized as follows: In Subsection 4.1, we provide a necessary and sufficient condition for an elliptic element to be an *Spr*-element (see Lemma 4.1.1). Subsection 4.2 is devoted to giving conditions which an *Spr*-element should satisfy (see Lemmas 4.2.1 through 4.2.4).

Remark 4.0.2. Throughout this section, a compact real form $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$ of $\tilde{\mathfrak{l}}$, an involutive automorphism θ of $\mathfrak{l}_u$, a real form $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\tilde{\mathfrak{l}}$, the set of non-zero roots of $\mathfrak{k}$ with respect to $\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$, and so forth obey the setting on Subsection 2.3.

4.1. **A necessary and sufficient condition.**

In this subsection, we will first explain that any *Spr*-element $S \in \mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ can be mapped into a fixed, Weyl chamber $\mathfrak{W}_{\mathfrak{k}}$ of $\mathfrak{k}$, and we will afterwards prove Lemma 4.1.1.

Let $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\gamma_j|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{j=1}^{t}$ be the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (ref. Remark 2.3.3), and let $\mathfrak{W}_{\mathfrak{k}}$ denote the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\gamma_j|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{j=1}^{t}$;

$$\mathfrak{W}_{\mathfrak{k}} = \{T \in \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\gamma_1(T) \geq 0, \cdots, -i\gamma_t(T) \geq 0\}. \tag{4.1.1}$$

Now, let us show the following (4.1.2):

(4.1.2) For any *Spr*-element S of a semisimple Lie algebra $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$, there exists an inner automorphism ϕ of $\mathfrak{l}$ satisfying $\phi(S) \in \mathfrak{W}_{\mathfrak{k}}$.

Since S is elliptic (cf. Remark 2.1.2), there exists a maximal compact subalgebra $\mathfrak{k}'$ of $\mathfrak{l}$ such that $S \in \mathfrak{k}'$. Thus, Theorem 7.2 in Helgason [He, pp. 183] implies that there exists an inner automorphism ϕ_1 of $\mathfrak{l}$ which maps $\mathfrak{k}'$ onto $\mathfrak{k}$; and hence $\phi_1(S) \in \mathfrak{k}$. Furthermore, there exists an element $K \in \mathfrak{k}$ such that $\exp \mathrm{ad}_{\mathfrak{k}}\, K(\phi_1(S)) \in \mathfrak{W}_{\mathfrak{k}}$ because $\mathfrak{k}$ is a compact Lie algebra. Accordingly, we define an inner automorphism ϕ of $\mathfrak{l}$ by $\phi := \exp \mathrm{ad}_{\mathfrak{l}}\, K \circ \phi_1$, and we conclude (4.1.2).

On account of (4.1.2) we will search a Weyl chamber $\mathfrak{W}_{\mathfrak{k}}$ of $\mathfrak{k}$ for *Spr*-elements, in the next section. In order to easily search, we are going to provide a condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}$ to be an *Spr*-element. Let us recall Remark 4.0.2, and prove the following:

Lemma 4.1.1. *For any non-zero element $T \in \mathfrak{W}_{\mathfrak{k}}$ (cf. (4.1.1)), the following three conditions are mutually equivalent:*

a) *T is an Spr-element of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$;*
b) *T is an Spr-element of $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$;*
c) *$\beta(T) = \pm i$ for each root $\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$.*

Here, $\triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) := \{\zeta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \,|\, \zeta(T) = 0\}$.

Proof. a) $\leftrightarrow$ c): We will prove that two conditions a) and c) are equivalent to each other. Since T belongs to $\mathfrak{W}_{\mathfrak{k}}$ ($\subset \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$), one perceives that the element T is a semisimple element of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$. Therefore, two conditions a) and c) are equivalent to each other if the condition c) is equivalent to the following condition:

$$\text{a}')\ \operatorname{ad}_{\tilde{\mathfrak{l}}} T|_{[T,\tilde{\mathfrak{l}}]} \text{ is a complex structure on the vector space } [T, \tilde{\mathfrak{l}}],$$

because it follows from $\tilde{\mathfrak{l}} = \mathfrak{l}^{\mathbb{C}}$ that $\operatorname{ad}_{\tilde{\mathfrak{l}}} T|_{[T,\tilde{\mathfrak{l}}]}$ is a complex structure on $[T, \tilde{\mathfrak{l}}]$ if and only if $\operatorname{ad}_{\mathfrak{l}} T|_{[T,\mathfrak{l}]}$ is a complex structure on $[T, \mathfrak{l}]$. For the reason, let us confirm that two conditions a$'$) and c) are equivalent to each other, from now on. In order to do so, we want to rewrite the root-space decomposition of $\tilde{\mathfrak{l}}$ with respect to $\tilde{\mathfrak{h}}$ as follows:

$$\begin{aligned}
\tilde{\mathfrak{l}} &= \tilde{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})} \operatorname{span}_{\mathbb{C}}\{X_\alpha\} \\
&= \tilde{\mathfrak{h}} \oplus \bigoplus_{\zeta \in \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})} \operatorname{span}_{\mathbb{C}}\{X_\zeta\} \oplus \bigoplus_{\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})} \operatorname{span}_{\mathbb{C}}\{X_\beta\} \\
&= \mathfrak{c}_{\tilde{\mathfrak{l}}}(T) \oplus \bigoplus_{\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})} \operatorname{span}_{\mathbb{C}}\{X_\beta\},
\end{aligned}$$

where X_α, $\alpha \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$, are given in Paragraph 2.3.1. This enables us to obtain

$$[T, \tilde{\mathfrak{l}}] = \bigoplus_{\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})} \operatorname{span}_{\mathbb{C}}\{X_\beta\} \tag{4.1.3}$$

because $[T, X_\beta] = \beta(T) \cdot X_\beta \neq 0$ for all $\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$ and the semisimple element T splits $\tilde{\mathfrak{l}}$ into $\mathfrak{c}_{\tilde{\mathfrak{l}}}(T) \oplus [T, \tilde{\mathfrak{l}}]$. Since (4.1.3) and since $(\operatorname{ad}_{\tilde{\mathfrak{l}}} T)^2(X_\beta) = (\beta(T))^2 \cdot X_\beta$, we conclude that $\operatorname{ad}_{\tilde{\mathfrak{l}}} T|_{[T,\tilde{\mathfrak{l}}]}$ is a complex structure on $[T, \tilde{\mathfrak{l}}]$ if and only if $\beta(T) = \pm i$ for all $\beta \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$. Consequently, two conditions a$'$) and c) are equivalent to each other.

b) $\leftrightarrow$ c): By arguments similar to the above ones, we can conclude that two conditions b) and c) are equivalent to each other, because of $\tilde{\mathfrak{l}} = \mathfrak{l}_u^{\mathbb{C}}$. Accordingly, we have completed the proof of Lemma 4.1.1. □

By use of Lemma 4.1.1, we can get Proposition 4.1.2.

Proposition 4.1.2. *The set of Spr-elements of a real semisimple Lie algebra $\mathfrak{l}$ is an empty set, in the case where $\mathfrak{l}$ is one of the following*:

$$\text{EVIII: } \mathfrak{e}_{8(8)}, \quad \text{EIX: } \mathfrak{e}_{8(-24)}, \quad \text{FI: } \mathfrak{f}_{4(4)}, \quad \text{FII: } \mathfrak{f}_{4(-20)}, \quad \text{G: } \mathfrak{g}_{2(2)}.$$

Proof. First, let us consider the case where $\mathfrak{l} = \mathfrak{g}_{2(2)}$. Suppose that $\mathfrak{g}_{2(2)}$ contains an *Spr*-element S. Then by (4.1.2) and Lemma 4.1.1, one deduces that the compact dual $\mathfrak{g}_2$ of $\mathfrak{g}_{2(2)}$ contains an *Spr*-element S'. Therefore, Lemma 3.1.1-(2) implies that $(\mathfrak{g}_2, \mathfrak{c}_{\mathfrak{g}_2}(S'))$ is a Hermitian symmetric Lie algebra of compact type. However, that is inconsistent with the result of Wolf [Wo]. Consequently, $\mathfrak{g}_{2(2)}$ contains no *Spr*-elements. In a similar way, we can confirm that the other Lie algebras also contain no *Spr*-elements. Thus, this proposition has been proved. □

Example 4.1.3. Let a Lie group G be one of the $E_{8(8)}$, $E_{8(-24)}$, $F_{4(4)}$, $F_{4(-20)}$ and $G_{2(2)}$. Remark 3.1.2 and Proposition 4.1.2 allow us to see that there exist no pseudo-Hermitian symmetric spaces on which G acts transitively. Hence, there exists an elliptic orbit which can not be a pseudo-Hermitian symmetric space—for example, $G_{2(2)}/U(2)$ is an elliptic orbit (cf. [Bm, Proposition 5.5]) but it can not be a pseudo-Hermitian symmetric space.

4.2. **$\mathfrak{k}$ is a direct sum of two simple ideals and the non-trivial center.** Let us recall Remark 4.0.2. In general, a maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ is a direct sum of compact simple ideals and the center. In the first half of this subsection, we assume $\mathfrak{k}$ to be the direct sum of two compact simple ideals $\mathfrak{k}_1$, $\mathfrak{k}_2$ and the center $\mathfrak{k}_3 \neq \{0\}$;

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3.$$

In this setting, we will provide a condition which an *Spr*-element $S \in \mathfrak{W}_\mathfrak{k}$ should satisfy (cf. Lemma 4.2.1).

For the simple root system $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R})} = \{-i\gamma_j|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R}}\}_{j=1}^t$ (given in Subsection 4.1), we assume that $\{-i\gamma_k|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R}}\}_{k=1}^s$ and $\{-i\gamma_l|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R}}\}_{l=s+1}^t$ are the set of simple roots in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_\mathbb{R})$ and in $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_\mathbb{R})$ respectively. Then for each $p = 1, 2$, we denote by $-i\mu_p$ the highest root in $\triangle(\mathfrak{k}_p, \mathfrak{k}_p \cap i\tilde{\mathfrak{h}}_\mathbb{R})$, and we describe them as

$$(4.2.1) \qquad \begin{cases} -i\mu_1 = -i(m_1 \cdot \gamma_1 + m_2 \cdot \gamma_2 + \cdots + m_s \cdot \gamma_s)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R}}, \\ -i\mu_2 = -i(m_{s+1} \cdot \gamma_{s+1} + m_{s+2} \cdot \gamma_{s+2} + \cdots + m_t \cdot \gamma_t)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R}}, \end{cases}$$

where m_a is a positive integer for each $a \in \{1, \ldots, t\}$. Note that for $p = 1, 2$, there exists a root $\mu'_p \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$ satisfying $-i\mu_p = -i\mu'_p|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_\mathbb{R}}$ (cf. Remark 2.3.3-(iii)).

4.2.1. *Case* $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3$. Now, let us verify Lemma 4.2.1.

Lemma 4.2.1. *Let S be an Spr-element of a semisimple Lie algebra $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ such that $S \in \mathfrak{W}_\mathfrak{k}$ (cf. (4.1.1)). Suppose that $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1$, $\mathfrak{k}_2$ and the non-trivial center $\mathfrak{k}_3$. Then one of the following four cases only occurs*:

(A-1) $-i\mu_2(S) = 0$, *and there exists an integer $k \in \{1, \ldots, s\}$ such that $m_k = 1$ and $-i\gamma_b(S) = \delta_{k,b}$ for any $1 \leq b \leq s$.*

(A-2) $-i\mu_1(S) = 0$, *and there exists an integer $l \in \{s+1, \ldots, t\}$ such that $m_l = 1$ and $-i\gamma_c(S) = \delta_{l,c}$ for any $s+1 \leq c \leq t$.*

(A-3) *There exist integers* $k \in \{1,\dots,s\}$ *and* $l \in \{s+1,\dots,t\}$ *such that* $m_k = m_l = 1$ *and* $-i\gamma_a(S) = \delta_{k,a} + \delta_{l,a}$ *for any* $1 \leq a \leq t$.
(A-4) S *is a non-zero, central element of* $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3$.

Here $-i\mu_1 = -i\sum_{k=1}^{s} m_k \cdot \gamma_k|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ *and* $-i\mu_2 = -i\sum_{l=s+1}^{t} m_l \cdot \gamma_l|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ *are the highest root in* $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ *and in* $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$*, respectively* (*cf.* (4.2.1)).

Proof. It is natural that one of the following four cases only occurs:

(A-1′) $-i\mu_1(S) \neq 0$ and $-i\mu_2(S) = 0$, (A-2′) $-i\mu_1(S) = 0$ and $-i\mu_2(S) \neq 0$,
(A-3′) $-i\mu_1(S) \neq 0$ and $-i\mu_2(S) \neq 0$, (A-4′) $-i\mu_1(S) = -i\mu_2(S) = 0$.

Let us consider Case (A-1′) $-i\mu_1(S) \neq 0$ and $-i\mu_2(S) = 0$, first. Naturally, it follows from $S \in \mathfrak{W}_{\mathfrak{k}}$ and (4.1.1) that

$$-i\gamma_1(S) \geq 0, \cdots, -i\gamma_s(S) \geq 0. \tag{4.2.2}$$

Hence we obtain $-i\mu_1(S) > 0$ because of (4.2.1) and $-i\mu_1(S) \neq 0$. There exists a root $\mu_1' \in \triangle(\tilde{\mathfrak{l}}, \tilde{\mathfrak{h}})$ such that $-i\mu_1 = -i\mu_1'|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$. Since $S \in \mathfrak{W}_{\mathfrak{k}}$ and S is an *Spr*-element of $\mathfrak{l}$, Lemma 4.1.1 allows us to have $\mu_1'(S) = \pm i$ or 0. Consequently, it follows from $-i\mu_1(S) > 0$ and $-i\mu_1 = -i\mu_1'|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ that

$$-i\mu_1(S) = 1.$$

This, together with (4.2.1) and (4.2.2), concludes that there exists an integer $k \in \{1,\dots,s\}$ satisfying $-i\gamma_k(S) > 0$. So, we have

$$-i\gamma_k(S) = 1$$

by using Lemma 4.1.1 again. From $-i\mu_1(S) = -i\gamma_k(S) = 1$, (4.2.1) and (4.2.2), it follows that $m_k = 1$ and $-i\gamma_d(S) \equiv 0$ for every $d \in \{1,\dots,k-1,k+1,\dots,s\}$. For the reasons, one perceives that there exists an integer $k \in \{1,\dots,s\}$ satisfying $m_k = 1$ and $-i\gamma_b(S) = \delta_{k,b}$ for any $1 \leq b \leq s$.

By arguments similar to those above, we can deduce that in Case (A-2′) there exists an integer $l \in \{s+1,\dots,t\}$ satisfying $m_l = 1$ and $-i\gamma_c(S) = \delta_{l,c}$ for any $s+1 \leq c \leq t$, and that in Case (A-3′) there exist integers $k \in \{1,\dots,s\}$ and $l \in \{s+1,\dots,t\}$ satisfying $m_k = m_l = 1$ and $-i\gamma_a(S) = \delta_{k,a} + \delta_{l,a}$ for any $1 \leq a \leq t$.

Let us consider Case (A-4′) $-i\mu_1(S) = -i\mu_2(S) = 0$, lastly. Since $S \in \mathfrak{W}_{\mathfrak{k}}$ and (4.2.1), we have $-i\gamma_a(S) \equiv 0$ for each $1 \leq a \leq t$. This shows that $-i\gamma(S) \equiv 0$ for every root $-i\gamma|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, because $\{-i\gamma_j|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{j=1}^{t}$ is the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$. Accordingly, S is a central element of $\mathfrak{k}$. Thus, we have got the conclusion. □

4.2.2. *Case* $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$. The following comes from Lemma 4.2.1:

Lemma 4.2.2. *Let* S *be an Spr-element of a semisimple Lie algebra* $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ *such that* $S \in \mathfrak{W}_{\mathfrak{k}}$ (*cf.* (4.1.1))*. Suppose that* $\mathfrak{k}$ *is the direct sum of two simple ideals* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$*. Then one of the following three cases only occurs*:

(B-1) $-i\mu_2(S) = 0$*, and there exists an integer* $k \in \{1,\dots,s\}$ *such that* $m_k = 1$ *and* $-i\gamma_b(S) = \delta_{k,b}$ *for any* $1 \leq b \leq s$.

(B-2) $-i\mu_1(S) = 0$, *and there exists an integer* $l \in \{s+1, \dots, t\}$ *such that* $m_l = 1$ *and* $-i\gamma_c(S) = \delta_{l,c}$ *for any* $s+1 \le c \le t$.

(B-3) *There exist integers* $k \in \{1, \dots, s\}$ *and* $l \in \{s+1, \dots, t\}$ *such that* $m_k = m_l = 1$ *and* $-i\gamma_a(S) = \delta_{k,a} + \delta_{l,a}$ *for any* $1 \le a \le t$.

Here $-i\mu_1 = -i\sum_{k=1}^{s} m_k \cdot \gamma_k|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ *and* $-i\mu_2 = -i\sum_{l=s+1}^{t} m_l \cdot \gamma_l|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ *are the highest root in* $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ *and in* $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, *respectively* (*cf.* (4.2.1)).

4.2.3. *Case* $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_{\mathfrak{z}}$. Recollect Remark 4.0.2. For the sake of Section 5, we also consider the case where $\mathfrak{k}$ is the direct sum of a compact simple ideal $\mathfrak{k}_1$ and the center $\mathfrak{k}_{\mathfrak{z}} \neq \{0\}$.

We denote by $-i\mu$ the highest root in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, and write it as follows:

$$-i\mu = -i(n_1 \cdot \gamma_1 + n_2 \cdot \gamma_2 + \cdots + n_t \cdot \gamma_t)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, \tag{4.2.3}$$

where n_a is a positive integer for each $1 \le a \le t$ and $\{-i\gamma_j|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{j=1}^{t}$ is the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (given in Subsection 4.1). In the setting, the proof of Lemma 4.2.1 enables us to deduce Lemma 4.2.3.

Lemma 4.2.3. *Let* S *be an* Spr*-element of a semisimple Lie algebra* $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ *such that* $S \in \mathfrak{W}_{\mathfrak{k}}$ (*cf.* (4.1.1)). *Suppose that* $\mathfrak{k}$ *is the direct sum of a simple ideal* $\mathfrak{k}_1$ *and the non-trivial center* $\mathfrak{k}_{\mathfrak{z}}$. *Then one of the following two cases only occurs*:

(C-1) *There exists an integer* $j \in \{1, \dots, t\}$ *such that* $n_j = 1$ *and* $-i\gamma_a(S) = \delta_{j,a}$ *for any* $1 \le a \le t$.

(C-2) S *is a non-zero, central element of* $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_{\mathfrak{z}}$.

Here, $\{-i\gamma_j|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{j=1}^{t}$ *is the simple root system of* $\mathfrak{k}$, *and* n_j $(1 \le j \le t)$ *is a coefficient of the highest root* $-i\mu \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (*cf.* (4.2.3)).

4.2.4. *Case* $\mathfrak{k} = \mathfrak{k}_1$. By Lemma 4.2.3, one has the following:

Lemma 4.2.4. *Let* S *be an* Spr*-element of a semisimple Lie algebra* $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ *such that* $S \in \mathfrak{W}_{\mathfrak{k}}$ (*cf.* (4.1.1)). *If* $\mathfrak{k}$ *is simple, then the following case only occurs*:

(D) *There exists an integer* $j \in \{1, \dots, t\}$ *such that* $n_j = 1$ *and* $-i\gamma_a(S) = \delta_{j,a}$ *for any* $1 \le a \le t$.

Here, $\{-i\gamma_j|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{j=1}^{t}$ *is the simple root system of* $\mathfrak{k}$, *and* n_j $(1 \le j \le t)$ *is a coefficient of the highest root* $-i\mu \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (*cf.* (4.2.3)).

By use of Lemma 4.2.4, we will prove Proposition 4.2.5.

Proposition 4.2.5. *The set of* Spr*-elements of* EIV: $\mathfrak{e}_{6(-26)}$ *is an empty set.*

Proof. It is known that $\mathfrak{f}_4$ is a maximal compact subalgebra of $\mathfrak{e}_{6(-26)}$ (cf. Helgason [He, Table V, pp. 518]) and that a coefficient of the highest root μ for $\mathfrak{f}_4$ is 2, 3 or 4 (cf. Bourbaki [Br, Plate VIII, pp. 287]). Hence, there exist no coefficients of μ whose values are 1. Accordingly, Lemma 4.2.4 assures that the set of Spr-elements of $\mathfrak{e}_{6(-26)}$ is empty. □

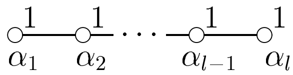
Finally we will state the following remark, and finish this section.

Remark 4.2.6. Let us comment on two Lemmas 4.2.1-(A-4) and 4.2.3-(C-2) in the case where $\mathfrak{l}^{\mathbb{C}}$ is simple. Let $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ be a real form of a complex simple Lie algebra, and let S be an Spr-element of $\mathfrak{g}$. If the center $\mathfrak{k}_{\mathfrak{z}}$ of $\mathfrak{k}$ is non-trivial and if S belongs to $\mathfrak{k}_{\mathfrak{z}}$, then the element S is the H-element of a Hermitian symmetric Lie algebra $(\mathfrak{g}, \mathfrak{k})$ because $\mathfrak{k} = \mathfrak{c}_{\mathfrak{g}}(S)$ follows from irreducibility of $(\mathfrak{g}, \mathfrak{k})$ (see Satake [Sa, pp. 54] for the definition of H-element). Consequently, an Spr-element S in Case (A-4) or (C-2) is the H-element of $(\mathfrak{l}, \mathfrak{k})$ when $\mathfrak{l}^{\mathbb{C}}$ is simple.

5. The classification of Spr-elements of each simple Lie algebra

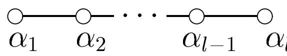
Our purpose in this section is to classify Spr-elements of each real form of all complex simple Lie algebras under our equivalence relation (defined in Section 1). This section consists of six subsections, and each of the six subsections is devoted to classifying Spr-elements of each real form of a complex simple Lie algebra. We finally collect the results obtained in every subsection (cf. Theorem 5.6.10), and we achieve the classification of simple irreducible pseudo-Hermitian symmetric Lie algebras without Berger's classification (cf. Corollary 5.6.11).

5.1. **Type** A_l $(l \geq 1)$**.** In this subsection, we deal with each real form of the complex simple Lie algebra $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$. First, let us introduce our setting. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{a}_l$, let $\{\alpha_a\}_{a=1}^l$ be the set of simple roots in $\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})$ whose Dynkin diagram is as follows:

$\mathfrak{a}_l$: 

(cf. Plate I in Bourbaki [Br, pp. 265]), and let $\mathfrak{g}_u$ be the compact real form of $\mathfrak{a}_l$ given by $\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})$ and (2.3.1). In addition, we define an element $Z_a \in \tilde{\mathfrak{h}}$ $(1 \leq a \leq l)$ by $\alpha_a(Z_b) = \delta_{a,b}$ for all $b \in \{1, \ldots, l\}$, namely $\{Z_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$. In the setting, we are going to classify Spr-elements of each real form of $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$.

Notation 5.1.1. In Subsection 5.1, we utilize the following notation:

- $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$.
- $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$. 

- $\mathfrak{g}_u$: the compact real form of $\mathfrak{a}_l$ given by $\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})$ and (2.3.1).
- $\{Z_a\}_{a=1}^l$: the dual basis of $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

5.1.1. *Case* AI $\mathfrak{sl}(2k+1, \mathbb{R})$: $l = 2k$ *and* $k \geq 1$. In this paragraph, we aim to classify Spr-elements of $\mathfrak{sl}(2k+1, \mathbb{R})$.

In the first place, we will construct an involutive automorphism θ_1 of $\mathfrak{a}_{2k} = \mathfrak{sl}(2k+1, \mathbb{C})$ such that (I) it satisfies the three conditions in Paragraph 2.3.2;

$$\text{(c1) } \theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u, \quad \text{(c2) } \theta_1(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}, \quad \text{(c3) } {}^t\theta_1(\Pi_{\triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})}$$

and (II) $\mathfrak{sl}(2k+1, \mathbb{R})$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_1. It is obvious from $\alpha_a(Z_b) = \delta_{a,b}$ that $\{Z_a\}_{a=1}^{2k}$ is a real basis of $\tilde{\mathfrak{h}}_{\mathbb{R}}$

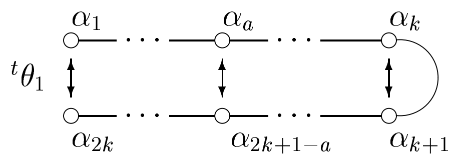
(see (2.3.2) for $\tilde{\mathfrak{h}}_{\mathbb{R}}$). Hence, $\{iZ_a\}_{a=1}^{2k}$ is a real basis of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$. So, we can define an involutive, linear isomorphism θ'_1 of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{2k}$ by

$$\theta'_1(iZ_a) := iZ_{2k+1-a} \quad \text{for } 1 \leq a \leq 2k. \tag{5.1.1}$$

From $\alpha_a(Z_b) = \delta_{a,b}$ and (5.1.1), it follows that ${}^t\theta'_{1\mathbb{C}}(\alpha_a) = \alpha_{2k+1-a}$ for all $1 \leq a \leq 2k$, where $\theta'_{1\mathbb{C}}$ denotes the complex linear extension of θ'_1 to $\tilde{\mathfrak{h}}$. Therefore, we conclude that

$${}^t\theta'_{1\mathbb{C}}(\triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})) = \triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})$$

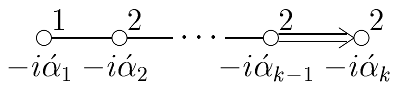
because $\triangle^+(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}}) = \{\sum_{p \leq q < r} \alpha_q \,|\, 1 \leq p < r \leq 2k+1\}$ (cf. Bourbaki [Br, pp. 265]). Then, Proposition 2.3.2 implies that there exists an involutive automorphism θ_1 of $\mathfrak{a}_{2k} = \mathfrak{sl}(2k+1, \mathbb{C})$ satisfying the three conditions (c1), (c2) and (c3). Here, the third condition (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})}$ has followed from $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and ${}^t\theta'_{1\mathbb{C}}(\alpha_a) = \alpha_{2k+1-a}$. Notice that θ_1 is the same involution as θ_ρ utilized in Murakami [Mu3, pp. 305, type AI].



Let us describe the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_1(K) = K\}$. By (c3), (5.1.1) and Remark 2.3.3-(ii), one perceives that $\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$ is a maximal abelian subalgebra of $\mathfrak{k}$ and it is as follows:

$$\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{i(Z_c + Z_{2k+1-c})\}_{c=1}^{k}. \tag{5.1.2}$$

The result of Murakami [Mu3, pp. 305, type AI] enables us to deduce that $\{-i\alpha_c|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$ is the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (ref. Remark 2.3.3-(iii), -(iv)), and that the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$ is as follows:

$\mathfrak{k} = \mathfrak{so}(2k+1)$: 

where $-i\acute{\alpha}_c := -i\alpha_c|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ for $1 \leq c \leq k$. Moreover, his result implies that the highest root $-i\mu$ in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is

$$-i\mu = -i(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_k)|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} \tag{5.1.3}$$

and that $\mathfrak{sl}(2k+1, \mathbb{R})$ is the real form $\mathfrak{g}$ of $\mathfrak{a}_{2k} = \mathfrak{sl}(2k+1, \mathbb{C})$ given by (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, where $\mathfrak{p}$ denotes the -1-eigenspace of θ_1 in $\mathfrak{g}_u$ ($\subset \mathfrak{a}_{2k}$). Now, let us describe the dual basis of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of the dual basis $\{Z_a\}_{a=1}^{2k}$ of $\Pi_{\triangle(\mathfrak{a}_{2k}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k}$. Its description will be utilized in the second place. Let $\{T_c\}_{c=1}^{k}$, $T_c \in \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$, be the dual basis of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$. Then by $\alpha_a(Z_b) = \delta_{a,b}$ and (5.1.2), we can describe the element T_c as follows:

$$T_c = i(Z_c + Z_{2k+1-c}) \quad \text{for } 1 \leq c \leq k. \tag{5.1.4}$$

In the second place, we will search Spr-elements of $\mathfrak{g} = \mathfrak{sl}(2k+1,\mathbb{R})$. Denote by $\mathfrak{W}^1_{\mathfrak{k}}$ the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^k$;

$$\mathfrak{W}^1_{\mathfrak{k}} = \{T \in \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, \cdots, -i\alpha_k(T) \geq 0\}.$$

On account of (4.1.2), we aim to search this Weyl chamber $\mathfrak{W}^1_{\mathfrak{k}}$ of $\mathfrak{k}$ for Spr-elements of $\mathfrak{g}$. First, let us provide a necessary condition for an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ to be an Spr-element of $\mathfrak{g}$. Suppose that an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$. Then by (5.1.3) and Lemma 4.2.4, we obtain $-i\alpha_c(T) = \delta_{1,c}$ for each $1 \leq c \leq k$. Accordingly, we have $T = T_1$ because $\{T_c\}_{c=1}^k$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^k$. Hence, it follows from (5.1.4) that

$$T = i(Z_1 + Z_{2k}). \tag{5.1.5}$$

Consequently, (5.1.5) is a necessary condition for an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ to be an Spr-element of $\mathfrak{g} = \mathfrak{sl}(2k+1,\mathbb{R})$. However, it is not the sufficient condition. Indeed, the element (5.1.5) $T = i(Z_1 + Z_{2k})$ can not be an Spr-element of $\mathfrak{g}$, because there exists a root $\beta = \sum_{q=1}^{2k}\alpha_q \in \triangle(\mathfrak{a}_{2k},\tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{a}_{2k},\tilde{\mathfrak{h}})$ and it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that $\beta(T) = \sum_{q=1}^{2k}\alpha_q(i(Z_1 + Z_{2k})) = 2i \neq \pm i$. This shows that the element $T = i(Z_1 + Z_{2k}) \in \mathfrak{W}^1_{\mathfrak{k}}$ can not be an Spr-element of $\mathfrak{g} = \mathfrak{sl}(2k+1,\mathbb{R})$ from Lemma 4.1.1. For the reasons, we get the following:

Proposition 5.1.2. *The set of Spr-elements of* AI: $\mathfrak{sl}(2k+1,\mathbb{R})$, $k \geq 1$, *is an empty set.*

5.1.2. *Case* AI $\mathfrak{sl}(2k,\mathbb{R})$: $l = 2k-1$ *and* $k \geq 2$. This paragraph deals with the classification of Spr-elements of $\mathfrak{sl}(2k,\mathbb{R})$ under our equivalence relation (defined in Section 1). Our result in this paragraph is Proposition 5.1.4.

We want to define an involutive automorphism θ_2 of $\mathfrak{g}_u$ such that $\mathfrak{sl}(2k,\mathbb{R})$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_2. In order to do so, we first construct an involutive automorphism θ_3 of $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k,\mathbb{C})$ which satisfies three conditions

$$\text{(c1) } \theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u, \quad \text{(c2) } \theta_3(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}, \quad \text{(c3) } {}^t\theta_3(\Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})}.$$

By use of θ_3, we will define an involutive automorphism θ_2 afterward. Since $\{Z_a\}_{a=1}^{2k-1}$ ($Z_a \in \tilde{\mathfrak{h}}$) is the dual basis of $\Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k-1}$, one deduces that $\{iZ_a\}_{a=1}^{2k-1}$ is a real basis of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ (see (2.3.2) for $\tilde{\mathfrak{h}}_{\mathbb{R}}$). Define an involutive, linear isomorphism θ'_3 of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{2k-1}$ by

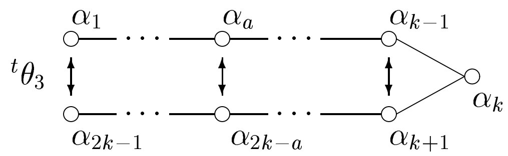
$$\theta'_3(iZ_a) := iZ_{2k-a} \quad \text{for } 1 \leq a \leq 2k-1. \tag{5.1.6}$$

Then since $\alpha_a(Z_b) = \delta_{a,b}$, we have ${}^t\theta'_{3\mathbb{C}}(\alpha_a) = \alpha_{2k-a}$ for all $1 \leq a \leq 2k-1$, where $\theta'_{3\mathbb{C}}$ denotes the complex linear extension of θ'_3 to $\tilde{\mathfrak{h}}$. Therefore, it follows that

$${}^t\theta'_{3\mathbb{C}}(\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})) = \triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})$$

because $\triangle^+(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}}) = \{\sum_{p\leq q<r}\alpha_q \mid 1 \leq p < r \leq 2k\}$ (cf. Bourbaki [Br, pp. 265]). Hence, Proposition 2.3.2 enables us to obtain an involutive automorphism θ_3 of $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k,\mathbb{C})$ satisfying conditions (i) $\theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\theta_3|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_3$ and (iii)

$\theta_3(X_{\pm\alpha_a}) = X_{\pm {}^t\theta_3(\alpha_a)}$. From $\alpha_a(Z_b) = \delta_{a,b}$, $\theta_3|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_3$ and (5.1.6), it follows that ${}^t\theta_3(\alpha_a) = \alpha_{2k-a}$ for every $1 \leq a \leq 2k-1$, so that ${}^t\theta_3(\Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})}$. Hence, we have constructed an involutive automorphism θ_3 of $\mathfrak{a}_{2k-1}$ satisfying the three conditions (c1), (c2) and (c3). Note that this involution θ_3 is the same as θ_ρ given in Murakami [Mu3, pp. 305, type AII].

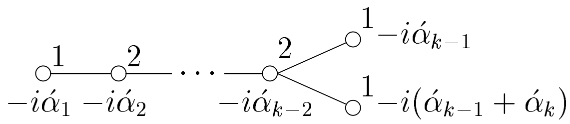


Now, let us define an automorphism θ_2 of $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k,\mathbb{C})$ by

$$\theta_2 := \theta_3 \circ \exp \pi \operatorname{ad}_{\mathfrak{a}_{2k-1}} iZ_k. \tag{5.1.7}$$

Notice that the restriction of θ_2 to $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ accords with $\theta_3|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_3$, and the automorphism θ_2 satisfies three conditions (c1) $\theta_2(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_2(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_2(\Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})}$. Further, notice that θ_2 is the same as the involution θ_1 in Murakami [Mu3, pp. 305, type AI]. Let $\mathfrak{k}$ denote the $+1$-eigenspace of θ_2 in $\mathfrak{g}_u$. The result of Murakami [Mu3, pp. 305, type AI] states that $\{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i(\alpha_{k-1}+\alpha_k)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{k-1}$ is the set of simple roots in $\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (ref. Remark 2.3.3-(iii), -(iv)) and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{so}(2k)$:



where $-i\acute{\alpha}_c := -i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ for $1 \leq c \leq k$; besides, his result implies that $\mathfrak{sl}(2k,\mathbb{R})$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k,\mathbb{C})$. Here, $\mathfrak{p}$ denotes the -1-eigenspace of θ_2. Remark that the highest root $-i\mu$ in $\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$-i\mu = -i\big(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{k-2} + \alpha_{k-1} + (\alpha_{k-1}+\alpha_k)\big)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} \quad \text{if } k \geq 3. \tag{5.1.8}$$

If $k = 2$, then $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1 := \mathfrak{su}(2)$ and $\mathfrak{k}_2 := \mathfrak{su}(2)$. In case of $k = 2$, we assume $\{-i\alpha_1|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$ and $\{-i(\alpha_1+\alpha_2)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$ to be the set of simple root in $\triangle(\mathfrak{k}_1,\mathfrak{k}_1\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $\triangle(\mathfrak{k}_2,\mathfrak{k}_2\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, respectively. Then, the highest root $-i\mu_1 \in \triangle(\mathfrak{k}_1,\mathfrak{k}_1\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2 \in \triangle(\mathfrak{k}_2,\mathfrak{k}_2\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$\begin{cases} -i\mu_1 = -i\alpha_1|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, \\ -i\mu_2 = -i(\alpha_1+\alpha_2)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} \quad \text{if } k = 2. \end{cases} \tag{5.1.9}$$

From now on, we are going to describe the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}$. The description will be needed later. Let $\{T_c\}_{c=1}^k$, $T_c \in \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$, be the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i(\alpha_{k-1}+\alpha_k)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{k-1}$, namely $-i\alpha_d(T_c) = \delta_{d,c}$ $(1 \leq d \leq k-1)$ and $-i(\alpha_{k-1}+\alpha_k)(T_c) = \delta_{k,c}$. We want to describe this basis $\{T_c\}_{c=1}^k$ in terms of the dual basis $\{Z_a\}_{a=1}^{2k-1}$ of $\Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k-1}$. It is immediate from $\theta_2|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_3$ and (5.1.6) that

$$\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{i(Z_d + Z_{2k-d}), iZ_k\}_{d=1}^{k-1}.$$

Accordingly, since $\alpha_a(Z_b)=\delta_{a,b}$, $-i\alpha_d(T_c)=\delta_{d,c}$ and $-i(\alpha_{k-1}+\alpha_k)(T_c)=\delta_{k,c}$, one can describe the element T_c $(1\leq c\leq k)$ as follows:

$$(5.1.10)\qquad \begin{cases} T_e=i(Z_e+Z_{2k-e}) & \text{for } 1\leq e\leq k-2,\\ T_{k-1}=i(Z_{k-1}-Z_k+Z_{k+1}), \\ T_k=iZ_k. \end{cases}$$

Now, let us search Spr-elements of $\mathfrak{g}=\mathfrak{sl}(2k,\mathbb{R})$. Denote by $\mathfrak{W}^2_{\mathfrak{k}}$ the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}=\{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}},-i(\alpha_{k-1}+\alpha_k)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{k-1}$;

$$\mathfrak{W}^2_{\mathfrak{k}}=\{T\in\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}\,|\,-i\alpha_1(T)\geq 0,\cdots,-i\alpha_{k-1}(T)\geq 0,-i(\alpha_{k-1}+\alpha_k)(T)\geq 0\}.$$

Taking (4.1.2) into consideration, we will search this Weyl chamber $\mathfrak{W}^2_{\mathfrak{k}}$ of $\mathfrak{k}$ for Spr-elements of $\mathfrak{g}=\mathfrak{sl}(2k,\mathbb{R})$. First, let us consider the case of $k\geq 3$, and provide a necessary condition for an element $T\in\mathfrak{W}^2_{\mathfrak{k}}$ to be an Spr-element of $\mathfrak{g}$. Suppose that an element $T\in\mathfrak{W}^2_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$. Then, Lemma 4.2.4, combined with (5.1.8), implies that $T=T_1$, T_{k-1} or T_k, because $\{T_c\}_{c=1}^k$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}=\{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}},-i(\alpha_{k-1}+\alpha_k)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{k-1}$. Therefore by (5.1.10), one has

$$(5.1.11)\qquad T=i(Z_1+Z_{2k-1}),\ i(Z_{k-1}-Z_k+Z_{k+1}) \text{ or } iZ_k.$$

This (5.1.11) is a necessary condition for an element $T\in\mathfrak{W}^2_{\mathfrak{k}}$ to be an Spr-element of $\mathfrak{g}=\mathfrak{sl}(2k,\mathbb{R})$. We will confirm whether (5.1.11) is the sufficient condition or not. There exists a root $\beta=\sum_{q=1}^{2k-1}\alpha_q\in\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})$, since $\triangle^+(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})=\{\sum_{p\leq q<r}\alpha_q\,|\,1\leq p<r\leq 2k\}$ (ref. Bourbaki [Br, pp. 265]). Accordingly, the element $T=i(Z_1+Z_{2k-1})\in\mathfrak{W}^2_{\mathfrak{k}}$ can not be an Spr-element of $\mathfrak{g}$ because $\alpha_a(Z_b)=\delta_{a,b}$ and $\beta(T)=\sum_{q=1}^{2k-1}\alpha_q(i(Z_1+Z_{2k-1}))=2i\neq\pm i$ (see Lemma 4.1.1). The other elements $T=i(Z_{k-1}-Z_k+Z_{k+1})$ and $T=iZ_k$ satisfy the condition c) in Lemma 4.1.1. Consequently, Lemma 4.1.1 assures that both $T=i(Z_{k-1}-Z_k+Z_{k+1})$ and $T=iZ_k$ are Spr-elements of $\mathfrak{g}=\mathfrak{sl}(2k,\mathbb{R})$. Thus, in case of $k\geq 3$, the condition "$T=i(Z_{k-1}-Z_k+Z_{k+1})$ or $T=iZ_k$" is a necessary and sufficient condition for an element $T\in\mathfrak{W}^2_{\mathfrak{k}}$ to be an Spr-element of $\mathfrak{g}=\mathfrak{sl}(2k,\mathbb{R})$. Next, let us consider the case of $k=2$. The arguments stated below are similar. Suppose that an element $T\in\mathfrak{W}^2_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$. Then, Lemma 4.2.2, together with (5.1.9), means that the element T equals T_1, T_2 or T_1+T_2, because $\{T_1,T_2\}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}=\{-i\alpha_1|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}},-i(\alpha_1+\alpha_2)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$. Therefore, $T=i(Z_1-Z_2+Z_3)$, iZ_2 or $i(Z_1+Z_3)$ (see (5.1.10)). Since T is an Spr-element of $\mathfrak{g}$, it must satisfy $\beta(T)=\pm i$ for any root $\beta\in\triangle(\mathfrak{a}_3,\tilde{\mathfrak{h}})\setminus\triangle_T(\mathfrak{a}_3,\tilde{\mathfrak{h}})$ (cf. Lemma 4.1.1); and hence it follows from $\alpha_a(Z_b)=\delta_{a,b}$ that the Spr-element T is either $i(Z_1-Z_2+Z_3)$ or iZ_2. Conversely, suppose that an element T' is either $i(Z_1-Z_2+Z_3)$ or iZ_2. Then, it belongs to $\mathfrak{W}^2_{\mathfrak{k}}$ and satisfies the condition c) in Lemma 4.1.1. So, the element T' is an Spr-element of $\mathfrak{g}$, due to Lemma 4.1.1. Accordingly, in case of $k=2$, an element $T\in\mathfrak{W}^2_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$ if and only if it is either $i(Z_1-Z_2+Z_3)$ or iZ_2. Summarizing above statements, we confirm that $Spr_{\mathfrak{g}}\cap\mathfrak{W}^2_{\mathfrak{k}}=\{i(Z_{k-1}-Z_k+Z_{k+1}),iZ_k\}$. So, it comes from (4.1.2) that

$$(5.1.12)\qquad Spr_{\mathfrak{g}}/(\{\pm 1\}\times\mathrm{Aut}(\mathfrak{g}))=\{\ [i(Z_{k-1}-Z_k+Z_{k+1})],\ [iZ_k]\ \}.$$

The following Lemma 5.1.3 enables us to see that the above *Spr*-element $i(Z_{k-1} - Z_k + Z_{k+1})$ is equivalent to iZ_k (see Section 1 for the definition of equivalent):

Lemma 5.1.3. *In the setting on Paragraph* 5.1.2; *there exists an automorphism* ϕ *of* $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k, \mathbb{C})$ *such that* $\phi(\mathfrak{g}) \subset \mathfrak{g}$ *and*

$$\begin{cases} \phi(iZ_p) = iZ_{2k-p} & \textit{for } 1 \leq p \leq 2k-1 \textit{ with } p \neq k, \\ \phi(iZ_k) = i(Z_{k-1} - Z_k + Z_{k+1}), \end{cases}$$

where $\mathfrak{g} = \mathfrak{sl}(2k, \mathbb{R})$ *and* $\{Z_a\}_{a=1}^{2k-1}$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k-1}$.

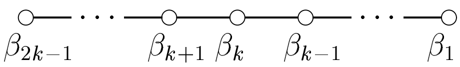
Proof. Let us construct such an automorphism ϕ of $\mathfrak{a}_{2k-1}$. Define an involutive, linear isomorphism ϕ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{2k-1}$ as follows:

$$\begin{cases} \phi'(iZ_p) := iZ_{2k-p} & \text{for } 1 \leq p \leq 2k-1 \text{ with } p \neq k, \\ \phi'(iZ_k) := i(Z_{k-1} - Z_k + Z_{k+1}). \end{cases} \tag{5.1.13}$$

Then, the complex linear extension $\phi'_{\mathbb{C}}$ of ϕ' to $\tilde{\mathfrak{h}}$ satisfies

$$\begin{cases} {}^t\phi'_{\mathbb{C}}(\alpha_q) = \alpha_{2k-q} & \text{for } 1 \leq q \leq k-2 \text{ and } k+2 \leq q \leq 2k-1, \\ {}^t\phi'_{\mathbb{C}}(\alpha_{k-1}) = \alpha_k + \alpha_{k+1}, \\ {}^t\phi'_{\mathbb{C}}(\alpha_k) = -\alpha_k, \\ {}^t\phi'_{\mathbb{C}}(\alpha_{k+1}) = \alpha_{k-1} + \alpha_k, \end{cases}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$. So, the Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2k-1}$ is as follows:



where $\beta_a := {}^t\phi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq 2k-1$. This implies that

$${}^t\phi'_{\mathbb{C}}(\triangle(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}})) = \triangle(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}})$$

since the Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2k-1}$ is the same as that of $\Pi_{\triangle(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k-1}$ (cf. Murakami [Mu3, Lemma 1, pp. 295]). Hence by Proposition 2.3.2, one gets an involutive automorphism $\bar{\phi}$ of $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k, \mathbb{C})$ satisfying (i) $\bar{\phi}(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (iii) $\bar{\phi}(X_{\pm\alpha_a}) = X_{\pm{}^t\bar{\phi}(\alpha_a)}$. Consequently by virtue of $\bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.1.13), the rest of proof is to demonstrate that the involution $\bar{\phi}$ of $\mathfrak{a}_{2k-1}$ satisfies the two conditions (a) and (b) in Proposition 2.3.4. Indeed, if $\bar{\phi}$ satisfies the two conditions, then there exists an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ such that $(\bar{\phi} \circ \exp \mathrm{ad}_{\mathfrak{a}_{2k-1}} iH)(\mathfrak{g}) = \mathfrak{g}$ (by Proposition 2.3.4). Since $\exp \mathrm{ad}_{\mathfrak{a}_{2k-1}} iH = \mathrm{id}$ on $\tilde{\mathfrak{h}}$, one has

$$(\bar{\phi} \circ \exp \mathrm{ad}_{\mathfrak{a}_{2k-1}} iH)|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'.$$

Defining ϕ by $\phi := \bar{\phi} \circ \exp \mathrm{ad}_{\mathfrak{a}_{2k-1}} iH$,[1] we can get the conclusion. So, the rest of proof is to demonstrate that $\bar{\phi}$ satisfies the two conditions (a) and (b) in Proposition 2.3.4. Let us show that, from now on. Since $\theta_2|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_3|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_3$ and (5.1.6), and since $\bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.1.13), we perceive that $\theta_2 \circ \bar{\phi} = \bar{\phi} \circ \theta_2$ on $i\tilde{\mathfrak{h}}_{\mathbb{R}}$. Thus, the

[1] This ϕ is an outer automorphism of $\mathfrak{sl}(2k, \mathbb{R})$.

involution $\bar{\phi}$ of $\mathfrak{a}_{2k-1}$ satisfies the condition (a) in Proposition 2.3.4. We want to show that the involution $\bar{\phi}$ also satisfies the condition (b) in Proposition 2.3.4. By the definition (5.1.7) of θ_2, we confirm that $\triangle_1(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}} : \theta_2)$ is an empty set (refer to (2.3.4) for $\triangle_1(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}} : \theta_2)$). Thus, it is natural that

$$^t\bar{\phi}\big(\triangle_1(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}} : \theta_2)\big) = \triangle_1(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}} : \theta_2),$$

namely $\bar{\phi}$ satisfies the condition (b). Accordingly, the involution $\bar{\phi}$ of $\mathfrak{a}_{2k-1}$ satisfies the two conditions (a), (b) in Proposition 2.3.4. For the reasons, we have verified Lemma 5.1.3. □

Lemma 5.1.3 and (5.1.12) allow us to lead the following:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [\,iZ_k\,]\ \}. \tag{5.1.14}$$

From Lemma 3.1.1-(1) and -(2), it follows that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(iZ_k))$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_k$. It is known that $\mathfrak{c}_{\mathfrak{g}}(iZ_k) = \mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{t}^1$ (cf. [Bm, Theorem 6.16]). That, together with (5.1.14), shows Proposition 5.1.4.

Proposition 5.1.4. *Under our equivalence relation, Spr-elements of* AI: $\mathfrak{g} = \mathfrak{sl}(2k, \mathbb{R})$, $k \geq 2$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [\,iZ_k\,]\ \}.$$

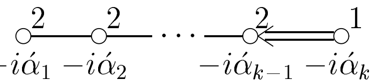
Besides, $(\mathfrak{g}, \mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{t}^1)$ *is the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_k$. *Here,* $\{Z_a\}_{a=1}^{2k-1}$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{a}_{2k-1}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k-1}$.

5.1.3. *Case* AII $\mathfrak{su}^*(2k)$: $l = 2k - 1$ *and* $k \geq 2$. In this paragraph, we classify Spr-elements of $\mathfrak{su}^*(2k)$ under our equivalence relation (see Proposition 5.1.5).

We use the involutive automorphism θ_3 of $\mathfrak{a}_{2k-1} = \mathfrak{sl}(2k, \mathbb{C})$ obtained in the previous paragraph. We again remark that this involution θ_3 is the same as θ_ρ given in Murakami [Mu3, pp. 305, type AII]. Let $\mathfrak{k}$ denote the $+1$-eigenspace of θ_3 in $\mathfrak{g}_u$ (see Notation 5.1.1 for $\mathfrak{g}_u$). Then, it follows from $\theta_3|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_3'$ and (5.1.6) that

$$\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{i(Z_d + Z_{2k-d}), iZ_k\}_{d=1}^{k-1}. \tag{5.1.15}$$

Due to Murakami's result [Mu3, pp. 305, type AII], one knows that $\{-i\alpha_c|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$ is the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{sp}(k)$:


where $-i\acute{\alpha}_c := -i\alpha_c|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ for $1 \leq c \leq k$ (ref. Remark 2.3.3). Moreover, by virtue of his result, one also sees that that $\mathfrak{su}^*(2k)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{a}_{2k-1}$ (where $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_3(P) = -P\}$) and that the highest root $-i\mu \in \triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$-i\mu = -i(2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{k-1} + \alpha_k)|_{\mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}. \tag{5.1.16}$$

First, let us describe the dual basis of $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of $\{Z_a\}_{a=1}^{2k-1}$. Let $\{T_c\}_{c=1}^{k}$, $T_c \in \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$, be the dual basis of $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$. Then by using $\alpha_a(Z_b) = \delta_{a,b}$ and (5.1.15), we obtain

$$(5.1.17)\qquad \begin{cases} T_d = i(Z_d + Z_{2k-d}) & \text{for } 1 \le d \le k-1, \\ T_k = iZ_k. \end{cases}$$

Next, we will provide a necessary and sufficient condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{su}^*(2k)$. Here, $\mathfrak{W}_{\mathfrak{k}}^3$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$;

$$\mathfrak{W}_{\mathfrak{k}}^3 = \{T \in \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \ge 0, \cdots, -i\alpha_{k-1}(T) \ge 0, -i\alpha_k(T) \ge 0\}.$$

Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ is an *Spr*-element of $\mathfrak{g}$. In this case, Lemma 4.2.4 and (5.1.16) enable us to have $-i\alpha_c(T) = \delta_{k,c}$ for each $c \in \{1,\ldots,k\}$. Therefore, we obtain $T = T_k$ because $\{T_c\}_{c=1}^{k}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\,\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{c=1}^{k}$. Hence, it is obvious from (5.1.17) that

$$(5.1.18)\qquad T = iZ_k.$$

This (5.1.18) is a necessary condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{su}^*(2k)$. On the other hand, (5.1.18) is also the sufficient condition, because it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that $\beta(T) = \beta(iZ_k) = \pm i$ for every root $\beta \in \triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})$ (cf. Lemma 4.1.1). Consequently, (5.1.18) is a necessary and sufficient condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{su}^*(2k)$. Accordingly by (4.1.2), we conclude that

$$(5.1.19)\qquad Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g})) = \{\ [\,iZ_k\,]\ \}.$$

Lemma 3.1.1-(1) and -(2) imply that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(iZ_k))$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_k$. In addition, Theorem 6.16 in [Bm] allows us to have $\mathfrak{c}_{\mathfrak{g}}(iZ_k) = \mathfrak{sl}(k,\mathbb{C})\oplus\mathfrak{t}^1$. Consequently since (5.1.19), we can assert the following:

Proposition 5.1.5. *Under our equivalence relation, Spr-elements of* AII: $\mathfrak{g} = \mathfrak{su}^*(2k)$, $k \ge 2$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g})) = \{\ [\,iZ_k\,]\ \}.$$

Besides, $(\mathfrak{g}, \mathfrak{sl}(k,\mathbb{C})\oplus\mathfrak{t}^1)$ *is the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_k$. *Here,* $\{Z_a\}_{a=1}^{2k-1}$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{a}_{2k-1},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2k-1}$.

5.1.4. *Case* AIII $\mathfrak{su}(j, l+1-j) : j = 1$. In this paragraph, we will classify *Spr*-elements of $\mathfrak{su}(1,l)$ (see Proposition 5.1.8).

First, we will give an involutive automorphism θ_4 of $\mathfrak{g}_u$ such that $\mathfrak{su}(1,l)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_4. Let us define an inner automorphism θ_4 of $\mathfrak{a}_l = \mathfrak{sl}(l+1,\mathbb{C})$ as follows:

$$(5.1.20)\qquad \theta_4 := \exp \pi\, \mathrm{ad}_{\mathfrak{a}_l}\, iZ_1.$$

Then, it follows from $iZ_1 \in \mathfrak{g}_u$ and $\theta_4|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ that the automorphism satisfies the three conditions in Paragraph 2.3.2; (c1) $\theta_4(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_4(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_4(\Pi_{\triangle(\mathfrak{a}_l,\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_l,\tilde{\mathfrak{h}})}$. Murakami's result [Mu3, pp. 297, type AIII] says that the automorphism θ_4 is involutive, the simple root system of $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_4(K) = K\}$ is $\{-i\alpha_c\}_{c=2}^{l}$ (ref. Remark 2.3.3), the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^{l}$ is

$$\mathfrak{k} = \mathfrak{su}(l) \oplus \mathfrak{t}^1\colon \quad \times \quad \underset{-i\alpha_2}{\overset{1}{\circ}} \cdots \!-\! \underset{-i\alpha_{l-1}}{\overset{1}{\circ}} \!-\!\!-\! \underset{-i\alpha_l}{\overset{1}{\circ}}$$

and $\mathfrak{su}(1,l)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{a}_l$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_4(P) = -P\}$. Remark that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$-i\mu = -i(\alpha_2 + \alpha_3 + \cdots + \alpha_l). \tag{5.1.21}$$

Now, let us search a Weyl chamber $\mathfrak{W}_{\mathfrak{k}}^4$ for *Spr*-elements of $\mathfrak{g} = \mathfrak{su}(1,l)$, where $\mathfrak{W}_{\mathfrak{k}}^4$ is given by

$$\mathfrak{W}_{\mathfrak{k}}^4 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \,|\, -i\alpha_2(T) \geq 0, -i\alpha_3(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0\}.$$

Lemma 5.1.6. *With the above notation; an element $T \in \mathfrak{W}_{\mathfrak{k}}^4$ is an Spr-element of $\mathfrak{g} = \mathfrak{su}(1,l)$ if and only if it is one of the following:*

$$iZ_c \text{ for } 2 \leq c \leq l, \quad i(-Z_1 + Z_c) \text{ for } 2 \leq c \leq l, \quad \pm iZ_1.$$

Proof. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^4$ is an *Spr*-element of $\mathfrak{g}$. Since $\alpha_a(Z_b) = \delta_{a,b}$ and $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^{l}$, one deduces that $-i\gamma(iZ_1) \equiv 0$ for all roots $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, and hence the element iZ_1 is a central element of $\mathfrak{k}$. The supposition and Lemma 4.2.3, combined with (5.1.21), assure that

$$(\text{c}'\text{-1})\ T = i(\lambda \cdot Z_1 + Z_c) \text{ for } 2 \leq c \leq l, \ \text{ or } \ (\text{c}'\text{-2})\ T = i\,\lambda \cdot Z_1,$$

where λ is a real number ($\lambda \neq 0$ in Case (c$'$-2)), because $T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$ and $\alpha_a(Z_b) = \delta_{a,b}$. Let us determine the value of the above λ. Since $T \in \mathfrak{W}_{\mathfrak{k}}^4$ is an *Spr*-element, Lemma 4.1.1 implies that the element must satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{a}_l, \tilde{\mathfrak{h}})$. Hence, the value of λ is as follows: $\lambda = -1$ or 0 in Case (c$'$-1), and $\lambda = \pm 1$ in Case (c$'$-2), because of $\alpha_a(Z_b) = \delta_{a,b}$ and $\triangle^+(\mathfrak{a}_l, \tilde{\mathfrak{h}}) = \{\sum_{p \leq q < r} \alpha_q \,|\, 1 \leq p < r \leq l+1\}$ (ref. Bourbaki [Br, pp. 265]). Therefore, if an element $T \in \mathfrak{W}_{\mathfrak{k}}^4$ is an *Spr*-element, then it is one of the following:

$$(\text{c-1.1})\ iZ_c \text{ for } 2 \leq c \leq l, \ (\text{c-1.2})\ i(-Z_1 + Z_c) \text{ for } 2 \leq c \leq l, \ (\text{c-2})\ \pm iZ_1.$$

Conversely, suppose that an element T' is one of the above elements. Then, it belongs to $\mathfrak{W}_{\mathfrak{k}}^4$, and satisfies $\beta(T') = \pm i$ for all roots $\beta \in \triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{a}_l, \tilde{\mathfrak{h}})$. Accordingly, the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{su}(1,l)$ (cf. Lemma 4.1.1). For the reasons, we have completed the proof of Lemma 5.1.6. □

By virtue of (4.1.2) and Lemma 5.1.6, we have

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{l} [iZ_{c_1}],\ [i(-Z_1 + Z_{c_2})], \\ [iZ_1] \end{array} \middle| \begin{array}{r} 2 \leq c_1 \leq l, \\ 2 \leq c_2 \leq l \end{array} \right\}, \tag{5.1.22}$$

where $\mathfrak{g} = \mathfrak{su}(1,l)$. The following lemma means that the above *Spr*-element iZ_c is equivalent to $i(-Z_1 + Z_{l+2-c})$.

Lemma 5.1.7. *In the above setting; there exists an involutive automorphism ϕ of $\mathfrak{g} = \mathfrak{su}(1, l)$ such that*

$$\begin{cases} \phi(iZ_1) = -iZ_1, \\ \phi(iZ_c) = i(-Z_1 + Z_{l+2-c}) \text{ for } 2 \leq c \leq l. \end{cases}$$

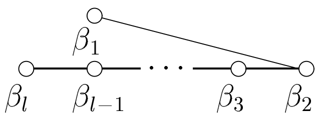
Proof. Define an involutive, linear isomorphism ϕ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^l$ by

$$\begin{cases} \phi'(iZ_1) := -iZ_1, \\ \phi'(iZ_c) := i(-Z_1 + Z_{l+2-c}) \text{ for } 2 \leq c \leq l. \end{cases} \tag{5.1.23}$$

Then, the complex linear extension $\phi'_{\mathbb{C}}$ of ϕ' to $\tilde{\mathfrak{h}}$ satisfies

$$\begin{cases} {}^t\phi'_{\mathbb{C}}(\alpha_1) = -(\alpha_1 + \alpha_2 + \cdots + \alpha_l), \\ {}^t\phi'_{\mathbb{C}}(\alpha_c) = \alpha_{l+2-c} \text{ for } 2 \leq c \leq l \end{cases} \tag{5.1.24}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$. Therefore, the Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^l$ is



where $\beta_a := {}^t\phi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq l$. So, it follows that

$${}^t\phi'_{\mathbb{C}}\big(\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})$$

(cf. Lemma 1 in Murakami [Mu3, pp. 295]). Consequently, Proposition 2.3.2 enables us to get an involutive automorphism ϕ of $\mathfrak{a}_l$ such that (i) $\phi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (iii) $\phi(X_{\pm\alpha_a}) = X_{\pm {}^t\phi(\alpha_a)}$. Therefore, the rest of proof is to verify that the involution ϕ is an automorphism of $\mathfrak{g} = \mathfrak{su}(1, l)$, because of $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.1.23). Since θ_4 is involutive and (5.1.20), and since $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.1.23), we are able to obtain

$$\phi \circ \theta_4 = \exp \pi \,\mathrm{ad}_{\mathfrak{a}_l} \phi(iZ_1) \circ \phi = \exp \pi \,\mathrm{ad}_{\mathfrak{a}_l}(-iZ_1) \circ \phi = \theta_4 \circ \phi.$$

This, together with $\phi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, shows that the involution ϕ is an automorphism of $\mathfrak{g} = \mathfrak{su}(1, l)$ (see Proposition 2.2.3).[2] Thus, we have proved Lemma 5.1.7. □

Lemma 5.1.7 and (5.1.22) imply that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\, [\, i(-Z_1 + Z_c)],\ [\, iZ_1\,] \mid 2 \leq c \leq l\}, \tag{5.1.25}$$

where $\mathfrak{g} = \mathfrak{su}(1, l)$. About the above Spr-elements, it is known that

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_c)) &= \mathfrak{su}(c-1) \oplus \mathfrak{su}(1, l+1-c) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(iZ_1) &= \mathfrak{su}(l) \oplus \mathfrak{t}^1 \end{aligned} \tag{5.1.26}$$

(cf. Theorem 6.16 in [Bm]). From this, we deduce that for any $2 \leq c, c' \leq l$,

$$i(-Z_1 + Z_c) \text{ is equivalent to } i(-Z_1 + Z_{c'}) \text{ if and only if } c = c'.$$

[2]This ϕ is an outer automorphism of $\mathfrak{su}(1, l)$.

Besides, it follows from $c \leq l$ and (5.1.26) that

$$i(-Z_1 + Z_c) \text{ is not equivalent to } iZ_1.$$

For the reasons, *Spr*-elements of $\mathfrak{g} = \mathfrak{su}(1, l)$ are classified as follows:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_c)],\ [iZ_1]\ \mid 2 \leq c \leq l\}.$$

Lemma 3.1.1-(1) and -(2), together with (5.1.26), yields that $(\mathfrak{g}, \mathfrak{su}(c-1) \oplus \mathfrak{su}(1, l+1-c) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{su}(l) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_c)$ and $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$, respectively. Hence, we conclude the following:

Proposition 5.1.8. *Under our equivalence relation, Spr-elements of* AIII: $\mathfrak{g} = su(j, l+1-j)$, $j = 1$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_c)],\ [iZ_1]\ \mid 2 \leq c \leq l\}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{su}(c-1) \oplus \mathfrak{su}(1, l+1-c) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{su}(l) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_c)$ *and* $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

5.1.5. *Case* AIII $\mathfrak{su}(j, l+1-j) : 2 \leq j \leq l-1$. In this paragraph, we achieve the classification of *Spr*-elements of $\mathfrak{su}(j, l+1-j)$ (see Proposition 5.1.12).

Let us give an involutive automorphism θ_5 of $\mathfrak{g}_u$ such that $\mathfrak{su}(j, l+1-j)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_5. Define an inner automorphism θ_5 of $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$ by

$$\theta_5 := \exp \pi \operatorname{ad}_{\mathfrak{a}_l} iZ_j. \tag{5.1.27}$$

Since $iZ_j \in \mathfrak{g}_u$ and $\theta_5|_{\tilde{\mathfrak{h}}} = \mathrm{id}$, it satisfies the conditions in Paragraph 2.3.2; (c1) $\theta_5(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_5(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_5(\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})}$. Furthermore, Murakami's result [Mu3, pp. 297, type AIII] implies that the automorphism θ_5 is involutive, $\{-i\alpha_f\}_{f=1}^{j-1} \cup \{-i\alpha_g\}_{g=j+1}^{l}$ is the simple root system $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})}$, its Dynkin diagram is

$$\mathfrak{k} = \mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1:\ \underset{-i\alpha_1}{\overset{1}{\circ}} \overset{1}{—\cdots—} \underset{-i\alpha_{j-1}}{\overset{1}{\circ}} \times \underset{-i\alpha_{j+1}}{\overset{1}{\circ}} \overset{1}{—\cdots—} \underset{-i\alpha_l}{\overset{1}{\circ}}$$

and $\mathfrak{su}(j, l+1-j)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \mid \theta_5(K) = K\}$ and $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_5(P) = -P\}$. Notice that $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1$, $\mathfrak{k}_2$ and the center $\mathfrak{k}_{\mathfrak{z}} = \mathfrak{t}^1$;

$$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_{\mathfrak{z}},$$

where $\mathfrak{k}_1$ and $\mathfrak{k}_2$ denote $\mathfrak{su}(j)$ and $\mathfrak{su}(l+1-j)$ respectively. Now, we assume $\{-i\alpha_f\}_{f=1}^{j-1}$ (resp. $\{-i\alpha_g\}_{g=j+1}^{l}$) to be the set of simple roots in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (resp. $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$). Then, the highest root $-i\mu_1$ in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2$ in $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$\begin{cases} -i\mu_1 = -i(\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1}), \\ -i\mu_2 = -i(\alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_l). \end{cases} \tag{5.1.28}$$

Let us prove Lemma 5.1.9.

Lemma 5.1.9. *With the assumptions and notation in Paragraph* 5.1.5; *an element* $T \in \mathfrak{W}^5_{\mathfrak{k}}$ *is an* Spr*-element of* $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$ *if and only if it is one of the following*:

$$\begin{array}{lll} iZ_f, & i(Z_f - Z_j), & iZ_{j+h}, \\ i(-Z_j + Z_{j+h}), & i(Z_f - Z_j + Z_{j+h}), & \pm iZ_j, \end{array}$$

where $1 \leq f \leq j-1$ *and* $1 \leq h \leq l-j$. *Here* $\mathfrak{W}^5_{\mathfrak{k}}$ *is the positive Weyl chamber with respect to* $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_f\}_{f=1}^{j-1} \cup \{-i\alpha_g\}_{g=j+1}^{l}$;

$$\mathfrak{W}^5_{\mathfrak{k}} = \left\{ T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\alpha_1(T) \geq 0, \cdots, -i\alpha_{j-1}(T) \geq 0, \\ \quad -i\alpha_{j+1}(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0 \end{array} \right\}.$$

Proof. Since $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$, any element $T \in \mathfrak{W}^5_{\mathfrak{k}}$ can be written as follows:

$$T = i(\lambda_1 \cdot Z_1 + \lambda_2 \cdot Z_2 + \cdots + \lambda_l \cdot Z_l), \quad \lambda_a \in \mathbb{R}.$$

Suppose that an element $T = \sum_{a=1}^{l} \lambda_a \cdot iZ_a \in \mathfrak{W}^5_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$. Then, Lemma 4.2.1, together with (5.1.28) and $\alpha_a(Z_b) = \delta_{a,b}$, means that one of the following four cases only occurs:

$$\text{(5.1.29)} \quad \begin{array}{ll} \text{(a}'\text{-1) } T = i(Z_f + \lambda_j \cdot Z_j); & \text{(a}'\text{-2) } T = i(\lambda_j \cdot Z_j + Z_{j+h}); \\ \text{(a}'\text{-3) } T = i(Z_f + \lambda_j \cdot Z_j + Z_{j+h}); & \text{(a}'\text{-4) } T = i\,\lambda_j \cdot Z_j, \end{array}$$

where $1 \leq f \leq j-1$ and $1 \leq h \leq l-j$. Here, $-i\gamma(iZ_j) \equiv 0$ for any root $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, because $\alpha_a(Z_b) = \delta_{a,b}$ and $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_f\}_{f=1}^{j-1} \cup \{-i\alpha_g\}_{g=j+1}^{l}$. Therefore, if an element $T \in \mathfrak{W}^5_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$, then it is one of the elements in (5.1.29). Let us determine the value of λ_j in (5.1.29). By Lemma 4.1.1, the Spr-element $T \in \mathfrak{W}^5_{\mathfrak{k}}$ must satisfy $\beta(T) = \pm i$ for each root $\beta \in \triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}}) \backslash \triangle_T(\mathfrak{a}_l, \tilde{\mathfrak{h}})$. Hence, $\lambda_j = 0$ or -1 in two Cases (a$'$-1) and (a$'$-2), $\lambda_j = -1$ in Case (a$'$-3), and $\lambda_j = \pm 1$ in Case (a$'$-4) because $\alpha_a(Z_b) = \delta_{a,b}$ and $\triangle^+(\mathfrak{a}_l, \tilde{\mathfrak{h}}) = \{\sum_{p \leq q < r} \alpha_q \,|\, 1 \leq p < r \leq l+1\}$ (ref. Bourbaki [Br, pp. 265]). Accordingly, if an element $T \in \mathfrak{W}^5_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$, then it is one of the following:

$$\text{(5.1.30)} \quad \begin{array}{ll} \text{(a-1) } iZ_f,\ i(Z_f - Z_j); & \text{(a-2) } iZ_{j+h},\ i(-Z_j + Z_{j+h}); \\ \text{(a-3) } i(Z_f - Z_j + Z_{j+h}); & \text{(a-4) } \pm iZ_j. \end{array}$$

Conversely, if an element T' is one of the elements in (5.1.30), then $\mathfrak{W}^5_{\mathfrak{k}} \ni T' \neq 0$ and $\beta(T') = \pm i$ for every root $\beta \in \triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{a}_l, \tilde{\mathfrak{h}})$; and hence the element T' is an Spr-element of $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$ (ref. Lemma 4.1.1). Accordingly, Lemma 5.1.9 has been shown. □

By (4.1.2) and Lemma 5.1.9, we perceive that

$$\text{(5.1.31)} \quad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$$
$$= \left\{ \begin{array}{l} [iZ_{f_1}],\ [i(Z_{f_2} - Z_j)], \\ [iZ_{j+h_1}],\ [i(-Z_j + Z_{j+h_2})], \\ [i(Z_{f_3} - Z_j + Z_{j+h_3})],\ [iZ_j] \end{array} \;\middle|\; \begin{array}{r} 1 \leq f_p \leq j-1, \\ 1 \leq h_p \leq l-j, \\ 1 \leq p \leq 3 \end{array} \right\},$$

where $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$. From now on, let us demonstrate that the above Spr-element iZ_f, iZ_{j+h} and $i(Z_f - Z_j + Z_{j+h})$ are equivalent to $i(Z_{j-f} - Z_j)$, $i(-Z_j + Z_{l+1-h})$ and $i(Z_{j-f} - Z_j + Z_{l+1-h})$, respectively.

Lemma 5.1.10. *In the setting on Paragraph* 5.1.5; *there exists an involutive automorphism* ψ *of* $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$ *satisfying*

$$\begin{cases} \psi(iZ_f) = i(Z_{j-f} - Z_j) & \text{for } 1 \leq f \leq j-1, \\ \psi(iZ_j) = -iZ_j, \\ \psi(iZ_{j+h}) = i(-Z_j + Z_{l+1-h}) & \text{for } 1 \leq h \leq l-j. \end{cases}$$

Here $\{Z_a\}_{a=1}^{l}$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$.

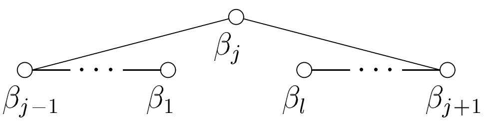
Proof. Since $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$, one can define an involutive, linear isomorphism ψ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ as follows:

$$\text{(5.1.32)} \qquad \begin{cases} \psi'(iZ_f) := i(Z_{j-f} - Z_j) & \text{for } 1 \leq f \leq j-1, \\ \psi'(iZ_j) := -iZ_j, \\ \psi'(iZ_{j+h}) := i(-Z_j + Z_{l+1-h}) & \text{for } 1 \leq h \leq l-j. \end{cases}$$

Then, it is immediate from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$\text{(5.1.33)} \qquad \begin{cases} {}^t\psi'_{\mathbb{C}}(\alpha_f) = \alpha_{j-f} & \text{for } 1 \leq f \leq j-1, \\ {}^t\psi'_{\mathbb{C}}(\alpha_j) = -(\alpha_1 + \alpha_2 + \cdots + \alpha_l), \\ {}^t\psi'_{\mathbb{C}}(\alpha_{j+h}) = \alpha_{l+1-h} & \text{for } 1 \leq h \leq l-j, \end{cases}$$

where $\psi'_{\mathbb{C}}$ denotes the complex linear extension of ψ' to $\tilde{\mathfrak{h}}$. Hence, the Dynkin diagram of $\{{}^t\psi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{l}$ is as follows:



where $\beta_a := {}^t\psi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq l$. Therefore, it follows that

$${}^t\psi'_{\mathbb{C}}(\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})) = \triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})$$

because the Dynkin diagram of $\{{}^t\psi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{l}$ is the same as that of $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$ (ref. Murakami [Mu3, Lemma 1, pp. 295]). Consequently, Proposition 2.3.2 enables us to obtain an involutive automorphism ψ of $\mathfrak{a}_l = \mathfrak{sl}(l+1, \mathbb{C})$ such that (i) $\psi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\psi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (iii) $\psi(X_{\pm\alpha_a}) = X_{\pm{}^t\psi(\alpha_a)}$. This involution ψ is an automorphism of $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$. Indeed, since $\psi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.1.32), and since θ_5 is involutive and (5.1.27), we have

$$\psi \circ \theta_5 = \psi \circ \exp \pi \operatorname{ad}_{\mathfrak{a}_l} iZ_j = \exp \pi \operatorname{ad}_{\mathfrak{a}_l} \psi(iZ_j) \circ \psi = \theta_5 \circ \psi.$$

Therefore, Proposition 2.2.3 means that ψ is an automorphism of $\mathfrak{g}$.[3] So, Lemma 5.1.10 comes from $\psi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.1.32). □

[3]This ψ is an outer automorphism of $\mathfrak{su}(j, l+1-j)$.

Lemma 5.1.10 and (5.1.31) state that

$$(5.1.34)\quad Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{l|l} [\,i(Z_{f_1}-Z_j)\,],\ [\,i(-Z_j+Z_{j+h_1})\,], & 1\le f_1\le j-1, 1\le h_1\le l-j, \\ [\,i(Z_{f_2}-Z_j+Z_{j+h_2})\,], & 1\le f_2\le [(j-2)/2]+1, \\ [\,iZ_j\,] & 1\le h_2\le l-j \end{array} \right\},$$

where $\mathfrak{g}=\mathfrak{su}(j,l+1-j)$. Now, we are going to investigate whether the above *Spr*-elements are mutually equivalent or not.

Lemma 5.1.11. *In the above setting; two Spr-elements $T=i(Z_f-Z_j+Z_{j+h})$ and $T'=i(Z_{f'}-Z_j+Z_{j+h'})$ of $\mathfrak{g}$ are equivalent to each other if and only if the following Case* (i) *or* (ii) *holds*:

(i) $f=f'$ *and* $h=h'$.
(ii) $f=h'$, $h=f'$ *and* $l+1=2j$.

Here, $1\le f,f'\le [(j-2)/2]+1$ *and* $1\le h,h'\le [(l-j-1)/2]+1$.

Proof. Theorem 6.16 in [Bm] implies that

$$(5.1.35)\qquad \mathfrak{c}_{\mathfrak{g}}(i(Z_f-Z_j+Z_{j+h}))=\mathfrak{su}(f,h)\oplus\mathfrak{su}(j-f,l+1-j-h)\oplus\mathfrak{t}^1.$$

Suppose that the *Spr*-element $T=i(Z_f-Z_j+Z_{j+h})$ is equivalent to $T'=i(Z_{f'}-Z_j+Z_{j+h'})$. Then, it is immediate from (5.1.35) that

$$\mathfrak{su}(f,h)\oplus\mathfrak{su}(j-f,l+1-j-h)=\mathfrak{su}(f',h')\oplus\mathfrak{su}(j-f',l+1-j-h').$$

Therefore, the following Case (1) or (2) occurs:

(1) $\mathfrak{su}(f,h)=\mathfrak{su}(f',h')$ and $\mathfrak{su}(j-f,l+1-j-h)=\mathfrak{su}(j-f',l+1-j-h')$.
(2) $\mathfrak{su}(f,h)=\mathfrak{su}(j-f',l+1-j-h')$ and $\mathfrak{su}(j-f,l+1-j-h)=\mathfrak{su}(f',h')$.

This shows that the following Case (i), (ii), (iii) or (iv) holds:

(i) $f=f'$ and $h=h'$.
(ii) $f=h'$, $h=f'$ and $l+1=2j$.
(iii) $f=j-f'$ and $h=l+1-j-h'$.
(iv) $f=l+1-j-h'$, $h=j-f'$ and $l+1=2j$.

The above two Cases (iii) and (iv) are contained in Case (i), because $1\le f,f'\le [(j-2)/2]+1$ and $1\le h,h'\le [(l-j-1)/2]+1$. Hence, each of the Cases (i) and (ii) holds when $T=i(Z_f-Z_j+Z_{j+h})$ is equivalent to $T'=i(Z_{f'}-Z_j+Z_{j+h'})$.

From now on, we will suppose that each of the Cases (i) and (ii) holds, and we will demonstrate that in each case, the *Spr*-element $T=i(Z_f-Z_j+Z_{j+h})$ is equivalent to $T'=i(Z_{f'}-Z_j+Z_{j+h'})$.

Case (i) $f=f'$ and $h=h'$: Suppose that Case (i) holds. Then, it is clear that $i(Z_f-Z_j+Z_{j+h})=i(Z_{f'}-Z_j+Z_{j+h'})$; and hence $T=i(Z_f-Z_j+Z_{j+h})$ is equivalent to $T'=i(Z_{f'}-Z_j+Z_{j+h'})$.

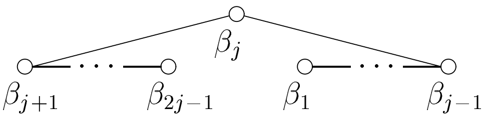
Case (ii) $f=h'$, $h=f'$ and $l+1=2j$: Suppose that Case (ii) holds. Let us show that there exists an automorphism ϕ of $\mathfrak{g}=\mathfrak{su}(j,l+1-j)=\mathfrak{su}(j,j)$ satisfying $\phi(T)=T'$. Since $i\tilde{\mathfrak{h}}_{\mathbb{R}}=\mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{2j-1}$, one can define an involutive linear isomorphism ϕ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ by

$$\text{(5.1.36)} \qquad \begin{cases} \phi'(iZ_s) := i(-Z_j + Z_{j+s}) \text{ for } 1 \leq s \leq j-1, \\ \phi'(iZ_j) := -iZ_j, \\ \phi'(iZ_{j+t}) := i(Z_t - Z_j) \text{ for } 1 \leq t \leq j-1. \end{cases}$$

Then, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$\text{(5.1.37)} \qquad \begin{cases} {}^t\phi'_{\mathbb{C}}(\alpha_s) = \alpha_{j+s} \text{ for } 1 \leq s \leq j-1, \\ {}^t\phi'_{\mathbb{C}}(\alpha_j) = -(\alpha_1 + \alpha_2 + \cdots + \alpha_{2j-1}), \\ {}^t\phi'_{\mathbb{C}}(\alpha_{j+t}) = \alpha_t \text{ for } 1 \leq t \leq j-1, \end{cases}$$

where $\phi'_{\mathbb{C}}$ denotes the complex linear extension of ϕ' to $\tilde{\mathfrak{h}}$. The Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2j-1}$ is as follows:



where $\beta_a := {}^t\phi'_{\mathbb{C}}(\alpha_a)$ for each $1 \leq a \leq 2j-1$. This yields that

$${}^t\phi'_{\mathbb{C}}\big(\triangle(\mathfrak{a}_{2j-1}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{a}_{2j-1}, \tilde{\mathfrak{h}})$$

because the Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2j-1}$ is the same as that of $\Pi_{\triangle(\mathfrak{a}_{2j-1}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2j-1}$ (cf. Lemma 1 in Murakami [Mu3, pp. 295]). Consequently, Proposition 2.3.2 assures that there exists an involutive automorphism ϕ of $\mathfrak{a}_{2j-1} = \mathfrak{sl}(2j, \mathbb{C})$ which satisfies $\phi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and $\phi(X_{\pm\alpha_a}) = X_{\pm {}^t\phi(\alpha_a)}$. We want to confirm that the involution ϕ is an automorphism of $\mathfrak{g}$. It follows from $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.1.36) that $\phi(iZ_j) = -iZ_j$. Therefore, we obtain

$$\phi \circ \theta_5 = \phi \circ \exp \pi \operatorname{ad}_{\mathfrak{a}_{2j-1}} iZ_j = \exp \pi \operatorname{ad}_{\mathfrak{a}_{2j-1}} \phi(iZ_j) \circ \phi = \theta_5 \circ \phi$$

because θ_5 is involutive and (5.1.27). Thus, Proposition 2.2.3 means that ϕ is an automorphism of $\mathfrak{g} = \mathfrak{su}(j,j)$.[4] Since $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.1.36), the supposition "$f = h'$, $h = f'$ and $l + 1 = 2j$" implies that

$$\phi(i(Z_f - Z_j + Z_{j+h})) = i(Z_{j+f} - Z_j + Z_h) = i(Z_{j+h'} - Z_j + Z_{f'}).$$

Consequently, if $f = h'$, $h = f'$ and $l + 1 = 2j$, then the *Spr*-element $T = i(Z_f - Z_j + Z_{j+h})$ is equivalent to $T' = i(Z_{f'} - Z_j + Z_{j+h'})$ via ϕ. For the reasons, we have completed the proof of Lemma 5.1.11. □

Let us continue to investigate whether the *Spr*-elements in (5.1.34) are mutually equivalent or not. It is known that

$$\mathfrak{c}_{\mathfrak{k}}(i(Z_f - Z_j)) = \mathfrak{su}(j-f) \oplus \mathfrak{su}(f) \oplus \mathfrak{su}(l+1-f) \oplus \mathfrak{t}^2,$$
$$\mathfrak{c}_{\mathfrak{k}}(i(-Z_j + Z_{j+h})) = \mathfrak{su}(h) \oplus \mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j-h) \oplus \mathfrak{t}^2,$$
$$\mathfrak{c}_{\mathfrak{k}}(i(Z_f - Z_j + Z_{j+h})) = \mathfrak{su}(f) \oplus \mathfrak{su}(j-f) \oplus \mathfrak{su}(h) \oplus \mathfrak{su}(l+1-j-h) \oplus \mathfrak{t}^3,$$

[4] This ϕ is an outer automorphism of $\mathfrak{su}(j,j)$.

$$\mathfrak{c}_{\mathfrak{k}}(iZ_j) = \mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1$$

(see Theorem 6.16 in [Bm]). Comparing the centers of the above centralizers, we are able to confirm that for any $1 \leq f_1 \leq j-1$, $1 \leq h_1 \leq l-j$, $1 \leq f_2 \leq [(j-2)/2]+1$ and $1 \leq h_2 \leq l-j$

$$\text{(5.1.38)} \quad \begin{cases} \text{both } i(Z_{f_1} - Z_j) \text{ and } i(-Z_j + Z_{j+h_1}) \text{ are} \\ \qquad \text{not equivalent to } i(Z_{f_2} - Z_j + Z_{j+h_2}) \text{ and } iZ_j; \\ i(Z_{f_2} - Z_j + Z_{j+h_2}) \text{ is not equivalent to } iZ_j. \end{cases}$$

It is also known that

$$\mathfrak{c}_{\mathfrak{g}}(i(Z_f - Z_j)) = \mathfrak{su}(j-f) \oplus \mathfrak{su}(l+1-j, f) \oplus \mathfrak{t}^1,$$
$$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+h})) = \mathfrak{su}(h) \oplus \mathfrak{su}(l+1-j-h, j) \oplus \mathfrak{t}^1$$

(see Theorem 6.16 in [Bm] again). Hence, for $1 \leq f, f' \leq j-1$ and $1 \leq h, h' \leq l-j$, one deduces that

$$\text{(5.1.39)} \quad \begin{cases} i(Z_f - Z_j) \text{ is equivalent to } i(Z_{f'} - Z_j) \text{ if and only if } f = f', \\ i(-Z_j + Z_{j+h}) \text{ is equivalent to } i(-Z_j + Z_{j+h'}) \text{ if and only if } h = h' \end{cases}$$

and that

$$\text{(5.1.40)} \quad \begin{aligned} &i(Z_f - Z_j) \text{ is not equivalent to } i(-Z_j + Z_{j+h}), \\ &\qquad\qquad \text{except for the case where } j - f = h \text{ and } 2j = l+1. \end{aligned}$$

Let us verify that the *Spr*-element $i(Z_f - Z_j)$ is equivalent to $i(-Z_j + Z_{2j-f})$ in case of "$l+1 = 2j$," where $1 \leq f \leq j-1$. In order to do so, we will show that the involution θ_3 of $\mathfrak{a}_{2j-1} = \mathfrak{sl}(2j, \mathbb{C})$ is an automorphism of $\mathfrak{g} = \mathfrak{su}(j, l+1-j) = \mathfrak{su}(j,j)$, where θ_3 was constructed in Paragraph 5.1.2. Since $\theta_3|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_3'$ and (5.1.6), one obtains $\theta_3(iZ_a) = iZ_{2j-a}$ for any $1 \leq a \leq 2j-1$; and hence $\theta_3(iZ_j) = iZ_j$. Therefore, it comes from (5.1.27) that

$$\theta_3 \circ \theta_5 = \theta_3 \circ \exp \pi \operatorname{ad}_{\mathfrak{a}_{2j-1}} iZ_j = \exp \pi \operatorname{ad}_{\mathfrak{a}_{2j-1}} \theta_3(iZ_j) \circ \theta_3 = \theta_5 \circ \theta_3.$$

In Paragraph 5.1.2, it is shown that $\theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u$. Consequently, Proposition 2.2.3 allows us to see that the involution θ_3 of $\mathfrak{a}_{2j-1}$ is an automorphism of $\mathfrak{g} = \mathfrak{su}(j,j)$.[5] It is natural that $\theta_3(i(Z_f - Z_j)) = i(Z_{2j-f} - Z_j)$, so that in case of $l+1 = 2j$, the *Spr*-element $i(Z_f - Z_j)$ is equivalent to $i(Z_{2j-f} - Z_j)$ via θ_3. For the reasons, one concludes that

$$\text{(5.1.41)} \quad \begin{aligned} &i(Z_f - Z_j) \text{ is equivalent to } i(-Z_j + Z_{j+h}) \\ &\qquad\qquad \text{in the case where } j - f = h \text{ and } 2j = l+1. \end{aligned}$$

By use of (5.1.34) and (5.1.38)–(5.1.41), we will prove Proposition 5.1.12.

Proposition 5.1.12. *Under our equivalence relation, Spr-elements of* AIII: $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$, $2 \leq j \leq l-1$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$$

[5] This θ_3 becomes an outer automorphism of $\mathfrak{su}(j,j)$.

$$= \begin{cases} \left\{ \begin{array}{l|l} [\,i(Z_{f_1} - Z_j)\,], & 1 \le f_1 \le j-1, \\ [\,i(-Z_j + Z_{j+h_1})\,], & 1 \le h_1 \le l-j, \\ [\,i(Z_{f_2} - Z_j + Z_{j+h_2})\,], & 1 \le f_2 \le [(j-2)/2]+1, \\ [\,iZ_j\,] & 1 \le h_2 \le l-j \end{array} \right\} & \textit{if } l+1 \ne 2j, \\ \left\{ \begin{array}{l|l} [\,i(Z_{f_1} - Z_j)\,], & 1 \le f_1 \le j-1, \\ [\,i(Z_{f_2} - Z_j + Z_{j+h_2})\,], & 1 \le f_2 \le [(j-2)/2]+1, \\ [\,iZ_j\,] & f_2 \le h_2 \le j - f_2 \end{array} \right\} & \textit{if } l+1 = 2j. \end{cases}$$

Besides, (1) $(\mathfrak{g}, \mathfrak{su}(j-f) \oplus \mathfrak{su}(l+1-j, f) \oplus \mathfrak{t}^1)$, (2) $(\mathfrak{g}, \mathfrak{su}(h) \oplus \mathfrak{su}(l+1-j-h, j) \oplus \mathfrak{t}^1)$, (3) $(\mathfrak{g}, \mathfrak{su}(f,h) \oplus \mathfrak{su}(j-f, l+1-j-h) \oplus \mathfrak{t}^1)$ *and* (4) $(\mathfrak{g}, \mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_f - Z_j)$, $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+h})$, $\rho_3 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_f - Z_j + Z_{j+h})$ *and* $\rho_4 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_j$*, respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{a}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. By (5.1.34), (5.1.38)–(5.1.41) and Lemma 5.1.11, we conclude the first half of statements on this proposition. Thus, we are going to show the latter half of the statements. Since $i(Z_f - Z_j)$ is an *Spr*-element of $\mathfrak{g} = \mathfrak{su}(j, l+1-j)$, Lemma 3.1.1-(1) and -(2) assure that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(Z_f - Z_j)))$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_f - Z_j)$. In addition, it is known that $\mathfrak{c}_{\mathfrak{g}}(i(Z_f - Z_j)) = \mathfrak{su}(j-f) \oplus \mathfrak{su}(l+1-j, f) \oplus \mathfrak{t}^1$ (ref. [Bm, Theorem 6.16]). Therefore, it follows that $(\mathfrak{g}, \mathfrak{su}(j-f) \oplus \mathfrak{su}(l+1-j, f) \oplus \mathfrak{t}^1)$ is the pseudo-Hermitian symmetric Lie algebra by $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_f - Z_j)$. By arguments similar to those mentioned above, we confirm that (2) $(\mathfrak{g}, \mathfrak{su}(h) \oplus \mathfrak{su}(l+1-j-h, j) \oplus \mathfrak{t}^1)$, (3) $(\mathfrak{g}, \mathfrak{su}(f,h) \oplus \mathfrak{su}(j-f, l+1-j-h) \oplus \mathfrak{t}^1)$ and (4) $(\mathfrak{g}, \mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+h})$, $\rho_3 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_f - Z_j + Z_{j+h})$ and $\rho_4 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_j$, respectively. Consequently, Proposition 5.1.12 has been proved. □

Collecting the results obtained in Subsection 5.1, we get the following:

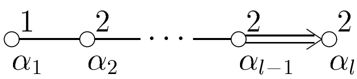
Table I.

		AI
1	$\mathfrak{g}$	$\mathfrak{sl}(2k+1, \mathbb{R})$: $k \ge 1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	None
2	$\mathfrak{g}$	$\mathfrak{sl}(2k, \mathbb{R})$: $k \ge 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_k\,]$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_k)$	$\mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{t}^1$

		AII
3	$\mathfrak{g}$	$\mathfrak{su}^*(2k)$: $k \ge 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_k\,]$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_k)$	$\mathfrak{sl}(k, \mathbb{C}) \oplus \mathfrak{t}^1$

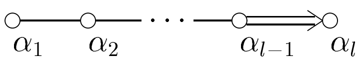
		AIII
4-1	$\mathfrak{g}$	$\mathfrak{su}(j, l+1-j)$: $l \geq 1$, $j = 1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_{j+b})\,]$, $[\,iZ_j\,]$, $j \leq b \leq l-j$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+b}))$	$\mathfrak{su}(b) \oplus \mathfrak{su}(l+1-j-b, j) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_j)$	$\mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1$
4-2	$\mathfrak{g}$	$\mathfrak{su}(j, l+1-j)$: $l \geq 3$, $2 \leq j \leq l-1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_a - Z_j)\,]$, $[\,i(-Z_j + Z_{j+b})\,]$, $[\,i(Z_c - Z_j + Z_{j+d})\,]$, $[\,iZ_j\,]$, $1 \leq a \leq j-1$, $1 \leq b \leq l-j$, $1 \leq c \leq [(j-2)/2] + 1$, $1 \leq d \leq l-j$: if $l+1 \neq 2j$
		$[\,i(Z_a - Z_j)\,]$, $[\,i(Z_c - Z_j + Z_{j+d})\,]$, $[\,iZ_j\,]$, $1 \leq a \leq j-1$, $1 \leq c \leq [(j-2)/2] + 1$, $c \leq d \leq j-c$: if $l+1 = 2j$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_a - Z_j))$	$\mathfrak{su}(j-a) \oplus \mathfrak{su}(l+1-j, a) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+b}))$	$\mathfrak{su}(b) \oplus \mathfrak{su}(l+1-j-b, j) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_c - Z_j + Z_{j+d}))$	$\mathfrak{su}(c, d) \oplus \mathfrak{su}(j-c, l+1-j-d) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_j)$	$\mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1$

5.2. **Type** B_l **($l \geq 2$).** In this subsection, we aim to classify Spr-elements of each real form of the complex simple Lie algebra $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$. First, let us introduce our setting. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{b}_l$, and let $\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})$ be the set of non-zero roots of $\mathfrak{b}_l$ with respect to $\tilde{\mathfrak{h}}$. We fix a linear order in $\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})$, and assume that the Dynkin diagram of $\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$ is as follows:

$\mathfrak{b}_l$: 

(ref. Bourbaki [Br, pp. 267, Plate II]). Then, we denote by $\mathfrak{g}_u$ the compact real form of $\mathfrak{b}_l$ which is given by $\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})$ and (2.3.1), and we denote by $\{Z_a\}_{a=1}^{l}$ ($Z_a \in \tilde{\mathfrak{h}}$) the dual basis of $\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$. In the above setting, we will classify Spr-elements of each real form of $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$.

Notation 5.2.1. In Subsection 5.2, we utilize the following notation:

- $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$.
- $\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$. 

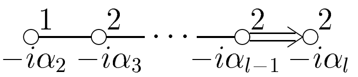
- $\mathfrak{g}_u$: the compact real form of $\mathfrak{b}_l$ given by $\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})$ and (2.3.1).
- $\{Z_a\}_{a=1}^l$: the dual basis of $\Pi_{\triangle(\mathfrak{b}_l,\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

5.2.1. *Case* BI $\mathfrak{so}(2j, 2l-2j+1) : j = 1$. Our aim in this paragraph is to classify *Spr*-elements of $\mathfrak{so}(2, 2l-1)$ (see Proposition 5.2.3).

Let us define an involutive automorphism θ_1 of $\mathfrak{g}_u$ such that $\mathfrak{so}(2, 2l-1)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_1. Define an inner automorphism θ_1 of $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$ by

$$\theta_1 := \exp \pi \operatorname{ad}_{\mathfrak{b}_l} iZ_1. \tag{5.2.1}$$

Then, it follows from $iZ_1 \in \mathfrak{g}_u$ and $\theta_1|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ that (c1) $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_1(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$, and (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{b}_l,\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{b}_l,\tilde{\mathfrak{h}})}$. Murakami's result [Mu3, pp. 297, type BI] enables us to see that the automorphism θ_1 is involutive, the simple root system of $\mathfrak{k}$ is $\{-i\alpha_c\}_{c=2}^l$, the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^l$ is

$\mathfrak{k} = \mathfrak{so}(2l-1) \oplus \mathfrak{t}^1$: $\times$ 

(cf. Remark 2.3.3), and $\mathfrak{so}(2, 2l-1)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{b}_l$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_1(K) = K\}$ and $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_1(P) = -P\}$. In addition, the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is

$$-i\mu = -i(\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_l). \tag{5.2.2}$$

From now on, we will verify the following:

Lemma 5.2.2. *In the above setting; an element $T \in \mathfrak{W}_{\mathfrak{k}}^1$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(2, 2l-1)$ if and only if it is one of the following:*

$$i(-Z_1 + Z_2), \qquad \pm iZ_1.$$

Here, $\mathfrak{W}_{\mathfrak{k}}^1$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^l$;

$$\mathfrak{W}_{\mathfrak{k}}^1 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \,|\, -i\alpha_2(T) \geq 0, -i\alpha_3(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0\}.$$

Proof. Since $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^l$, we can describe an element $T \in \mathfrak{W}_{\mathfrak{k}}^1$ as

$$T = i(\lambda_1 \cdot Z_1 + \lambda_2 \cdot Z_2 + \cdots + \lambda_l \cdot Z_l), \quad \lambda_a \in \mathbb{R}.$$

Suppose that an element $T = \sum_{a=1}^l \lambda_a \cdot iZ_a \in \mathfrak{W}_{\mathfrak{k}}^1$ is an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(2, 2l-1)$. Then, Lemma 4.2.3 implies that one of the following two cases only occurs:

$$\text{(c}'\text{-1) } T = i(\lambda_1 \cdot Z_1 + Z_2), \qquad \text{(c}'\text{-2) } T = i\,\lambda_1 \cdot Z_1, \tag{5.2.3}$$

since $\alpha_a(Z_b) = \delta_{a,b}$ and (5.2.2). Here, $-i\gamma(iZ_1) \equiv 0$ for every root $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ because of $\alpha_a(Z_b) = \delta_{a,b}$ and $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^l$. Accordingly, either Case (c$'$-1) or (c$'$-2) occurs when $T \in \mathfrak{W}_{\mathfrak{k}}^1$ is an *Spr*-element of $\mathfrak{g}$. Let us determine the value of λ_1 in (5.2.3). Lemma 4.1.1 means that the *Spr*-element $T \in \mathfrak{W}_{\mathfrak{k}}^1$ must satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{b}_l, \tilde{\mathfrak{h}})$. Therefore, we conclude that $\lambda_1 = -1$ in Case (c$'$-1), and $\lambda_1 = \pm 1$ in Case (c$'$-2), because $\alpha_a(Z_b) = \delta_{a,b}$ and $\triangle^+(\mathfrak{b}_l, \tilde{\mathfrak{h}})$ is as follows:

(5.2.4) $\triangle^+(\mathfrak{b}_l, \tilde{\mathfrak{h}})$

$$= \left\{ \sum_{a \le p \le l} \alpha_p, \sum_{b \le q < c} \alpha_q, \sum_{b \le q < c} \alpha_q + 2 \sum_{c \le r \le l} \alpha_r \,\middle|\, 1 \le a \le l, 1 \le b < c \le l \right\}$$

(see Plate II in Bourbaki [Br, pp. 267]). Thus, the *Spr*-element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is one of the following:

$$\text{(c-1) } i(-Z_1 + Z_2), \qquad \text{(c-2) } \pm iZ_1. \tag{5.2.5}$$

Conversely, if an element T' is one of the elements in (5.2.5), then it follows that $T' \neq 0$ and $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{b}_l, \tilde{\mathfrak{h}})$; and therefore the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(2, 2l-1)$ (cf. Lemma 4.1.1). For the reasons, we have verified Lemma 5.2.2. □

Now, we will prove Proposition 5.2.3.

Proposition 5.2.3. *Under our equivalence relation, Spr-elements of* BI: $\mathfrak{g} = \mathfrak{so}(2j, 2l - 2j + 1)$, $j = 1$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ \, [i(-Z_1 + Z_2)], \ [iZ_1] \, \}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(2, 2l-3) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{so}(2l-1) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(-Z_1 + Z_2)$ *and* $\rho_2 = \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, iZ_1$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. By (4.1.2) and Lemma 5.2.2, we comprehend

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ \, [i(-Z_1 + Z_2)], \ [iZ_1] \, \}.$$

Lemma 3.1.1-(1) and -(2) imply that the pair $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_2)))$ and $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(iZ_1))$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(-Z_1 + Z_2)$ and $\rho_2 := \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, iZ_1$, respectively. By Theorem 6.16 in [Bm], we see that

$$\mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_2)) = \mathfrak{so}(2, 2l-3) \oplus \mathfrak{t}^1, \quad \mathfrak{c}_{\mathfrak{g}}(iZ_1) = \mathfrak{so}(2l-1) \oplus \mathfrak{t}^1. \tag{5.2.6}$$

This (5.2.6) shows that $(\mathfrak{g}, \mathfrak{so}(2, 2l-3) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}(2l-1) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by $\rho_1 = \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(-Z_1 + Z_2)$ and $\rho_2 = \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, iZ_1$, respectively. Besides, this (5.2.6) also shows that the *Spr*-elements $i(-Z_1 + Z_2)$ and iZ_1 can not be equivalent to each other. Consequently, we have proved Proposition 5.2.3. □

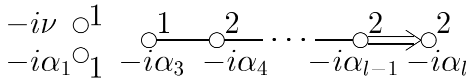
5.2.2. *Case* BI $\mathfrak{so}(2j, 2l-2j+1) : j = 2$. This paragraph is devoted to classifying *Spr*-elements of $\mathfrak{so}(4, 2l-3)$ (cf. Proposition 5.2.5).

First, we will give an involutive automorphism θ_2 of $\mathfrak{g}_u$ such that $\mathfrak{so}(4, 2l-3)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_2. Define an inner automorphism θ_2 of $\mathfrak{b}_l$ by

$$\theta_2 := \exp \pi \, \mathrm{ad}_{\mathfrak{b}_l} \, iZ_2. \tag{5.2.7}$$

Then, it follows from $iZ_2 \in \mathfrak{g}_u$ and $\theta_2|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ that (c1) $\theta_2(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_2(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_2(\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})}$—that is, θ_2 satisfies the three conditions in

Paragraph 2.3.2. By Murakami's result [Mu3, pp. 297, type BI], one knows that the automorphism θ_2 is involutive, $\{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^{l}$ is the simple root system of $\mathfrak{k} := \{K \in \mathfrak{g}_u \mid \theta_2(K) = K\}$ (ref. Remark 2.3.3) and the Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(2l-3)$: 

Here, $-i\nu$ denotes the lowest root $i(\alpha_1 + 2\sum_{c=2}^{l}\alpha_c) \in \triangle(\mathfrak{g}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (ref. Remark 2.3.1). Besides, one also knows that $\mathfrak{so}(4, 2l-3)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{b}_l$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_2(P) = -P\}$. Now, $\mathfrak{k}$ is the direct sum of three simple ideals $\mathfrak{k}_1 := \mathfrak{su}(2)$, $\mathfrak{k}_2 := \mathfrak{su}(2)$ and $\mathfrak{k}_3 := \mathfrak{so}(2l-3)$, where $\mathfrak{k}_3 = \{0\}$ in case of $l = 2$. We assume that $\{-i\alpha_1\}$, $\{-i\nu\}$ and $\{-i\alpha_k\}_{k=3}^{l}$ are the set of simple roots in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $\triangle(\mathfrak{k}_3, \mathfrak{k}_3 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, respectively. Then for $p = 1, 2, 3$, the highest root $-i\mu_p \in \triangle(\mathfrak{k}_p, \mathfrak{k}_p \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$(5.2.8)\qquad \begin{cases} -i\mu_1 = -i\alpha_1, \\ -i\mu_2 = -i\nu, \\ -i\mu_3 = -i(\alpha_3 + 2\alpha_4 + 2\alpha_5 + \cdots + 2\alpha_l). \end{cases}$$

Denote by $\mathfrak{W}_{\mathfrak{k}}^2$ a Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^{l}$;

$$\mathfrak{W}_{\mathfrak{k}}^2 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, -i\nu(T) \geq 0, -i\alpha_3(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0\}.$$

We will search this Weyl chamber for Spr-elements of $\mathfrak{g} = \mathfrak{so}(4, 2l-3)$. First, let us describe the dual basis $\{T_a\}_{a=1}^{l}$ of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^{l}$ in terms of $\{Z_a\}_{a=1}^{l}$. Let T_a be an element of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ determined by $-i\alpha_1(T_a) = \delta_{1,a}$, $-i\nu(T_a) = \delta_{2,a}$ and $-i\alpha_k(T_a) = \delta_{k,a}$ $(3 \leq k \leq l)$. Then, since $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$, and since $\alpha_a(Z_b) = \delta_{a,b}$ and $-i\nu = i(\alpha_1 + 2\sum_{c=2}^{l}\alpha_c)$, we obtain

$$(5.2.9)\qquad \begin{cases} T_1 = i(Z_1 - \frac{1}{2}Z_2), \\ T_2 = -\frac{i}{2}Z_2, \\ T_k = i(-Z_2 + Z_k) \text{ for } 3 \leq k \leq l. \end{cases}$$

From now on, we are going to search the chamber $\mathfrak{W}_{\mathfrak{k}}^2$ for Spr-elements of $\mathfrak{g}$.

Lemma 5.2.4. *In the above setting, an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(4, 2l-3)$ if and only if it is either $i(-Z_2 + Z_3)$ or $i(Z_1 - Z_2)$ when $l \geq 3$, and it is $i(Z_1 - Z_2)$ when $l = 2$.*

Proof. In the first place, we consider the case of $l \geq 3$. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an Spr-element of $\mathfrak{g}$. By arguments similar to those on the proof of Lemma 4.2.1, we comprehend that the element T satisfies one of the following seven conditions:

(1) $-i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 0$;
(2) $-i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 0$;
(3) $-i\mu_1(T) = 0, -i\mu_2(T) = 0, -i\mu_3(T) = 1$;
(4) $-i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 0$;

$$(5)\quad -i\mu_1(T)=1, -i\mu_2(T)=0, -i\mu_3(T)=1;$$
$$(6)\quad -i\mu_1(T)=1, -i\mu_2(T)=1, -i\mu_3(T)=1;$$
$$(7)\quad -i\mu_1(T)=0, -i\mu_2(T)=1, -i\mu_3(T)=1.$$

Hence, it follows from (5.2.8) that the element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is one of the following seven elements:

$$\begin{array}{lll} (1)\ T_1, & (2)\ T_2, & (3)\ T_3, \\ (4)\ T_1+T_2, & (5)\ T_1+T_3, & (6)\ T_1+T_2+T_3, \\ (7)\ T_2+T_3, & & \end{array}$$

where we note that $\{T_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^l$. By use of (5.2.9), one can rewrite the above seven elements as follows:

$$\begin{array}{lll} (1)\ i(Z_1 - \frac{1}{2}Z_2), & (2)\ -\frac{i}{2}Z_2, & (3)\ i(-Z_2+Z_3), \\ (4)\ i(Z_1 - Z_2), & (5)\ i(Z_1 - \frac{3}{2}Z_2 + Z_3), & (6)\ i(Z_1 - 2Z_2 + Z_3), \\ (7)\ i(-\frac{3}{2}Z_2 + Z_3). & & \end{array}$$

Since T belongs to $\mathfrak{W}^2_{\mathfrak{k}}$ and is an Spr-element, it must satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}}) \backslash \triangle_T(\mathfrak{b}_l, \tilde{\mathfrak{h}})$ (cf. Lemma 4.1.1). Therefore, the Spr-element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is one of the following two elements:

$$(3)\ i(-Z_2+Z_3), \qquad (4)\ i(Z_1 - Z_2)$$

because of $\alpha_a(Z_b) = \delta_{a,b}$ and (5.2.4). Conversely, if an element T' is one of the above two elements, then it satisfies $\beta(T') = \pm i$ for each root $\beta \in \triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{b}_l, \tilde{\mathfrak{h}})$; and hence Lemma 4.1.1 means that the element T' is an Spr-element of $\mathfrak{g}$. For the reasons, we have got the conclusion in case of $l \geq 3$.

In the second place, let us consider the case of $l = 2$. In this case, $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1 = \mathfrak{su}(2)$ and $\mathfrak{k}_2 = \mathfrak{su}(2)$, and one sees that $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}}) = \Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, i(\alpha_1 + 2\alpha_2)\}$. Moreover, $-i\alpha_1$ (resp. $i(\alpha_1 + 2\alpha_2)$) is the highest root in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (resp. $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$). Therefore, if an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(4,1)$, then Lemma 4.2.2 implies that it is one of the following:

$$\text{(b-1)}\ i(Z_1 - \tfrac{1}{2}Z_2), \quad \text{(b-2)}\ -\tfrac{i}{2}Z_2, \quad \text{(b-3)}\ i(Z_1 - Z_2)$$

because $\{i(Z_1 - (1/2)\cdot Z_2), -(i/2)\cdot Z_2\}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, i(\alpha_1 + 2\alpha_2)\}$. So, the arguments stated above allow us to conclude that in case of $l = 2$, an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$ if and only if it is $i(Z_1 - Z_2)$. Accordingly, we have completed the proof of Lemma 5.2.4. □

Now, let us demonstrate Proposition 5.2.5.

Proposition 5.2.5. *Under our equivalence relation, Spr-elements of* BI: $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j+1)$, $j = 2$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \begin{cases} \{\ [i(Z_1 - Z_2)],\ [i(-Z_2 + Z_3)]\ \} & \text{if } 3 \leq l, \\ \{\ [i(Z_1 - Z_2)]\ \} & \text{if } 2 = l. \end{cases}$$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(2, 2l-3) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{so}(4, 2l-5) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_1 - Z_2)$ *and*

$\rho_2 = \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(-Z_2 + Z_3)$, *respectively. Here,* $\{Z_a\}_{a=1}^{l}$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$.

Proof. Lemma 5.2.4 and (4.1.2) assure that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \begin{cases} \{\, [\, i(Z_1 - Z_2)\,],\ [\, i(-Z_2 + Z_3)\,] \,\} & \text{if } 3 \leq l, \\ \{\, [\, i(Z_1 - Z_2)\,] \,\} & \text{if } 2 = l. \end{cases}$$

Theorem 6.16 in [Bm] implies that

$$\mathfrak{c}_{\mathfrak{g}}(i(Z_1 - Z_2)) = \mathfrak{so}(2, 2l-3) \oplus \mathfrak{t}^1,$$
$$\mathfrak{c}_{\mathfrak{g}}(i(-Z_2 + Z_3)) = \mathfrak{so}(4, 2l-5) \oplus \mathfrak{t}^1.$$

This enables us to confirm that the Spr-element $i(Z_1 - Z_2)$ is not equivalent to $i(-Z_2 + Z_3)$, and that $(\mathfrak{g}, \mathfrak{so}(2, 2l-3) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}(4, 2l-5) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(Z_1 - Z_2)$ and $\rho_2 := \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(-Z_2 + Z_3)$, respectively (see Lemma 3.1.1). Therefore, Proposition 5.2.5 has been deduced. □

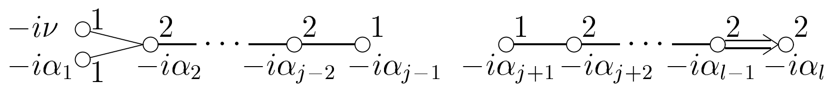
5.2.3. *Case* BI $\mathfrak{so}(2j, 2l-2j+1) : 3 \leq j \leq l$. In this paragraph, we assert Proposition 5.2.7, which is the classification of Spr-elements of $\mathfrak{so}(2j, 2l-2j+1)$ under our equivalence relation.

First, let us define an involutive automorphism θ_3 of $\mathfrak{g}_u$ such that $\mathfrak{so}(2j, 2l-2j+1)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_3. Define an inner automorphism θ_3 of $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$ by

$$\theta_3 := \exp \pi \, \mathrm{ad}_{\mathfrak{b}_l} \, i Z_j.$$

Then, we have (c1) $\theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_3(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_3(\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})}$ because $iZ_j \in \mathfrak{g}_u$ and $\theta_3|_{\tilde{\mathfrak{h}}} = \mathrm{id}$. The result of Murakami [Mu3, pp. 297, type BI] implies that the automorphism θ_3 is involutive. Besides, his result also implies that $\{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$ is the set of simple roots in $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, its Dynkin diagram is

$\mathfrak{k} = \mathfrak{so}(2j) \oplus \mathfrak{so}(2l-2j+1)$:



and $\mathfrak{so}(2j, 2l-2j+1)$ is the real form of $\mathfrak{b}_l = \mathfrak{so}(2l+1, \mathbb{C})$ given by (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$. Here, we denote by $-i\nu$ the lowest root $i(\alpha_1 + 2\sum_{c=2}^{l} \alpha_c) \in \triangle(\mathfrak{g}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (ref. Remark 2.3.1), and we denote by $\mathfrak{k}$ (resp. $\mathfrak{p}$) the $+1$ (resp. -1)-eigenspace of θ_3 in $\mathfrak{g}_u$. Now, $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1 := \mathfrak{so}(2j)$ and $\mathfrak{k}_2 := \mathfrak{so}(2l-2j+1)$, where we note that $\mathfrak{k}_2 = \{0\}$ in case of $j = l$. Let us assume $\{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\}$ (resp. $\{-i\alpha_t\}_{t=j+1}^{l}$) to be the simple root system of $\mathfrak{k}_1$ (resp. $\mathfrak{k}_2$). Then, the highest root $-i\mu_1 \in \triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2 \in \triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$(5.2.10) \qquad \begin{cases} -i\mu_1 = -i(\alpha_1 + 2\sum_{v=2}^{j-2} \alpha_v + \alpha_{j-1} + \nu), \\ -i\mu_2 = -i(\alpha_{j+1} + 2\sum_{w=j+2}^{l} \alpha_w). \end{cases}$$

Let $\mathfrak{W}^3_{\mathfrak{k}}$ denote the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$;

$$\mathfrak{W}^3_{\mathfrak{k}} = \left\{ T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\alpha_1(T) \geq 0, \cdots, -i\alpha_{j-1}(T) \geq 0, -i\nu(T) \geq 0, \\ -i\alpha_{j+1}(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0 \end{array} \right\}.$$

Let us provide a necessary and sufficient condition for an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j+1)$. In order to do so, we are going to describe the dual basis of $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of $\{Z_a\}_{a=1}^l$. Let $\{T_a\}_{a=1}^l$, $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}}$, be the dual basis of $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$—that is, T_a is an element of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ which satisfies $-i\alpha_s(T_a) = \delta_{s,a}$ $(1 \leq s \leq j-1)$, $-i\nu(T_a) = \delta_{j,a}$, and $-i\alpha_t(T_a) = \delta_{t,a}$ $(j+1 \leq t \leq l)$. Then, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and $-i\nu = i(\alpha_1 + 2\sum_{c=2}^{l} \alpha_c)$ that

$$(5.2.11) \qquad \begin{cases} T_1 = i(Z_1 - \frac{1}{2}Z_j), \\ T_u = i(Z_u - Z_j) \text{ for } 2 \leq u \leq j-1, \\ T_j = -\frac{i}{2}Z_j, \\ T_t = i(-Z_j + Z_t) \text{ for } j+1 \leq t \leq l. \end{cases}$$

Now, let us provide a necessary and sufficient condition for an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ to be an *Spr*-element of $\mathfrak{g}$.

Lemma 5.2.6. *With the above assumptions; an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j+1)$ if and only if it is either $i(Z_{j-1} - Z_j)$ or $i(-Z_j + Z_{j+1})$ when $j \leq l-1$, and it is $i(Z_{j-1} - Z_j)$ when $j = l$.*

Proof. First, let us consider the case of $j \leq l-1$. Suppose that an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an *Spr*-element. Then, Lemma 4.2.2 and (5.2.10) mean that the *Spr*-element T is one of the following:

(b′-1.1) T_1, (b′-1.2) T_{j-1}, (b′-1.3) T_j,
(b′-2) T_{j+1},
(b′-3.1) $T_1 + T_{j+1}$, (b′-3.2) $T_{j-1} + T_{j+1}$, (b′-3.3) $T_j + T_{j+1}$

because $\{T_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$. By means of (5.2.11), we can rewrite the above description as follows:

(b-1.1) $i(Z_1 - \frac{1}{2}Z_j)$, (b-1.2) $i(Z_{j-1} - Z_j)$, (b-1.3) $-\frac{i}{2}Z_j$,
(b-2) $i(-Z_j + Z_{j+1})$,
(b-3.1) $i(Z_1 - \frac{3}{2}Z_j + Z_{j+1})$, (b-3.2) $i(Z_{j-1} - 2Z_j + Z_{j+1})$,
(b-3.3) $i(-\frac{3}{2}Z_j + Z_{j+1})$.

Any *Spr*-element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ must satisfy $\beta(T) = \pm i$ for each root $\beta \in \triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{b}_l, \tilde{\mathfrak{h}})$ (cf. Lemma 4.1.1). Consequently, since $\alpha_a(Z_b) = \delta_{a,b}$ and (5.2.4), we deduce that the *Spr*-element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is either (b-1.2) $i(Z_{j-1} - Z_j)$ or (b-2) $i(-Z_j + Z_{j+1})$. Conversely, suppose that an element T' is either $i(Z_{j-1} - Z_j)$ or $i(-Z_j + Z_{j+1})$. Then, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and (5.2.4) that $\beta(T') = \pm i$ for all roots $\beta \in \triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{b}_l, \tilde{\mathfrak{h}})$. Hence, Lemma 4.1.1 assures that the element T' is an

Spr-element of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j+1)$. Therefore, we have completed the proof of Lemma 5.2.6 in case of $j \leq l-1$.

From now on, let us consider the case of $j = l$. In this case, it follows that $\mathfrak{k}_2 = \mathfrak{so}(2l-2j+1) = \{0\}$, so that $\mathfrak{k} = \mathfrak{k}_1 = \mathfrak{so}(2l)$ is a simple Lie algebra and the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows: $-i\mu = -i(\alpha_1 + 2\sum_{v=2}^{l-2}\alpha_v + \alpha_{l-1} + \nu)$. So, Lemma 4.2.4 implies that if an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an Spr-element, then it satisfies one of the following three conditions:

1) $-i\alpha_1(T) = 1$, $-i\alpha_d(T) = 0$ for $2 \leq d \leq l-1$, and $-i\nu(T) = 0$;
2) $-i\alpha_e(T) = 0$ for $1 \leq e \leq l-2$, $-i\alpha_{l-1}(T) = 1$, and $-i\nu(T) = 0$;
3) $-i\alpha_f(T) = 0$ for $1 \leq f \leq l-1$, and $-i\nu(T) = 1$;

and therefore, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and $-i\nu = i(\alpha_1 + 2\sum_{c=2}^{l}\alpha_c)$ that the element T is equal to $i(Z_1 - (1/2)\cdot Z_l)$, $i(Z_{l-1} - Z_l)$ or $-(i/2)\cdot Z_l$. By arguments stated first, we conclude that in case of $j = l$, an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g}$ if and only if it is $i(Z_{l-1} - Z_l)$. For the reasons, we have got the conclusion. □

By use of Lemma 5.2.6, we will demonstrate Proposition 5.2.7.

Proposition 5.2.7. *Under our equivalence relation, Spr-elements of* BI: $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j+1)$, $3 \leq j \leq l$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \begin{cases} \{\ [i(Z_{j-1} - Z_j)],\ [i(-Z_j + Z_{j+1})]\ \} & \text{if } j \leq l-1, \\ \{\ [i(Z_{j-1} - Z_j)]\ \} & \text{if } j = l. \end{cases}$$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(2j-2, 2l-2j+1) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{so}(2j, 2l-2j-1) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_{j-1} - Z_j)$ and $\rho_2 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(-Z_j + Z_{j+1})$, respectively. Here, $\{Z_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{b}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.*

Proof. By (4.1.2) and Lemma 5.2.6, one confirms that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \begin{cases} \{\ [i(Z_{j-1} - Z_j)],\ [i(-Z_j + Z_{j+1})]\ \} & \text{if } j \leq l-1, \\ \{\ [i(Z_{j-1} - Z_j)]\ \} & \text{if } j = l. \end{cases}$$

Lemma 3.1.1-(1) and -(2) allow us to see that the pair $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j)))$ and $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1})))$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_{j-1} - Z_j)$ and $\rho_2 := \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(-Z_j + Z_{j+1})$, respectively. Moreover, Theorem 6.16 in [Bm] implies that

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j)) &= \mathfrak{so}(2j-2, 2l-2j+1) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1})) &= \mathfrak{so}(2j, 2l-2j-1) \oplus \mathfrak{t}^1. \end{aligned} \tag{5.2.12}$$

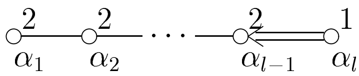
Therefore, $(\mathfrak{g}, \mathfrak{so}(2j-2, 2l-2j+1) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}(2j, 2l-2j-1) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by $\rho_1 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_{j-1} - Z_j)$ and $\rho_2 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(-Z_j + Z_{j+1})$, respectively. From (5.2.12), it is natural that the Spr-elements $i(Z_{j-1} - Z_j)$ and $i(-Z_j + Z_{j+1})$ are not equivalent to each other. Consequently, we have proved Proposition 5.2.7. □

The results obtained in Subsection 5.2 are as follows (see three Propositions 5.2.3, 5.2.5 and 5.2.7):

Table II.

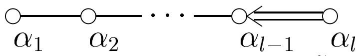
		BI
5	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j+1)$: $l \geq 2$, $1 \leq j \leq l$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1} - Z_j)\,]$, $[\,i(-Z_j + Z_{j+1})\,]$ where $Z_0 = 0$: if $1 \leq j \leq l-1$
		$[\,i(Z_{j-1} - Z_j)\,]$: if $j = l$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j))$	$\mathfrak{so}(2j-2, 2l-2j+1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j, 2l-2j-1) \oplus \mathfrak{t}^1$

5.3. **Type** C_l $(l \geq 3)$**.** This subsection is devoted to classifying Spr-elements of each real form of the complex simple Lie algebra $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$. Let us introduce our setting. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{c}_l$, let $\{\alpha_a\}_{a=1}^l$ be the set of simple roots in $\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})$ whose Dynkin diagram is as follows:

$\mathfrak{c}_l$: 

(ref. Plate III in Bourbaki [Br, pp. 269]), and let $\mathfrak{g}_u$ be the compact real form of $\mathfrak{c}_l$ given by $\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})$ and (2.3.1). We denote by $\{Z_a\}_{a=1}^l$ $(Z_a \in \tilde{\mathfrak{h}})$ the dual basis of $\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$. In the setting, let us classify Spr-elements of each real form of $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$ under our equivalence relation.

Notation 5.3.1. In Subsection 5.3, we utilize the following notation:

- $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$.
- $\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$. 

- $\mathfrak{g}_u$: the compact real form of $\mathfrak{c}_l$ given by $\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})$ and (2.3.1).
- $\{Z_a\}_{a=1}^l$: the dual basis of $\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

5.3.1. *Case* CI $\mathfrak{sp}(l, \mathbb{R})$. In this paragraph, we will classify Spr-elements of $\mathfrak{sp}(l, \mathbb{R})$—that is, we will assert Proposition 5.3.4.

First, let us define an involutive automorphism θ_1 of $\mathfrak{g}_u$ such that $\mathfrak{sp}(l, \mathbb{R})$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_1. We define an inner automorphism θ_1 of $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$ by

$$\theta_1 := \exp \pi \,\mathrm{ad}_{\mathfrak{c}_l} iZ_l. \tag{5.3.1}$$

Then, it follows from $iZ_l \in \mathfrak{g}_u$ and $\theta_1|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ that it satisfies (c1) $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_1(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})}$. It is known that the automorphism θ_1 of $\mathfrak{c}_l$ is involutive (cf. Murakami [Mu3, pp. 297, type CI]). In addition, it is also known that $\{-i\alpha_d\}_{d=1}^{l-1}$ is the set of simple roots in $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d\}_{d=1}^{l-1}$ is

$$\mathfrak{k} = \mathfrak{su}(l) \oplus \mathfrak{t}^1 \colon \underset{-i\alpha_1}{\overset{1}{\circ}} - \underset{-i\alpha_2}{\overset{1}{\circ}} \cdots - \underset{-i\alpha_{l-1}}{\overset{1}{\circ}} \quad \times$$

and $\mathfrak{sp}(l,\mathbb{R})$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{c}_l$ (see Murakami [Mu3, pp. 297, type CI] again). Here, $\mathfrak{k} := \{K \in \mathfrak{g}_u \mid \theta_1(K) = K\}$ and $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_1(P) = -P\}$. Remark that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$-i\mu = -i(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}). \tag{5.3.2}$$

Next, we will verify the following:

Lemma 5.3.2. *In the above setting; an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{sp}(l,\mathbb{R})$ if and only if it is one of the following:*

$$i(Z_d - Z_l) \text{ for } 1 \leq d \leq l-1, \qquad \pm iZ_l.$$

Here, $\mathfrak{W}^1_{\mathfrak{k}}$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d\}_{d=1}^{l-1}$;

$$\mathfrak{W}^1_{\mathfrak{k}} = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, -i\alpha_2(T) \geq 0, \cdots, -i\alpha_{l-1}(T) \geq 0\}.$$

Proof. Any element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ can be written as follows:

$$T = i(\lambda_1 \cdot Z_1 + \lambda_2 \cdot Z_2 + \cdots + \lambda_l \cdot Z_l), \quad \lambda_a \in \mathbb{R}$$

because $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$. Suppose that an element $T = \sum_{a=1}^{l} \lambda_a \cdot iZ_a \in \mathfrak{W}^1_{\mathfrak{k}}$ is an *Spr*-element. Since $\alpha_a(Z_b) = \delta_{a,b}$ and $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d\}_{d=1}^{l-1}$, we perceive that $-i\gamma(iZ_l) \equiv 0$ for any root $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$. Therefore, the supposition and Lemma 4.2.3 enable us to deduce that one of the following two cases only occurs:

$$\text{(c}'\text{-1) } T = i(Z_d + \lambda_l \cdot Z_l) \text{ for } 1 \leq d \leq l-1, \quad \text{(c}'\text{-2) } T = i\,\lambda_l \cdot Z_l$$

because (5.3.2) and $\alpha_a(Z_b) = \delta_{a,b}$. Let us determine the value of λ_l in each of the Cases (c′-1) and (c′-2). The positive root system $\triangle^+(\mathfrak{c}_l, \tilde{\mathfrak{h}})$ is

$$\begin{aligned} &\triangle^+(\mathfrak{c}_l, \tilde{\mathfrak{h}}) \\ &= \left\{ \sum_{b \leq f < c} \alpha_f,\ \sum_{b \leq f < c} \alpha_f + 2\sum_{c \leq g < l} \alpha_g + \alpha_l,\ 2\sum_{a \leq h < l} \alpha_h + \alpha_l \;\middle|\; 1 \leq b < c \leq l, 1 \leq a \leq l \right\} \end{aligned} \tag{5.3.3}$$

(cf. Bourbaki [Br, pp. 269, Plate III][6]). Any *Spr*-element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ has to satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{c}_l, \tilde{\mathfrak{h}})$ (see Lemma 4.1.1). Therefore, it follows from (5.3.3) that $\lambda_l = -1$ in Case (c′-1), and $\lambda_l = \pm 1$ in Case (c′-2). For the reasons, we conclude that an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is one of the following:

$$\text{(c-1) } i(Z_d - Z_l) \text{ for } 1 \leq d \leq l-1, \quad \text{(c-2) } \pm iZ_l$$

when it is an *Spr*-element of $\mathfrak{g} = \mathfrak{sp}(l,\mathbb{R})$. Conversely, suppose that an element T' is either $i(Z_d - Z_l)$ or $\pm iZ_l$. Then, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and (5.3.3) that $\beta(T') = \pm i$ for each root $\beta \in \triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{c}_l, \tilde{\mathfrak{h}})$; and so the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{sp}(l,\mathbb{R})$ (cf. Lemma 4.1.1). Consequently, Lemma 5.3.2 has been shown. □

[6]Erratum: pp. 269, line 10 on [Br], read "$2\epsilon_i = 2\sum_{i \leq k < l} \alpha_k + \alpha_l$" instead of "$2\epsilon_i = \sum_{i \leq k < l} \alpha_k + \alpha_l$".

Let us investigate whether the *Spr*-element $i(Z_d - Z_l)$ in Lemma 5.3.2 is equivalent to $i(Z_{d'} - Z_l)$ or not.

Lemma 5.3.3. *With the above assumptions; two Spr-elements* $T = i(Z_d - Z_l)$ *and* $T' = i(Z_{d'} - Z_l)$ *of* $\mathfrak{g} = \mathfrak{sp}(l, \mathbb{R})$ *are equivalent to each other if and only if Case* (i) $d = d'$ *or* (ii) $d = l - d'$ *holds. Here,* $1 \leq d, d' \leq l - 1$.

Proof. It is known that

$$\mathfrak{c}_{\mathfrak{g}}(i(Z_d - Z_l)) = \mathfrak{su}(d, l - d) \oplus \mathfrak{t}^1$$

(ref. Theorem 6.16 in [Bm]). Hence, if the element $T = i(Z_d - Z_l)$ is equivalent to $T' = i(Z_{d'} - Z_l)$, then it follows that $\mathfrak{su}(d, l - d) = \mathfrak{su}(d', l - d')$. Therefore, one sees that Case (i) $d = d'$ or (ii) $d = l - d'$ holds when $T = i(Z_d - Z_l)$ is equivalent to $T' = i(Z_{d'} - Z_l)$.

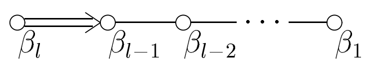
Now, let us confirm that the converse also true. If Case (i) $d = d'$ holds, then $i(Z_d - Z_l) = i(Z_{d'} - Z_l)$; and so $T = i(Z_d - Z_l)$ is equivalent to $T' = i(Z_{d'} - Z_l)$. Henceforth, we devote ourselves to confirming that the *Spr*-element $T = i(Z_d - Z_l)$ is equivalent to $T' = i(Z_{d'} - Z_l)$ in Case (ii) $d = l - d'$. For the confirmation, we will construct an automorphism φ of $\mathfrak{g} = \mathfrak{sp}(l, \mathbb{R})$ such that $\varphi(i(Z_d - Z_l)) = i(Z_{l-d} - Z_l)$. Let us define an involutive linear isomorphism φ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^l$ by

$$(5.3.4) \qquad \begin{cases} \varphi'(iZ_d) := i(Z_{l-d} - 2Z_l) \text{ for } 1 \leq d \leq l - 1, \\ \varphi'(iZ_l) := -iZ_l. \end{cases}$$

Then since $\alpha_a(Z_b) = \delta_{a,b}$, one obtains

$$(5.3.5) \qquad \begin{cases} {}^t\varphi'_{\mathbb{C}}(\alpha_d) = \alpha_{l-d} \text{ for } 1 \leq d \leq l - 1, \\ {}^t\varphi'_{\mathbb{C}}(\alpha_l) = -(2\sum_{d=1}^{l-1} \alpha_d + \alpha_l), \end{cases}$$

where $\varphi'_{\mathbb{C}}$ denotes the complex linear extension of φ' to $\tilde{\mathfrak{h}}$. From (5.3.5), we comprehend that the Dynkin diagram of $\{{}^t\varphi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^l$ is as follows:



where $\beta_a := {}^t\varphi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq l$. This shows that

$${}^t\varphi'_{\mathbb{C}}\big(\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})$$

because the Dynkin diagram of $\{{}^t\varphi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^l$ is the same as that of $\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$ (cf. Murakami [Mu3, Lemma 1, pp. 295]). Accordingly, there exists an involutive automorphism φ of $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$ such that $\varphi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and $\varphi(X_{\pm\alpha_a}) = X_{\pm{}^t\varphi(\alpha_a)}$ (cf. Proposition 2.3.2). Now, let us show that φ is an automorphism of $\mathfrak{g} = \mathfrak{sp}(l, \mathbb{R})$. Since $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.3.4), one has $\varphi(iZ_l) = -iZ_l$. Thus from θ_1 being involutive and (5.3.1), it follows that

$$\varphi \circ \theta_1 = \varphi \circ \exp \pi\, \mathrm{ad}_{\mathfrak{c}_l} iZ_l = \exp \pi\, \mathrm{ad}_{\mathfrak{c}_l} \varphi(iZ_l) \circ \varphi = \theta_1 \circ \varphi.$$

Therefore, φ is an automorphism of $\mathfrak{g}$ (cf. Proposition 2.2.3).[7] Using φ, we will confirm that the Spr-element $T = i(Z_d - Z_l)$ is equivalent to $T' = i(Z_{d'} - Z_l)$ in Case (ii) $d = l - d'$. Suppose that Case (ii) $d = l - d'$ holds. Then since $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.3.4), we have

$$\varphi(iZ_d - Z_l) = i(Z_{l-d} - 2Z_l + Z_l) = i(Z_{d'} - Z_l).$$

Hence, $T = i(Z_d - Z_l)$ is equivalent to $T' = i(Z_{d'} - Z_l)$ in Case (ii) $d = l - d'$. Therefore, we have completed the proof of Lemma 5.3.3. □

Now, we will prove Proposition 5.3.4.

Proposition 5.3.4. *Under our equivalence relation, Spr-elements of* CI: $\mathfrak{g} = \mathfrak{sp}(l, \mathbb{R})$ *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(Z_d - Z_l)],\ [iZ_l]\ \mid 1 \le d \le [l/2]\}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{su}(d, l-d) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{su}(l) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_d - Z_l)$ *and* $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_l$*, respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. By virtue of (4.1.2) and two Lemmas 5.3.2 and 5.3.3, one concludes that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(Z_d - Z_l)],\ [iZ_l]\ \mid 1 \le d \le [l/2]\}.$$

Lemma 3.1.1-(1) and -(2) imply that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(Z_d - Z_l)))$ and $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(iZ_l))$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_d - Z_l)$ and $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_l$, respectively. On the other hand, Theorem 6.16 in [Bm] allows us to have

$$\mathfrak{c}_{\mathfrak{g}}(i(Z_d - Z_l)) = \mathfrak{su}(d, l-d) \oplus \mathfrak{t}^1, \qquad \mathfrak{c}_{\mathfrak{g}}(iZ_l) = \mathfrak{su}(l) \oplus \mathfrak{t}^1. \tag{5.3.6}$$

Consequently, $(\mathfrak{g}, \mathfrak{su}(d, l-d) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{su}(l) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_d - Z_l)$ and $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_l$, respectively. Moreover, it comes from $l \ge 3$, $1 \le d \le [l/2]$ and (5.3.6) that the Spr-elements $i(Z_d - Z_l)$ and iZ_l are not equivalent to each other. Consequently, we have proved Proposition 5.3.4. □

5.3.2. *Case* CII $\mathfrak{sp}(j, l-j) : 1 \le j \le l-1$. This paragraph is devoted to classifying Spr-elements of $\mathfrak{sp}(j, l-j)$ (see Proposition 5.3.6).

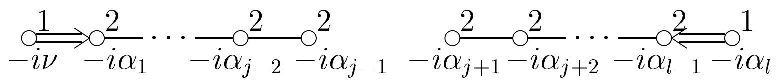
Let us define an involutive automorphism θ_2 of $\mathfrak{g}_u$ such that $\mathfrak{sp}(j, l-j)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_2. Let θ_2 be an inner automorphism of $\mathfrak{c}_l = \mathfrak{sp}(l, \mathbb{C})$ defined by

$$\theta_2 := \exp \pi \operatorname{ad}_{\mathfrak{c}_l} iZ_j.$$

Since $iZ_j \in \mathfrak{g}_u$ and $\theta_2|_{\tilde{\mathfrak{h}}} = \mathrm{id}$, we have (c1) $\theta_2(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_2(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_2(\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})}$. Murakami's result [Mu3, pp. 297, type CII] states that the automorphism θ_2 is involutive. Denote by $\mathfrak{k}$ and $\mathfrak{p}$ the $+1$ and -1-eigenspace of θ_2 in $\mathfrak{g}_u$, respectively. Then, from the result of Murakami [Mu3, pp. 297, type CII], it also follows that $\{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^l$ is the simple root system of $\mathfrak{k}$, its Dynkin diagram is

[7] This φ is an outer automorphism of $\mathfrak{sp}(l, \mathbb{R})$.

$\mathfrak{k} = \mathfrak{sp}(j) \oplus \mathfrak{sp}(l-j)$:



and $\mathfrak{sp}(j, l-j)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{c}_l$. Here, $-i\nu$ denotes the lowest root $i(2\sum_{d=1}^{l-1} \alpha_d + \alpha_l)$ in $\triangle(\mathfrak{g}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (cf. Remark 2.3.1). Now, we assume $\{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\}$ (resp. $\{-i\alpha_t\}_{t=j+1}^{l}$) to be the simple root system of $\mathfrak{k}_1$ (resp. $\mathfrak{k}_2$), where $\mathfrak{k}_1 := \mathfrak{sp}(j)$ and $\mathfrak{k}_2 := \mathfrak{sp}(l-j)$. In this case, the highest root $-i\mu_1$ in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2$ in $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$\text{(5.3.7)} \qquad \begin{cases} -i\mu_1 = -i(2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{j-1} + \nu), \\ -i\mu_2 = -i(2\alpha_{j+1} + 2\alpha_{j+2} + \cdots + 2\alpha_{l-1} + \alpha_l). \end{cases}$$

We aim to provide a necessary and sufficient condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{sp}(j, l-j)$, where $\mathfrak{W}_{\mathfrak{k}}^2$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$;

$$\mathfrak{W}_{\mathfrak{k}}^2 = \left\{ T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\alpha_1(T) \geq 0, \cdots, -i\alpha_{j-1}(T) \geq 0, -i\nu(T) \geq 0, \\ -i\alpha_{j+1}(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0 \end{array} \right\}.$$

For the aim, we will describe the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of $\{Z_a\}_{a=1}^{l}$. Let $\{T_a\}_{a=1}^{l}$, $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}}$, be the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$, namely T_a is an element of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ which satisfies $-i\alpha_s(T_a) = \delta_{s,a}$ $(1 \leq s \leq j-1)$, $-i\nu(T_a) = \delta_{j,a}$, and $-i\alpha_t(T_a) = \delta_{t,a}$ $(j+1 \leq t \leq l)$. Then, one obtains

$$\text{(5.3.8)} \qquad \begin{cases} T_s = i(Z_s - Z_j) \text{ for } 1 \leq s \leq j-1, \\ T_j = -\frac{i}{2} Z_j, \\ T_u = i(-Z_j + Z_u) \text{ for } j+1 \leq u \leq l-1, \\ T_l = i(-\frac{1}{2} Z_j + Z_l) \end{cases}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$ and $-i\nu = i(2\sum_{d=1}^{l-1} \alpha_d + \alpha_l)$. Now, let us provide a necessary and sufficient condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{sp}(j, l-j)$.

Lemma 5.3.5. *In the setting on Paragraph* 5.3.2; *an element* $T \in \mathfrak{W}_{\mathfrak{k}}^2$ *is an Spr-element of* $\mathfrak{g} = \mathfrak{sp}(j, l-j)$ *if and only if* $T = i(-Z_j + Z_l)$.

Proof. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an *Spr*-element of $\mathfrak{g}$. Then, Lemma 4.2.2 and (5.3.7) mean that one of the following three cases only occurs:

$$\text{(b}'\text{-1) } T = T_j, \quad \text{(b}'\text{-2) } T = T_l, \quad \text{(b}'\text{-3) } T = T_j + T_l$$

because $\{T_a\}_{a=1}^{l}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$. By (5.3.8), the above description can be rewritten as

$$\text{(b-1) } T = -\frac{i}{2} Z_j, \quad \text{(b-2) } T = i(-\frac{1}{2} Z_j + Z_l), \quad \text{(b-3) } T = i(-Z_j + Z_l).$$

The two elements (b-1) $T = -(i/2) \cdot Z_j$ and (b-2) $T = i(-(1/2) \cdot Z_j + Z_l)$ can not be *Spr*-elements. Indeed, there exists a root $\alpha_j \in \triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})$, and it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that $\alpha_j(T) = -i/2 \neq \pm i$ for $T = -(i/2) \cdot Z_j$ and $i(-(1/2) \cdot Z_j + Z_l)$. Hence, Lemma 4.1.1 assures that the two elements $-(i/2) \cdot Z_j$ and $i(-(1/2) \cdot Z_j + Z_l)$ can not be *Spr*-elements. On the other hand, since $\alpha_a(Z_b) = \delta_{a,b}$ and (5.3.3), the other element (b-3) $T = i(-Z_j + Z_l)$ satisfies $\beta(T) = \pm i$ for each root $\beta \in \triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{c}_l, \tilde{\mathfrak{h}})$; so that the element $T = i(-Z_j + Z_l)$ is an *Spr*-element of $\mathfrak{g} = \mathfrak{sp}(j, l-j)$ (ref. Lemma 4.1.1). Therefore, if an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is an *Spr*-element, then $T = i(-Z_j + Z_l)$. On the above arguments, the converse has been confirmed—that is, if $T = i(-Z_j + Z_l)$, then it is an *Spr*-element. For the reasons, we have demonstrated Lemma 5.3.5. □

Now, we are going to prove the following:

Proposition 5.3.6. *Under our equivalence relation, Spr-elements of* CII: $\mathfrak{g} = \mathfrak{sp}(j, l-j)$, $1 \leq j \leq l-1$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ \ [i(-Z_j + Z_l)] \ \}.$$

Besides, $(\mathfrak{g}, \mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1)$ *is the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$. *Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{c}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. Lemma 5.3.5, together with (4.1.2), implies that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ \ [i(-Z_j + Z_l)] \ \}.$$

Hence, the first half of statements on this proposition has been shown. Lemma 3.1.1-(1) and -(2) imply that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_l)))$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$. In addition, it is known that $\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_l)) = \mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$ (cf. Theorem 6.16 in [Bm]). Accordingly, $(\mathfrak{g}, \mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1)$ is the pseudo-Hermitian symmetric Lie algebra by $\rho = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$. Thus, we have proved Proposition 5.3.6. □

The following table comes from two Propositions 5.3.4 and 5.3.6:

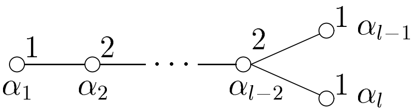
Table III.

		CI
6	$\mathfrak{g}$	$\mathfrak{sp}(l, \mathbb{R})$: $l \geq 3$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[i(Z_d - Z_l)]$, $[iZ_l]$, $1 \leq d \leq [l/2]$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_d - Z_l))$	$\mathfrak{su}(d, l-d) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_l)$	$\mathfrak{su}(l) \oplus \mathfrak{t}^1$

		CII
7	$\mathfrak{g}$	$\mathfrak{sp}(j, l-j)$: $l \geq 3$, $1 \leq j \leq l-1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[i(-Z_j + Z_l)]$

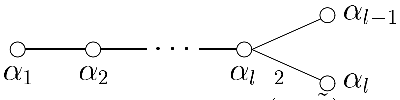
		$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j+Z_l))$	$\mathfrak{su}(j,l-j)\oplus\mathfrak{t}^1$

5.4. **Type** D_l $(l \geq 4)$**.** Our purpose in this subsection is to classify *Spr*-elements of each real form of the complex simple Lie algebra $\mathfrak{d}_l = \mathfrak{so}(2l,\mathbb{C})$. First, we will introduce our setting. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{d}_l$, and let $\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})$ be the set of non-zero roots of $\mathfrak{d}_l$ with respect to $\tilde{\mathfrak{h}}$. Denote by $\{\alpha_a\}_{a=1}^l$ the set of simple roots in $\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})$, and assume that the Dynkin diagram of $\Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$ is as follows:

$\mathfrak{d}_l$: 

(cf. Plate IV in Bourbaki [Br, pp. 271]). Then, we denote by $\mathfrak{g}_u$ the compact real form of $\mathfrak{d}_l$ which is given by $\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})$ and (2.3.1), and we denote by $\{Z_a\}_{a=1}^l$ $(Z_a \in \tilde{\mathfrak{h}})$ the dual basis of $\Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$. In the setting, let us classify *Spr*-elements of each real form of $\mathfrak{d}_l = \mathfrak{so}(2l,\mathbb{C})$.

Notation 5.4.1. In Subsection 5.4, we utilize the following notation:

- $\mathfrak{d}_l = \mathfrak{so}(2l,\mathbb{C})$.
- $\Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$. 

- $\mathfrak{g}_u$: the compact real form of $\mathfrak{d}_l$ given by $\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})$ and (2.3.1).
- $\{Z_a\}_{a=1}^l$: the dual basis of $\Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

5.4.1. *Case* DI $\mathfrak{so}(2j+1,2l-2j-1) : j = 0$. Our aim in this paragraph is to classify *Spr*-elements of $\mathfrak{so}(1,2l-1)$ (see Proposition 5.4.3).

In the first place, let us construct an involutive automorphism θ_1 of $\mathfrak{d}_l = \mathfrak{so}(2l,\mathbb{C})$ such that (I) it satisfies the three conditions in Paragraph 2.3.2;

$$\text{(c1) } \theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u, \quad \text{(c2) } \theta_1(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}, \quad \text{(c3) } {}^t\theta_1(\Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})}$$

and (II) $\mathfrak{so}(1,2l-1)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_1. From $Z_a \in \tilde{\mathfrak{h}}$ and $\alpha_a(Z_b) = \delta_{a,b}$, it is natural that $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^l$ (recall (2.3.2) for $\tilde{\mathfrak{h}}_{\mathbb{R}}$). This enables us to define an involutive linear isomorphism θ'_1 of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ by

$$\text{(5.4.1)} \qquad \begin{cases} \theta'_1(iZ_k) := iZ_k \text{ for } 1 \leq k \leq l-2, \\ \theta'_1(iZ_{l-1}) := iZ_l, \\ \theta'_1(iZ_l) := iZ_{l-1}. \end{cases}$$

Then, since $\alpha_a(Z_b) = \delta_{a,b}$, one obtains

$$\text{(5.4.2)} \qquad \begin{cases} {}^t\theta'_{1\mathbb{C}}(\alpha_k) = \alpha_k \text{ for } 1 \leq k \leq l-2, \\ {}^t\theta'_{1\mathbb{C}}(\alpha_{l-1}) = \alpha_l, \\ {}^t\theta'_{1\mathbb{C}}(\alpha_l) = \alpha_{l-1}, \end{cases}$$

where $\theta'_{1\mathbb{C}}$ denotes the complex linear extension of θ'_1 to $\tilde{\mathfrak{h}}$. It comes from (5.4.2) that

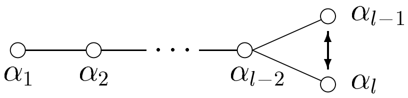
$$ {}^t\theta'_{1\mathbb{C}}\big(\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}}) $$

because $\triangle^+(\mathfrak{d}_l,\tilde{\mathfrak{h}})$ is as follows (cf. Bourbaki [Br, pp. 271, Plate IV][8]):

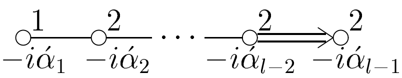
(5.4.3) $\triangle^+(\mathfrak{d}_l,\tilde{\mathfrak{h}})$

$$ = \left\{ \begin{matrix} \sum_{p\leq f<q}\alpha_f,\ \sum_{r\leq f\leq l-2}\alpha_f+\alpha_l, \\ \sum_{s\leq f<t}\alpha_f + 2\sum_{t\leq h<l-1}\alpha_h+\alpha_{l-1}+\alpha_l \end{matrix} \;\middle|\; \begin{matrix} 1\leq p<q\leq l,\ 1\leq r<l, \\ 1\leq s<t<l \end{matrix} \right\}. $$

Therefore, there exists an involutive automorphism θ_1 of $\mathfrak{d}_l$ such that (i) $\theta_1(\mathfrak{g}_u)\subset\mathfrak{g}_u$, (ii) $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (iii) $\theta_1(X_{\pm\alpha_a}) = X_{\pm {}^t\theta_1(\alpha_a)}$ (see Proposition 2.3.2). This involution θ_1 satisfies the condition (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l,\tilde{\mathfrak{h}})}$, since $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (5.4.2). Accordingly, the involution θ_1 of $\mathfrak{d}_l = \mathfrak{so}(2l,\mathbb{C})$ satisfies the three conditions (c1), (c2) and (c3). Here, we remark that θ_1 is the same involution as θ_ρ in Murakami [Mu3, pp. 305, type DI].

${}^t\theta_1$ 

Let us enumerate the simple root system of $\mathfrak{k}$, the highest root in $\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and so on, where $\mathfrak{k}$ denotes the $+1$-eigenspace of θ_1 in $\mathfrak{g}_u$. The result of Murakami [Mu3, pp. 305, type DI] implies that $\{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{l-1}$ is the simple root system of $\mathfrak{k}$ (cf. Remark 2.3.3) and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{so}(2l-1)$: 

where $-i\acute{\alpha}_d := -i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ for $1\leq d\leq l-1$. Moreover, it follows that $\mathfrak{so}(1,2l-1)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k}\oplus i\mathfrak{p}$ of $\mathfrak{d}_l$, where $\mathfrak{p}$ denotes the -1-eigenspace of θ_1 in $\mathfrak{g}_u$. Remark that the highest root $-i\mu\in\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

(5.4.4) $$ -i\mu = -i(\alpha_1+2\alpha_2+2\alpha_3+\cdots+2\alpha_{l-1})|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}. $$

In the second place, let us describe the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of $\{Z_a\}_{a=1}^l$. Its description will be useful in the third place. It is immediate from $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (5.4.1) that

(5.4.5) $$ \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_k, i(Z_{l-1}+Z_l)\}_{k=1}^{l-2}. $$

Now, let $\{T_d\}_{d=1}^{l-1}$, $T_d\in\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$, be the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{l-1}$. Then by $\alpha_a(Z_b) = \delta_{a,b}$ and (5.4.5), one deduces that

(5.4.6) $$ T_k = iZ_k \text{ for } 1\leq k\leq l-2, \quad T_{l-1} = i(Z_{l-1}+Z_l). $$

In the third place, let us prove the following:

[8]Erratum: pp. 271, line 8 on [Br], read "$\epsilon_i-\epsilon_j = \sum_{i\leq k<j}\alpha_k$" instead of "$\epsilon_i-\epsilon_j = \sum_{i<k<j}\alpha_k$".

Lemma 5.4.2. *In the above setting; an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(1, 2l-1)$ if and only if $T = iZ_1$. Here, $\mathfrak{W}^1_{\mathfrak{k}}$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},\, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{l-1}$;*

$$\mathfrak{W}^1_{\mathfrak{k}} = \{T \in \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, -i\alpha_2(T) \geq 0, \cdots, -i\alpha_{l-1}(T) \geq 0\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. Then, Lemma 4.2.4 and (5.4.4) allow us to have $T = T_1$ because $\{T_d\}_{d=1}^{l-1}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{d=1}^{l-1}$. So, $T = iZ_1$ follows from (5.4.6). Hence, if an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an *Spr*-element, then one has $T = iZ_1$.

Conversely, suppose that $T = iZ_1$. Since $\alpha_a(Z_b) = \delta_{a,b}$ and (5.4.3), we confirm that the element $T = iZ_1$ satisfies $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{d}_l, \tilde{\mathfrak{h}})$; and hence the element $T = iZ_1$ is an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(1, 2l-1)$ (see Lemma 4.1.1). Therefore, we have got the conclusion. □

By virtue of (4.1.2) and Lemma 5.4.2, one concludes that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [iZ_1]\ \},$$

where $\mathfrak{g} = \mathfrak{so}(1, 2l-1)$. From Lemma 3.1.1-(1) and -(2), it follows that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(iZ_1))$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_1$. Besides, it is known that $\mathfrak{c}_{\mathfrak{g}}(iZ_1) = \mathfrak{so}(1, 2l-3) \oplus \mathfrak{t}^1$ (cf. Theorem 6.16 in [Bm]). For the reasons, we conclude the following:

Proposition 5.4.3. *Under our equivalence relation, Spr-elements of* DI: $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$, $j = 0$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [iZ_1]\ \}.$$

Besides, $(\mathfrak{g}, \mathfrak{so}(1, 2l-3) \oplus \mathfrak{t}^1)$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_1$. Here, $\{Z_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

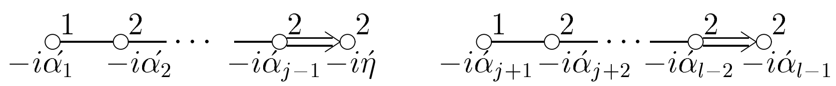
5.4.2. *Case* DI $\mathfrak{so}(2j+1, 2l-2j-1) : 1 \leq j \leq l-3$. In this paragraph, we devote ourselves to classifying *Spr*-elements of $\mathfrak{so}(2j+1, 2l-2j-1)$. The result in this paragraph is Proposition 5.4.6.

By use of the involution θ_1 in the previous paragraph, we define an automorphism θ_2 of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ as follows:

$$\theta_2 := \theta_1 \circ \exp \pi\, \mathrm{ad}_{\mathfrak{d}_l}\, iZ_j. \tag{5.4.7}$$

This θ_2 is involutive, since it is the same as θ_i in Murakami [Mu3, pp. 305, type DI]. Note that the involution θ_2 of $\mathfrak{d}_l$ satisfies the three conditions in Paragraph 2.3.2; (c1) $\theta_2(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_2(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_2(\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}$. Denote by $\mathfrak{k}$ the $+1$-eigenspace of θ_2 in $\mathfrak{g}_u$. Due to the result of Murakami [Mu3, pp. 305, type DI], one comprehends that $\{-i\alpha_s|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{s=1}^{j-1} \cup \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\} \cup \{-i\alpha_t|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{t=j+1}^{l-1}$ is the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, where $\eta := \sum_{p=j}^{l-1} \alpha_p$, and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{so}(2j+1) \oplus \mathfrak{so}(2l-2j-1)$:



where $-i\acute{\alpha}_h := -i\alpha_h|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ for each $1 \le h \le j-1$ and $j+1 \le h \le l-1$, and $-i\acute{\eta} := -i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$. His result also assures that $\mathfrak{so}(2j+1, 2l-2j-1)$ is the real form of $\mathfrak{d}_l$ given by (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_2(P) = -P\}$. Now, $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1 := \mathfrak{so}(2j+1)$ and $\mathfrak{k}_2 := \mathfrak{so}(2l-2j-1)$. Then, let us assume that $\{-i\alpha_s|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{s=1}^{j-1} \cup \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$ (resp. $\{-i\alpha_t|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{t=j+1}^{l-1}$) is the set of simple roots in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (resp. $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$). From the assumption, it follows that the highest root $-i\mu_1 \in \triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2 \in \triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are

$$(5.4.8)\qquad \begin{cases} -i\mu_1 = -i(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{j-1} + 2\eta)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, \\ -i\mu_2 = -i(\alpha_{j+1} + 2\alpha_{j+2} + 2\alpha_{j+3} + \cdots + 2\alpha_{l-1})|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}. \end{cases}$$

We will provide a necessary and sufficient condition for an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$, where $\mathfrak{W}^2_{\mathfrak{k}}$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{s=1}^{j-1} \cup \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\} \cup \{-i\alpha_t|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{t=j+1}^{l-1}$;

$$\mathfrak{W}^2_{\mathfrak{k}} = \left\{ T \in \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\alpha_1(T) \ge 0, \cdots, -i\alpha_{j-1}(T) \ge 0, -i\eta(T) \ge 0, \\ -i\alpha_{j+1}(T) \ge 0, \cdots, -i\alpha_{l-1}(T) \ge 0 \end{array} \right\}.$$

In order to do so, we want to describe the dual basis of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of $\{Z_a\}_{a=1}^{l}$. The definition (5.4.7) of θ_2 enables us obtain $\theta_2|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ because $\exp \pi\, \mathrm{ad}_{\mathfrak{d}_l} iZ_j = \mathrm{id}$ on $\tilde{\mathfrak{h}}$. Thus since $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (5.4.1), any element $T \in \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$ can be described as follows:

$$(5.4.9)\qquad T = i(\lambda_1 \cdot Z_1 + \cdots + \lambda_{l-2} \cdot Z_{l-2} + \lambda_{l-1} \cdot (Z_{l-1} + Z_l)), \quad \lambda_d \in \mathbb{R}.$$

Let $\{T_d\}_{d=1}^{l-1}$ ($T_d \in \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$) be the dual basis of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{s=1}^{j-1} \cup \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\} \cup \{-i\alpha_t|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{t=j+1}^{l-1}$. Then, by (5.4.9), $\eta = \sum_{p=j}^{l-1} \alpha_p$ and $\alpha_a(Z_b) = \delta_{a,b}$, we see that

$$(5.4.10)\qquad \begin{cases} T_q = iZ_q \text{ for } 1 \le q \le j, \\ T_u = i(-Z_j + Z_u) \text{ for } j+1 \le u \le l-2, \\ T_{l-1} = i(-Z_j + Z_{l-1} + Z_l). \end{cases}$$

Now, let us provide a necessary and sufficient condition for an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$.

Lemma 5.4.4. *In the setting on Paragraph* 5.4.2; *an element* $T \in \mathfrak{W}^2_{\mathfrak{k}}$ *is an Spr-element of* $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$ *if and only if it is either* iZ_1 *or* $i(-Z_j + Z_{j+1})$.

Proof. Suppose that an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. In this case, Lemma 4.2.2 and (5.4.8) imply that the element T equals T_1, T_{j+1} or $T_1 + T_{j+1}$, because $\{T_d\}_{d=1}^{l-1}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{s=1}^{j-1} \cup \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\} \cup \{-i\alpha_t|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}_{t=j+1}^{l-1}$. Therefore, it follows from (5.4.10) that one of the following three cases only occurs:

(b′-1) $T = iZ_1$, (b′-2) $T = i(-Z_j + Z_{j+1})$, (b′-3) $T = i(Z_1 - Z_j + Z_{j+1})$.

There exists a root $\beta = \sum_{1 \le f < j+1} \alpha_f + 2\sum_{j+1 \le h < l-1} \alpha_h + \alpha_{l-1} + \alpha_l \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})$ (ref. (5.4.3)), and it comes from $\alpha_a(Z_b) = \delta_{a,b}$ that $\beta(i(Z_1 - Z_j + Z_{j+1})) = 2i \ne \pm i$.

Therefore, Lemma 4.1.1 means that the element $T = i(Z_1 - Z_j + Z_{j+1})$ in Case (b$'$-3) is not an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$. The other elements (b$'$-1) $T = iZ_1$ and (b$'$-2) $T = i(-Z_j + Z_{j+1})$ satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \backslash \triangle_T(\mathfrak{d}_l, \tilde{\mathfrak{h}})$. So, the elements in Case (b$'$-1) and (b$'$-2) are *Spr*-elements of $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$ (cf. Lemma 4.1.1). Hence, if an element $T \in \mathfrak{W}^2_{\mathfrak{k}}$ is an *Spr*-element, then one of the following two cases only occurs:

$$\text{(b-1) } T = iZ_1, \quad \text{(b-2) } T = i(-Z_j + Z_{j+1}).$$

On the other hand, in the arguments stated above, it has been already confirmed that the elements in Case (b-1) and (b-2) are *Spr*-elements of $\mathfrak{g}$. Hence, we have verified Lemma 5.4.4. □

Let us investigate whether the *Spr*-elements iZ_1 and $i(-Z_j + Z_{j+1})$ in Lemma 5.4.4 are equivalent to each other or not.

Lemma 5.4.5. *In the above setting; the Spr-element $T_1 = iZ_1$ of $\mathfrak{g}$ is equivalent to $T_{j+1} = i(-Z_j + Z_{j+1})$ if and only if $2j+1 = l$. Here, $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$.*

Proof. It is known that

$$\text{(5.4.11)} \qquad \begin{aligned} \mathfrak{c}_{\mathfrak{g}}(iZ_1) &= \mathfrak{so}(2j-1, 2l-2j-1) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1})) &= \mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1 \end{aligned}$$

(cf. Theorem 6.16 in [Bm]). This asserts that $2j-1 = 2l-2j-3$ when the *Spr*-element $T_1 = iZ_1$ is equivalent to $T_{j+1} = i(-Z_j + Z_{j+1})$. Hence, it follows that $2j+1 = l$ if the element T_1 is equivalent to T_{j+1}.

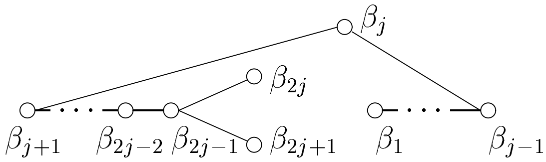
Now, we will verify that the element $T_1 = iZ_1$ is equivalent to $T_{j+1} = i(-Z_j + Z_{j+1})$ in case of $2j+1 = l$. Suppose that $2j+1 = l$. Let us construct an automorphism ϕ of $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1) = \mathfrak{so}(2j+1, 2j+1)$ satisfying $\phi(T_1) = T_{j+1}$. Define an involutive linear isomorphism ϕ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{2j+1}$ by

$$\text{(5.4.12)} \qquad \begin{cases} \phi'(iZ_s) := i(-Z_j + Z_{j+s}) \text{ for } 1 \le s \le j-1, \\ \phi'(iZ_j) := i(-Z_j + Z_{2j} + Z_{2j+1}), \\ \phi'(iZ_u) := i(Z_{u-j} - Z_j + Z_{2j} + Z_{2j+1}) \text{ for } j+1 \le u \le 2j-1, \\ \phi'(iZ_n) := iZ_n \text{ for } n = 2j,\ 2j+1. \end{cases}$$

Then we have

$$\text{(5.4.13)} \qquad \begin{cases} {}^t\phi'_{\mathbb{C}}(\alpha_s) = \alpha_{j+s} \text{ for } 1 \le s \le j-1, \\ {}^t\phi'_{\mathbb{C}}(\alpha_j) = -\sum_{v=1}^{2j-1} \alpha_v, \\ {}^t\phi'_{\mathbb{C}}(\alpha_u) = \alpha_{u-j} \text{ for } j+1 \le u \le 2j-1, \\ {}^t\phi'_{\mathbb{C}}(\alpha_n) = \sum_{w=j}^{2j-1} \alpha_w + \alpha_n \text{ for } n = 2j,\ 2j+1, \end{cases}$$

by virtue of $\alpha_a(Z_b) = \delta_{a,b}$. Here, $\phi'_{\mathbb{C}}$ denotes the complex linear extension of ϕ' to $\tilde{\mathfrak{h}}$. The Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2j+1}$ is as follows:



where $\beta_a := {}^t\phi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq 2j+1$. Therefore, ϕ' satisfies

$$ {}^t\phi'_{\mathbb{C}}\big(\triangle(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}}) $$

because the Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2j+1}$ is the same as that of $\Pi_{\triangle(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2j+1}$ (cf. Murakami [Mu3, Lemma 1, pp. 295]). Thus, Proposition 2.3.2 implies that there exists an involutive automorphism $\bar{\phi}$ of $\mathfrak{d}_{2j+1} = \mathfrak{so}(4j+2, \mathbb{C})$ satisfying three conditions (i) $\bar{\phi}(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (iii) $\bar{\phi}(X_{\pm\alpha_a}) = X_{\pm{}^t\bar{\phi}(\alpha_a)}$. From now on, we aim to have an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ such that $\bar{\phi} \circ \exp \mathrm{ad}_{\mathfrak{d}_{2j+1}} iH$ is an automorphism of $\mathfrak{g} = \mathfrak{so}(2j+1, 2j+1)$. For the aim, let us show that $\bar{\phi}$ satisfies the two conditions (a) and (b) in Proposition 2.3.4. From $\bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.4.13), it is obvious that ${}^t\bar{\phi}(\alpha_s) = \alpha_{j+s}$ $(1 \leq s \leq j-1)$, ${}^t\bar{\phi}(\alpha_j) = -\sum_{v=1}^{2j-1}\alpha_v$, ${}^t\bar{\phi}(\alpha_u) = \alpha_{u-j}$ $(j+1 \leq u \leq 2j-1)$, and ${}^t\bar{\phi}(\alpha_n) = \sum_{w=j}^{2j-1}\alpha_w + \alpha_n$ $(n = 2j,\ 2j+1)$. On the other hand, the definition (5.4.7) of θ_2 and (5.4.3) state that

$$ \begin{aligned} &\triangle_1^+(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}} : \theta_2) \\ &= \left\{ \begin{array}{l|l} \sum_{p_1 \leq f < q_1} \alpha_f, & 1 \leq p_1 < q_1 \leq j, \\ \sum_{p_2 \leq f < q_2} \alpha_f, & j+1 \leq p_2 < q_2 \leq 2j, \\ \sum_{s_1 \leq f < t_1} \alpha_f + 2\sum_{t_1 \leq h < 2j} \alpha_h + \alpha_{2j} + \alpha_{2j+1}, & 1 \leq s_1 < t_1 \leq j, \\ \sum_{s_2 \leq f < t_2} \alpha_f + 2\sum_{t_2 \leq h < 2j} \alpha_h + \alpha_{2j} + \alpha_{2j+1} & j+1 \leq s_2 < t_2 < 2j+1 \end{array} \right\} \end{aligned} $$

(see (2.3.4) for $\triangle_1^+(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}} : \theta_2)$). So, the involution $\bar{\phi}$ of $\mathfrak{d}_{2j+1}$ satisfies the condition (b) in Proposition 2.3.4, namely ${}^t\bar{\phi}\big(\triangle_1(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}} : \theta_2)\big) = \triangle_1(\mathfrak{d}_{2j+1}, \tilde{\mathfrak{h}} : \theta_2)$. Moreover, since $\bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.4.12), and since $\theta_2|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (5.4.1), one obtains $\theta_2 \circ \bar{\phi} = \bar{\phi} \circ \theta_2$ on $i\tilde{\mathfrak{h}}_{\mathbb{R}}$; and therefore the involution $\bar{\phi}$ also satisfies the condition (a) in Proposition 2.3.4. For the reasons, Proposition 2.3.4 enables us to have an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ such that $\bar{\phi} \circ \exp \mathrm{ad}_{\mathfrak{d}_{2j+1}} iH$ is an automorphism of $\mathfrak{g}$. Defining ϕ by $\phi := \bar{\phi} \circ \exp \mathrm{ad}_{\mathfrak{d}_{2j+1}} iH$,[9] we deduce that

$$ \phi(T_1) = \phi(iZ_1) = i(-Z_j + Z_{j+1}) = T_{j+1} $$

because $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \bar{\phi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.4.12). Accordingly, the *Spr*-element T_1 is equivalent to T_{j+1} via ϕ. Hence, we have proved Lemma 5.4.5. □

Now, let us demonstrate Proposition 5.4.6.

Proposition 5.4.6. *Under our equivalence relation, Spr-elements of* DI: $\mathfrak{g} = \mathfrak{so}(2j+1, 2l-2j-1)$, $1 \leq j \leq l-3$, *are classified as follows*:

$$ Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{ll} \{\ [\,iZ_1\,],\ [\,i(-Z_j + Z_{j+1})\,]\ \} & \textit{if } 2j+1 \neq l, \\ \{\ [\,iZ_1\,]\ \} & \textit{if } 2j+1 = l. \end{array} \right. $$

[9] This ϕ is an outer automorphism of $\mathfrak{so}(2j+1, 2j+1)$.

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(2j-1, 2l-2j-1) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$ *and* $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. It is immediate from (4.1.2) and two Lemmas 5.4.4 and 5.4.5 that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \begin{cases} \{\ [iZ_1],\ [i(-Z_j + Z_{j+1})]\ \} & \text{if } 2j+1 \neq l, \\ \{\ [iZ_1]\ \} & \text{if } 2j+1 = l. \end{cases}$$

Lemma 3.1.1-(1) and -(2) imply that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(iZ_1))$ and $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1})))$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$ and $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$, respectively. Thus, it follows from (5.4.11) that $(\mathfrak{g}, \mathfrak{so}(2j-1, 2l-2j-1) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$ and $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$, respectively. Therefore, Proposition 5.4.6 has been shown. □

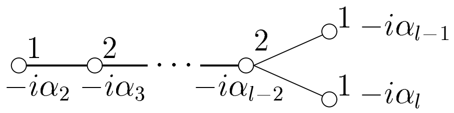
5.4.3. *Case* DI $\mathfrak{so}(2j, 2l-2j) : j = 1$. In this paragraph, we will achieve the classification of *Spr*-elements of $\mathfrak{so}(2, 2l-2)$ (see Proposition 5.4.9).

Let us define an involutive automorphism θ_3 of $\mathfrak{g}_u$ such that $\mathfrak{so}(2, 2l-2)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_3. Let θ_3 be an inner automorphism of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ defined by

(5.4.14) $$\theta_3 := \exp \pi \operatorname{ad}_{\mathfrak{d}_l} iZ_1$$

(recall Notation 5.4.1 for Z_1 and for later). This θ_3 is involutive and satisfies the three conditions in Paragraph 2.3.2; (c1) $\theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_3(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_3(\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}$ (cf. Murakami [Mu3, pp. 297, type DI]). It is shown by Murakami [Mu3] that $\{-i\alpha_c\}_{c=2}^l$ is the set of simple roots in $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, its Dynkin diagram is

$\mathfrak{k} = \mathfrak{so}(2l-2) \oplus \mathfrak{t}^1$: ×



and $\mathfrak{so}(2, 2l-2)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{d}_l$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_3(K) = K\}$ and $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_3(P) = -P\}$. Remark that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

(5.4.15) $$-i\mu = -i(\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l).$$

Now, let us prove Lemma 5.4.7.

Lemma 5.4.7. *In the above setting; an element* $T \in \mathfrak{W}_{\mathfrak{k}}^3$ *is an Spr-element of* $\mathfrak{g} = \mathfrak{so}(2, 2l-2)$ *if and only if it is one of the following*:

$$\begin{array}{lll} i(-Z_1 + Z_2), & iZ_{l-1}, & i(-Z_1 + Z_{l-1}), \\ iZ_l, & i(-Z_1 + Z_l), & \pm iZ_1. \end{array}$$

Here, $\mathfrak{W}_{\mathfrak{k}}^3$ *is the positive Weyl chamber with respect to* $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^l$;

$$\mathfrak{W}_{\mathfrak{k}}^3 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \,|\, -i\alpha_2(T) \geq 0, -i\alpha_3(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. For any root $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, one obtains $-i\gamma(iZ_1) \equiv 0$ because of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^{l}$ and $\alpha_a(Z_b) = \delta_{a,b}$. Therefore, Lemma 4.2.3, together with (5.4.15), $\alpha_a(Z_b) = \delta_{a,b}$ and $T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$, implies that one of the following cases only occurs:

$$\text{(c}'\text{-1.1) } T = i(\lambda \cdot Z_1 + Z_2), \quad \text{(c}'\text{-1.2) } T = i(\lambda \cdot Z_1 + Z_{l-1}),$$
$$\text{(c}'\text{-1.3) } T = i(\lambda \cdot Z_1 + Z_l), \quad \text{(c}'\text{-2) } T = i\,\lambda \cdot Z_1,$$

where λ is a real number ($\lambda \neq 0$ in Case (c′-2)). By the supposition and Lemma 4.1.1, the element T must satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{d}_l, \tilde{\mathfrak{h}})$. So, since (5.4.3) and $\alpha_a(Z_b) = \delta_{a,b}$, we have $\lambda = -1$ in Case (c′-1.1), $\lambda = 0$ or -1 in two Cases (c′-1.2) and (c′-1.3), and $\lambda = \pm 1$ in Case (c′-2). Accordingly, if an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an *Spr*-element, then it is one of the following:

$$\begin{array}{lll} \text{(c-1.1) } i(-Z_1 + Z_2), & \text{(c-1.2) } iZ_{l-1}, & i(-Z_1 + Z_{l-1}), \\ \text{(c-1.3) } iZ_l, \quad i(-Z_1 + Z_l), & \text{(c-2) } \pm iZ_1. & \end{array}$$

Conversely, if an element T' is one of the above elements, then it follows from (5.4.3) and $\alpha_a(Z_b) = \delta_{a,b}$ that the element T' satisfies $\beta(T') = \pm i$ for every root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{d}_l, \tilde{\mathfrak{h}})$; and hence it is an *Spr*-element of $\mathfrak{g}$ (ref. Lemma 4.1.1). Hence, Lemma 5.4.7 has been proved. □

Lemma 5.4.7 and (4.1.2) imply that

$$\text{(5.4.16)} \quad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{lll} [i(-Z_1 + Z_2)], & [iZ_{l-1}], & [i(-Z_1 + Z_{l-1})], \\ [iZ_l], & [i(-Z_1 + Z_l)], & [iZ_1] \end{array} \right\},$$

where $\mathfrak{g} = \mathfrak{so}(2, 2l - 2)$. From now on, we are going to verify that the above *Spr*-element iZ_{l-1} (resp. iZ_l) is equivalent to $i(-Z_1 + Z_{l-1})$ (resp. $i(-Z_1 + Z_l)$).

Lemma 5.4.8. *There exists an involutive automorphism φ of $\mathfrak{d}_l$ which satisfies $\varphi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, $\varphi(\mathfrak{g}) \subset \mathfrak{g}$ and*

$$\begin{cases} \varphi(iZ_1) = -iZ_1, \\ \varphi(iZ_k) = i(-2Z_1 + Z_k) \quad \text{for } 2 \leq k \leq l-2, \\ \varphi(iZ_n) = i(-Z_1 + Z_n) \quad \text{for } n = l-1,\ l. \end{cases}$$

Here, $\mathfrak{g} = \mathfrak{so}(2, 2l - 2)$, and $\{Z_a\}_{a=1}^{l}$ is the dual basis of $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$.

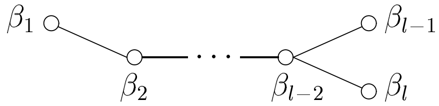
Proof. We aim to construct such an automorphism φ. Let us define an involutive linear isomorphism φ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$ by

$$\text{(5.4.17)} \quad \begin{cases} \varphi'(iZ_1) := -iZ_1, \\ \varphi'(iZ_k) := i(-2Z_1 + Z_k) \quad \text{for } 2 \leq k \leq l-2, \\ \varphi'(iZ_n) := i(-Z_1 + Z_n) \quad \text{for } n = l-1,\ l. \end{cases}$$

Then, it comes from $\alpha_a(Z_b) = \delta_{a,b}$ that

$$
\begin{aligned}
{}^t\varphi'_{\mathbb{C}}(\alpha_1) &= -(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l),\\
{}^t\varphi'_{\mathbb{C}}(\alpha_c) &= \alpha_c \quad \text{for } 2 \le c \le l,
\end{aligned}
$$

where $\varphi'_{\mathbb{C}}$ denotes the complex linear extension of φ' to $\tilde{\mathfrak{h}}$. Therefore, the Dynkin diagram of $\{{}^t\varphi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{l}$ is as follows:



Here, $\beta_a := {}^t\varphi'_{\mathbb{C}}(\alpha_a)$ for $1 \le a \le l$. Consequently, the linear involution φ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ satisfies

$$
{}^t\varphi'_{\mathbb{C}}\big(\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})
$$

(cf. Murakami [Mu3, Lemma 1, pp. 295]). For the reasons, Proposition 2.3.2 assures that there exists an involutive automorphism φ of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ satisfying three conditions (i) $\varphi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (iii) $\varphi(X_{\pm\alpha_a}) = X_{\pm{}^t\varphi(\alpha_a)}$. Furthermore, since θ_3 is involutive and (5.4.14), and since (5.4.17) and $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$, we deduce that

$$
\varphi \circ \theta_3 = \varphi \circ \exp \pi \operatorname{ad}_{\mathfrak{d}_l} iZ_1 = \exp \pi \operatorname{ad}_{\mathfrak{d}_l}(-iZ_1) \circ \varphi = \theta_3 \circ \varphi.
$$

Thus, Proposition 2.2.3 means that the involution φ of $\mathfrak{d}_l$ is an automorphism of $\mathfrak{g} = \mathfrak{so}(2, 2l-2)$.[10] Hence, Lemma 5.4.8 follows from $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.4.17). □

By (5.4.16) and Lemma 5.4.8, one concludes that

(5.4.18)
$$
\begin{aligned}
&Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g}))\\
&\qquad = \{\,[i(-Z_1 + Z_2)],\ [i(-Z_1 + Z_{l-1})],\ [i(-Z_1 + Z_l)],\ [iZ_1]\,\},
\end{aligned}
$$

where $\mathfrak{g} = \mathfrak{so}(2, 2l-2)$. Let us confirm that the above *Spr*-element $i(-Z_1 + Z_{l-1})$ is equivalent to $i(-Z_1 + Z_l)$. The involution θ_1 in Paragraph 5.4.1 satisfies $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$, and satisfies $\theta_1(iZ_1) = iZ_1$ since $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (5.4.1). Accordingly, it follows from (5.4.14) that $\theta_3 \circ \theta_1 = \theta_1 \circ \theta_3$, so that Proposition 2.2.3 means that the involution θ_1 is an automorphism of $\mathfrak{g} = \mathfrak{so}(2, 2l-2)$.[11] In addition, we have $\theta_1(i(-Z_1 + Z_{l-1})) = i(-Z_1 + Z_l)$ by virtue of $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$ and (5.4.1). Hence, the *Spr*-element $i(-Z_1+Z_{l-1})$ is equivalent to $i(-Z_1+Z_l)$ via θ_1. Therefore by (5.4.18), we see that

(5.4.19)
$$
Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \{\,[i(-Z_1 + Z_2)],\ [i(-Z_1 + Z_l)],\ [iZ_1]\,\}.
$$

Now, we will demonstrate Proposition 5.4.9.

Proposition 5.4.9. *Under our equivalence relation, Spr-elements of* DI: $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$, $j = 1$, *are classified as follows*:

$$
Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \{\,[i(-Z_1 + Z_2)],\ [i(-Z_1 + Z_l)],\ [iZ_1]\,\}.
$$

[10] This φ is an inner automorphism of $\mathfrak{so}(2, 2l-2)$.

[11] This θ_1 becomes an outer automorphism of $\mathfrak{so}(2, 2l-2)$.

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(2, 2l-4) \oplus \mathfrak{t}^1)$, (2) $(\mathfrak{g}, \mathfrak{su}(1, l-1) \oplus \mathfrak{t}^1)$ *and* (3) $(\mathfrak{g}, \mathfrak{so}(2l-2) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_2)$, $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_l)$ *and* $\rho_3 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. It has been shown that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_2)],\ [i(-Z_1 + Z_l)],\ [iZ_1]\ \}$$

(see (5.4.19)). It is necessary to investigate whether the above Spr-elements are mutually equivalent or not. By Theorem 6.16 in [Bm], one gets

$$\begin{aligned} &\mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_2)) = \mathfrak{so}(2, 2l-4) \oplus \mathfrak{t}^1, \\ &\mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_l)) = \mathfrak{su}(1, l-1) \oplus \mathfrak{t}^1, \\ &\mathfrak{c}_{\mathfrak{g}}(iZ_1) = \mathfrak{so}(2l-2) \oplus \mathfrak{t}^1. \end{aligned} \tag{5.4.20}$$

Therefore since $l \geq 4$, the Spr-elements $i(-Z_1 + Z_2)$, $i(-Z_1 + Z_l)$ and iZ_1 are not mutually equivalent. So, the first half of statements on this proposition has been shown. Lemma 3.1.1 and (5.4.20) allow us to deduce the conclusion. □

5.4.4. *Case* DI $\mathfrak{so}(2j, 2l-2j)$: $l = 4$ *and* $j = 2$. Our aim in this paragraph is to achieve the classification of Spr-elements of $\mathfrak{so}(4,4)$ (cf. Proposition 5.4.12).

Let us give an involutive automorphism θ_4 of $\mathfrak{g}_u$ such that $\mathfrak{so}(4,4)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_4. Define an inner automorphism θ_4 of $\mathfrak{d}_4 = \mathfrak{so}(8, \mathbb{C})$ by

$$\theta_4 := \exp \pi \operatorname{ad}_{\mathfrak{d}_4} iZ_2. \tag{5.4.21}$$

Since $iZ_2 \in \mathfrak{g}_u$ and $\theta_4|_{\tilde{\mathfrak{h}}} = \mathrm{id}$, one obtains (c1) $\theta_4(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_4(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_4(\Pi_{\triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}})}$, which means that the automorphism θ_4 satisfies the three conditions in Paragraph 2.3.2. The result of Murakami [Mu3, pp. 297, type DI] implies that θ_4 is involutive, $\{-i\alpha_1, -i\nu, -i\alpha_3, -i\alpha_4\}$ is the set of simple roots in $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \mid \theta_4(K) = K\}$, and its Dynkin diagram is as follows:

$$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2): \quad \begin{array}{cc} -i\nu \ \circ^1 & \circ^1 \ -i\alpha_3 \\ -i\alpha_1 \ \circ^1 & \circ^1 \ -i\alpha_4 \end{array}$$

where $-i\nu := i(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)$. Furthermore, his result enables us to deduce that $\mathfrak{so}(4,4)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{d}_4$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_4(P) = -P\}$. Now, we want to describe the dual basis $\{T_a\}_{a=1}^4$ of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_3, -i\alpha_4\}$, in terms of the dual basis $\{Z_a\}_{a=1}^4$ of $\Pi_{\triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^4$. The description will be useful at once. By virtue of $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^4$ and $\alpha_a(Z_b) = \delta_{a,b}$, it is easy to see that

$$\begin{cases} T_1 = i(Z_1 - \frac{1}{2}Z_2), & T_2 = -\frac{i}{2}Z_2, \\ T_3 = i(-\frac{1}{2}Z_2 + Z_3), & T_4 = i(-\frac{1}{2}Z_2 + Z_4). \end{cases} \tag{5.4.22}$$

By use of (5.4.22), we will prove the following:

Lemma 5.4.10. *With the above assumptions; an element $T \in \mathfrak{W}^4_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(4,4)$ if and only if it is one of the following:*

$$\begin{array}{lll} i(-Z_2+Z_3+Z_4), & i(-Z_2+Z_4), & i(-Z_2+Z_3), \\ i(Z_1-Z_2+Z_4), & i(Z_1-Z_2+Z_3), & i(Z_1-Z_2). \end{array}$$

Here, $\mathfrak{W}^4_{\mathfrak{k}}$ is a Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_3, -i\alpha_4\}$;

$$\mathfrak{W}^4_{\mathfrak{k}} = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, -i\nu(T) \geq 0, -i\alpha_3(T) \geq 0, -i\alpha_4(T) \geq 0\}.$$

Proof. Notice that $\mathfrak{k}$ is the direct sum of four simple ideals $\mathfrak{k}_a := \mathfrak{su}(2)$ $(1 \leq a \leq 4)$, and that the highest root $-i\mu_a \in \triangle(\mathfrak{k}_a, \mathfrak{k}_a \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$(5.4.23) \qquad \begin{cases} -i\mu_1 = -i\alpha_1, & -i\mu_2 = -i\nu \quad (= i(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)), \\ -i\mu_3 = -i\alpha_3, & -i\mu_4 = -i\alpha_4, \end{cases}$$

where we assume $\triangle(\mathfrak{k}_a, \mathfrak{k}_a \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ to be a subset of $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ $(1 \leq a \leq 4)$. Now, suppose that an element $T \in \mathfrak{W}^4_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. Arguments similar to those on the proof of Lemma 4.2.1 allow us to confirm that the *Spr*-element $T \in \mathfrak{W}^4_{\mathfrak{k}}$ satisfies one of the following fifteen conditions:

(1) $-i\mu_1(T) = 0, -i\mu_2(T) = 0, -i\mu_3(T) = 0, -i\mu_4(T) = 1$;
(2) $-i\mu_1(T) = 0, -i\mu_2(T) = 0, -i\mu_3(T) = 1, -i\mu_4(T) = 0$;
(3) $-i\mu_1(T) = 0, -i\mu_2(T) = 0, -i\mu_3(T) = 1, -i\mu_4(T) = 1$;
(4) $-i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 0, -i\mu_4(T) = 0$;
(5) $-i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 0, -i\mu_4(T) = 1$;
(6) $-i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 1, -i\mu_4(T) = 0$;
(7) $-i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 1, -i\mu_4(T) = 1$;
(8) $-i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 0, -i\mu_4(T) = 0$;
(9) $-i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 0, -i\mu_4(T) = 1$;
(10) $-i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 1, -i\mu_4(T) = 0$;
(11) $-i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 1, -i\mu_4(T) = 1$;
(12) $-i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 0, -i\mu_4(T) = 0$;
(13) $-i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 0, -i\mu_4(T) = 1$;
(14) $-i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 1, -i\mu_4(T) = 0$;
(15) $-i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 1, -i\mu_4(T) = 1$.

Then, T becomes one of the following fifteen elements:

(1) $T_4 = i(-\frac{1}{2}Z_2 + Z_4)$,
(2) $T_3 = i(-\frac{1}{2}Z_2 + Z_3)$,
(3) $T_3 + T_4 = i(-Z_2 + Z_3 + Z_4)$,
(4) $T_2 = -\frac{i}{2}Z_2$,
(5) $T_2 + T_4 = i(-Z_2 + Z_4)$,

(6) $T_2 + T_3 = i(-Z_2 + Z_3)$,
(7) $T_2 + T_3 + T_4 = i(-\frac{3}{2}Z_2 + Z_3 + Z_4)$,
(8) $T_1 = i(Z_1 - \frac{1}{2}Z_2)$,
(9) $T_1 + T_4 = i(Z_1 - Z_2 + Z_4)$,
(10) $T_1 + T_3 = i(Z_1 - Z_2 + Z_3)$,
(11) $T_1 + T_3 + T_4 = i(Z_1 - \frac{3}{2}Z_2 + Z_3 + Z_4)$,
(12) $T_1 + T_2 = i(Z_1 - Z_2)$,
(13) $T_1 + T_2 + T_4 = i(Z_1 - \frac{3}{2}Z_2 + Z_4)$,
(14) $T_1 + T_2 + T_3 = i(Z_1 - \frac{3}{2}Z_2 + Z_3)$,
(15) $T_1 + T_2 + T_3 + T_4 = i(Z_1 - 2Z_2 + Z_3 + Z_4)$

because (5.4.22), (5.4.23) and $\{T_a\}_{a=1}^4$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_3, -i\alpha_4\}$. Lemma 4.1.1 states that the *Spr*-element T must satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{d}_4, \tilde{\mathfrak{h}})$. Therefore, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and (5.4.3) that the element T is one of the following:

$$\begin{array}{llllll} (3) & i(-Z_2+Z_3+Z_4), & (5) & i(-Z_2+Z_4), & (6) & i(-Z_2+Z_3), \\ (9) & i(Z_1-Z_2+Z_4), & (10) & i(Z_1-Z_2+Z_3), & (12) & i(Z_1-Z_2). \end{array}$$

Conversely, if an element T' is one of the above six elements, then it satisfies $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{d}_4, \tilde{\mathfrak{h}})$. So, the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{so}(4,4)$, due to Lemma 4.1.1. Therefore, we have demonstrated Lemma 5.4.10. □

Lemma 5.4.10 and (4.1.2) allow us to lead the following:

$$(5.4.24)\quad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{lll} [\,i(-Z_2+Z_3+Z_4)\,], & [\,i(-Z_2+Z_4)\,], & [\,i(-Z_2+Z_3)\,], \\ [\,i(Z_1-Z_2+Z_4)\,], & [\,i(Z_1-Z_2+Z_3)\,], & [\,i(Z_1-Z_2)\,] \end{array} \right\},$$

where $\mathfrak{g} = \mathfrak{so}(4,4)$. The following lemma implies that the above *Spr*-elements $i(-Z_2 + Z_4)$, $i(-Z_2 + Z_3)$ and $i(Z_1 - Z_2)$ are mutually equivalent, and that the above *Spr*-elements $i(-Z_2 + Z_3 + Z_4)$, $i(Z_1 - Z_2 + Z_4)$ and $i(Z_1 - Z_2 + Z_3)$ are mutually equivalent.

Lemma 5.4.11. *In the above setting; there exists an automorphism ϕ of $\mathfrak{g} = \mathfrak{so}(4,4)$ such that $\phi(iZ_1) = iZ_3$, $\phi(iZ_2) = iZ_2$, $\phi(iZ_3) = iZ_4$ and $\phi(iZ_4) = iZ_1$.*

Proof. Define a linear isomorphism ϕ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^4$ as follows:

$$(5.4.25)\qquad \phi'(iZ_1) := iZ_3, \quad \phi'(iZ_2) := iZ_2, \quad \phi'(iZ_3) := iZ_4, \quad \phi'(iZ_4) := iZ_1.$$

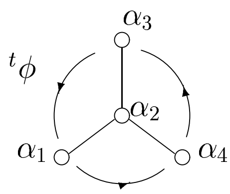
Then, it is immediate from $\alpha_a(Z_b) = \delta_{a,b}$ that

$${}^t\phi'_{\mathbb{C}}(\alpha_1) = \alpha_4, \quad {}^t\phi'_{\mathbb{C}}(\alpha_2) = \alpha_2, \quad {}^t\phi'_{\mathbb{C}}(\alpha_3) = \alpha_1, \quad {}^t\phi'_{\mathbb{C}}(\alpha_4) = \alpha_3,$$

where $\phi'_{\mathbb{C}}$ denotes the complex linear extension of ϕ' to $\tilde{\mathfrak{h}}$. Therefore, the linear isomorphism ϕ' satisfies

$$ {}^t\phi'_{\mathbb{C}}\big(\triangle(\mathfrak{d}_4,\tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{d}_4,\tilde{\mathfrak{h}}) $$

(see (5.4.3) for $\triangle(\mathfrak{d}_4,\tilde{\mathfrak{h}})$). Hence, Proposition 2.3.2 enables us to obtain an automorphism ϕ of $\mathfrak{d}_4 = \mathfrak{so}(8,\mathbb{C})$ such that (i) $\phi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (iii) $\phi(X_{\pm\alpha_a}) = X_{\pm {}^t\phi^{-1}(\alpha_a)}$.



From (5.4.21) and (5.4.25), one gets

$$ \phi\circ\theta_4 = \phi\circ\exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} iZ_2 = \exp\pi\,\mathrm{ad}_{\mathfrak{d}_4}\phi(iZ_2)\circ\phi = \theta_4\circ\phi. $$

Thus, Proposition 2.2.3 implies that ϕ is an automorphism of $\mathfrak{g} = \mathfrak{so}(4,4)$.[12] Therefore, Lemma 5.4.11 comes from $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.4.25). □

By Lemma 5.4.11 and (5.4.24), one deduces that for $\mathfrak{g} = \mathfrak{so}(4,4)$

(5.4.26) $$ Spr_{\mathfrak{g}}/(\{\pm1\}\times\mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_2+Z_3)],\ [i(Z_1-Z_2+Z_3)]\ \}. $$

Now, let us show that the above Spr-element $i(-Z_2+Z_3)$ is equivalent to $i(Z_1-Z_2+Z_3)$. The involution φ in Lemma 5.4.8 ($l=4$) satisfies $\varphi(iZ_2) = i(-2Z_1+Z_2)$ and $\varphi(iZ_3) = i(-Z_1+Z_3)$, so that $\varphi(i(-Z_2+Z_3)) = i(Z_1-Z_2+Z_3)$. Consequently, if the involution φ is an automorphism of $\mathfrak{g}$, then the Spr-element $i(-Z_2+Z_3)$ is equivalent to $i(Z_1-Z_2+Z_3)$ via φ. So, we devote ourselves to confirming that the involution φ is an automorphism of $\mathfrak{g} = \mathfrak{so}(4,4)$. Since $\varphi(iZ_2) = i(-2Z_1+Z_2)$ and (5.4.21), one perceives that

$$ \varphi\circ\theta_4 = \varphi\circ\exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} iZ_2 = \exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} i(-2Z_1+Z_2)\circ\varphi. $$

Since $\exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} iZ_1$ $(=\theta_3)$ is involutive (ref. Paragraph 5.4.3), one can deduce that $\exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} i(-2Z_1) = \mathrm{id}$ on $\mathfrak{d}_4 = \mathfrak{so}(8,\mathbb{C})$. Accordingly, it follows from $[Z_1,Z_2]=0$ that $\exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} i(-2Z_1+Z_2) = \exp\pi\,\mathrm{ad}_{\mathfrak{d}_4} iZ_2 = \theta_4$, so that $\varphi\circ\theta_4 = \theta_4\circ\varphi$. This, together with $\varphi(\mathfrak{g}_u)\subset\mathfrak{g}_u$, asserts that the involution φ in Lemma 5.4.8 is an automorphism of $\mathfrak{g} = \mathfrak{so}(4,4)$ (see Proposition 2.2.3).[13] By (5.4.26) and the above arguments, we conclude that

$$ Spr_{\mathfrak{g}}/(\{\pm1\}\times\mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_2+Z_3)]\ \}, $$

where $\mathfrak{g} = \mathfrak{so}(4,4)$. Lemma 3.1.1 enables us to see that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(-Z_2+Z_3)))$ is the pseudo-Hermitian symmetric Lie algebra by an involutive automorphism $\rho := \exp\pi\,\mathrm{ad}_{\mathfrak{g}} i(-Z_2+Z_3)$. Theorem 6.16 in [Bm] states that $\mathfrak{c}_{\mathfrak{g}}(i(-Z_2+Z_3)) = \mathfrak{so}(2,4)\oplus\mathfrak{t}^1$. Hence, $(\mathfrak{g}, \mathfrak{so}(2,4)\oplus\mathfrak{t}^1)$ is the pseudo-Hermitian symmetric Lie algebra by $\rho = \exp\pi\,\mathrm{ad}_{\mathfrak{g}} i(-Z_2+Z_3)$. For the reasons, we have got the following:

[12] This ϕ is an outer automorphism of $\mathfrak{so}(4,4)$.

[13] This φ becomes an outer automorphism of $\mathfrak{so}(4,4)$.

Proposition 5.4.12. *Under our equivalence relation, Spr-elements of* DI: $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$, $l = 4$ *and* $j = 2$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [\,i(-Z_2 + Z_3)\,]\ \}.$$

Besides, $(\mathfrak{g}, \mathfrak{so}(2,4) \oplus \mathfrak{t}^1)$ *is the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_2 + Z_3)$. *Here,* $\{Z_a\}_{a=1}^4$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^4$.

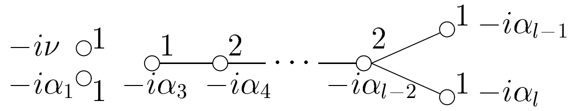
5.4.5. *Case* DI $\mathfrak{so}(2j, 2l-2j)$: $l \geq 5$ *and* $j = 2$. In this paragraph, we will classify *Spr*-elements of $\mathfrak{so}(4, 2l-4)$ (see Proposition 5.4.14).

First, we are going to give an involutive automorphism θ_5 of $\mathfrak{g}_u$ such that $\mathfrak{so}(4, 2l-4)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_5. Define an inner automorphism θ_5 of $\mathfrak{d}_l$ by

$$\theta_5 := \exp \pi \operatorname{ad}_{\mathfrak{d}_l} iZ_2. \tag{5.4.27}$$

Then, it satisfies (c1) $\theta_5(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_5(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_5(\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}$, since $iZ_2 \in \mathfrak{g}_u$ and $\theta_5|_{\tilde{\mathfrak{h}}} = \mathrm{id}$. By Murakami's result [Mu3, pp. 297, type DI], one knows that the automorphism θ_5 is involutive, the simple root system of $\mathfrak{k}$ is $\{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^l$ and the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^l$ is as follows:

$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(2l-4)$:



Here, $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_5(K) = K\}$ and $-i\nu := i(\alpha_1 + 2\sum_{d=2}^{l-2} \alpha_d + \alpha_{l-1} + \alpha_l)$. Besides, one also knows that $\mathfrak{so}(4, 2l-4)$ is the real form of $\mathfrak{d}_l$ given by (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, where $\mathfrak{p}$ denotes the -1-eigenspace of θ_5 in $\mathfrak{g}_u$. Now, $\mathfrak{k}$ is the direct sum of three simple ideals $\mathfrak{k}_1$, $\mathfrak{k}_2$ and $\mathfrak{k}_3$, where $\mathfrak{k}_1 := \mathfrak{su}(2)$, $\mathfrak{k}_2 := \mathfrak{su}(2)$ and $\mathfrak{k}_3 := \mathfrak{so}(2l-4)$. We assume that $\{-i\alpha_1\}$, $\{-i\nu\}$ and $\{-i\alpha_k\}_{k=3}^l$ are the set of simple roots in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $\triangle(\mathfrak{k}_3, \mathfrak{k}_3 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, respectively. Then for $p = 1, 2, 3$, the highest root $-i\mu_p \in \triangle(\mathfrak{k}_p, \mathfrak{k}_p \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$\begin{cases} -i\mu_1 = -i\alpha_1, \\ -i\mu_2 = -i\nu, \\ -i\mu_3 = -i(\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l). \end{cases} \tag{5.4.28}$$

Now, let T_a $(1 \leq a \leq l)$ be an element of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ defined by $-i\alpha_1(T_a) = \delta_{1,a}$, $-i\nu(T_a) = \delta_{2,a}$ and $-i\alpha_k(T_a) = \delta_{k,a}$ $(3 \leq k \leq l)$—that is, $\{T_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^l$. Since $\alpha_a(Z_b) = \delta_{a,b}$, $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^l$ and $-i\nu = i(\alpha_1 + 2\sum_{d=2}^{l-2} \alpha_d + \alpha_{l-1} + \alpha_l)$, the element T_a can be written as follows:

$$\begin{cases} T_1 = i(Z_1 - \frac{1}{2}Z_2), \\ T_2 = -\frac{i}{2}Z_2, \\ T_h = i(-Z_2 + Z_h) \quad \text{for } 3 \leq h \leq l-2, \\ T_n = i(-\frac{1}{2}Z_2 + Z_n) \quad \text{for } n = l-1, l. \end{cases} \tag{5.4.29}$$

By use of (5.4.29), we will deduce the following lemma:

Lemma 5.4.13. *In the above setting; an element $T \in \mathfrak{W}^5_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(4, 2l-4)$ if and only if it is one of the following:*

$$\begin{array}{lll} i(-Z_2+Z_3), & i(-Z_2+Z_{l-1}), & i(-Z_2+Z_l), \\ i(Z_1-Z_2+Z_{l-1}), & i(Z_1-Z_2+Z_l), & i(Z_1-Z_2). \end{array}$$

Here, $\mathfrak{W}^5_{\mathfrak{k}}$ is a Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^l$;

$$\mathfrak{W}^5_{\mathfrak{k}} = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, -i\nu(T) \geq 0, -i\alpha_3(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}^5_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. Then, arguments similar to those on the proof of Lemma 4.2.1 enable us to confirm that it satisfies one of the following seven conditions:

$$\begin{array}{ll}
\text{(A)} & -i\mu_1(T) = 0, -i\mu_2(T) = 0, -i\mu_3(T) = 1; \\
\text{(B)} & -i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 0; \\
\text{(C)} & -i\mu_1(T) = 0, -i\mu_2(T) = 1, -i\mu_3(T) = 1; \\
\text{(D)} & -i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 0; \\
\text{(E)} & -i\mu_1(T) = 1, -i\mu_2(T) = 0, -i\mu_3(T) = 1; \\
\text{(F)} & -i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 0; \\
\text{(G)} & -i\mu_1(T) = 1, -i\mu_2(T) = 1, -i\mu_3(T) = 1.
\end{array}$$

Therefore, since (5.4.28), (5.4.29) and $\{T_a\}_{a=1}^l$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_k\}_{k=3}^l$, the *Spr*-element T is one of the following fifteen elements:

(A.1) $T_3 = i(-Z_2 + Z_3)$,
(A.2) $T_{l-1} = i(-\frac{1}{2}Z_2 + Z_{l-1})$,
(A.3) $T_l = i(-\frac{1}{2}Z_2 + Z_l)$;
(B) $T_2 = -\frac{i}{2}Z_2$;
(C.1) $T_2 + T_3 = i(-\frac{3}{2}Z_2 + Z_3)$,
(C.2) $T_2 + T_{l-1} = i(-Z_2 + Z_{l-1})$,
(C.3) $T_2 + T_l = i(-Z_2 + Z_l)$;
(D) $T_1 = i(Z_1 - \frac{1}{2}Z_2)$;
(E.1) $T_1 + T_3 = i(Z_1 - \frac{3}{2}Z_2 + Z_3)$,
(E.2) $T_1 + T_{l-1} = i(Z_1 - Z_2 + Z_{l-1})$,
(E.3) $T_1 + T_l = i(Z_1 - Z_2 + Z_l)$;
(F) $T_1 + T_2 = i(Z_1 - Z_2)$;
(G.1) $T_1 + T_2 + T_3 = i(Z_1 - 2Z_2 + Z_3)$,
(G.2) $T_1 + T_2 + T_{l-1} = i(Z_1 - \frac{3}{2}Z_2 + Z_{l-1})$,
(G.3) $T_1 + T_2 + T_l = i(Z_1 - \frac{3}{2}Z_2 + Z_l)$.

The Spr-element T has to satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{d}_l, \tilde{\mathfrak{h}})$ (cf. Lemma 4.1.1). So, it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and (5.4.3) that the element T is one of the following six elements:

$$\begin{array}{lll} \text{(A.1) } i(-Z_2+Z_3), & \text{(C.2) } i(-Z_2+Z_{l-1}), & \text{(C.3) } i(-Z_2+Z_l), \\ \text{(E.2) } i(Z_1-Z_2+Z_{l-1}), & \text{(E.3) } i(Z_1-Z_2+Z_l), & \text{(F) } i(Z_1-Z_2). \end{array}$$

Conversely, if an element T' is one of the above six elements, then it belongs to $\mathfrak{W}^5_{\mathfrak{k}}$ and satisfies the condition c) in Lemma 4.1.1. By Lemma 4.1.1, the element T' is an Spr-element of $\mathfrak{g}$. Therefore, we have proved Lemma 5.4.13. □

Lemma 5.4.13, combined with (4.1.2), yields that

$$(5.4.30) \quad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{lll} [i(-Z_2+Z_3)], & [i(-Z_2+Z_{l-1})], & [i(-Z_2+Z_l)], \\ [i(Z_1-Z_2+Z_{l-1})], & [i(Z_1-Z_2+Z_l)], & [i(Z_1-Z_2)] \end{array} \right\},$$

where $\mathfrak{g} = \mathfrak{so}(4, 2l-4)$. From now on, we will demonstrate that the above Spr-element $i(-Z_2+Z_{l-1})$ (resp. $i(Z_1-Z_2+Z_{l-1})$) is equivalent to $i(-Z_2+Z_l)$ (resp. $i(Z_1-Z_2+Z_l)$). Using the involution θ_1 in Paragraph 5.4.1, one can obtain $\theta_1(i(-Z_2+Z_{l-1})) = i(-Z_2+Z_l)$ and $\theta_1(i(Z_1-Z_2+Z_{l-1})) = i(Z_1-Z_2+Z_l)$ because of (5.4.1) and $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_1$. Hence, it suffices to confirm that the involution θ_1 of $\mathfrak{d}_l$ is an automorphism of $\mathfrak{g} = \mathfrak{so}(4, 2l-4)$. The involution θ_1 satisfies $\theta_1(iZ_2) = iZ_2$, so that it follows from (5.4.27) that

$$\theta_1 \circ \theta_5 = \exp \pi \operatorname{ad}_{\mathfrak{d}_l} \theta_1(iZ_2) \circ \theta_1 = \theta_5 \circ \theta_1.$$

Accordingly, Proposition 2.2.3 assures that the involution θ_1 is an automorphism of $\mathfrak{g}$.[14] Here, we recall that $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$ (see Paragraph 5.4.1). For the reasons, the Spr-element $i(-Z_2+Z_{l-1})$ (resp. $i(Z_1-Z_2+Z_{l-1})$) is equivalent to $i(-Z_2+Z_l)$ (resp. $i(Z_1-Z_2+Z_l)$) via θ_1. Hence, by (5.4.30) we deduce that

$$(5.4.31) \qquad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{ll} [i(-Z_2+Z_3)], & [i(-Z_2+Z_l)], \\ [i(Z_1-Z_2+Z_l)], & [i(Z_1-Z_2)] \end{array} \right\}.$$

Let us show that the above Spr-element $i(-Z_2+Z_l)$ is equivalent to $i(Z_1-Z_2+Z_l)$. The involution φ in Lemma 5.4.8 satisfies $\varphi(i(-Z_2+Z_l)) = i(Z_1-Z_2+Z_l)$. Hence, if φ is an automorphism of $\mathfrak{g} = \mathfrak{so}(4, 2l-4)$, then the element $i(-Z_2+Z_l)$ is equivalent to $i(Z_1-Z_2+Z_l)$ via φ. Therefore, we are going to verify that the involution φ is an automorphism of $\mathfrak{g}$. Since $\varphi(iZ_2) = i(-2Z_1+Z_2)$ and (5.4.27), one has

$$\varphi \circ \theta_5 = \exp \pi \operatorname{ad}_{\mathfrak{d}_l} \varphi(iZ_2) \circ \varphi = \exp \pi \operatorname{ad}_{\mathfrak{d}_l} i(-2Z_1+Z_2) \circ \varphi.$$

In Paragraph 5.4.3, we saw that an inner automorphism $\exp \pi \operatorname{ad}_{\mathfrak{d}_l} iZ_1$ $(= \theta_3)$ was involutive. From that, one deduces $\exp \pi \operatorname{ad}_{\mathfrak{d}_l} i(-2Z_1) = \mathrm{id}$. Hence, it follows from $[Z_1, Z_2] = 0$ that $\exp \pi \operatorname{ad}_{\mathfrak{d}_l} i(-2Z_1+Z_2) = \exp \pi \operatorname{ad}_{\mathfrak{d}_l} iZ_2 = \theta_5$. Accordingly, the involution φ in Lemma 5.4.8 is commutative with θ_5; and thus Proposition 2.2.3

[14]This θ_1 is an outer automorphism of $\mathfrak{so}(4, 2l-4)$.

implies that the involution φ is an automorphism of $\mathfrak{g} = \mathfrak{so}(4, 2l-4)$.[15] Therefore, the *Spr*-element $i(-Z_2 + Z_l)$ is equivalent to $i(Z_1 - Z_2 + Z_l)$ via φ, and it follows from (5.4.31) that

$$\begin{aligned} &Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) \\ &\qquad = \{\, [\, i(-Z_2 + Z_3)\,],\ [\, i(-Z_2 + Z_l)\,],\ [\, i(Z_1 - Z_2)\,]\, \}. \end{aligned} \tag{5.4.32}$$

Now, we will prove Proposition 5.4.14.

Proposition 5.4.14. *Under our equivalence relation, Spr-elements of* DI: $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$, $l \geq 5$ *and* $j = 2$, *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\, [\, i(-Z_2 + Z_3)\,],\ [\, i(-Z_2 + Z_l)\,],\ [\, i(Z_1 - Z_2)\,]\, \}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(4, 2l-6) \oplus \mathfrak{t}^1)$, (2) $(\mathfrak{g}, \mathfrak{su}(2, l-2) \oplus \mathfrak{t}^1)$ *and* (3) $(\mathfrak{g}, \mathfrak{so}(2, 2l-4) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_2 + Z_3)$, $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_2 + Z_l)$ *and* $\rho_3 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_1 - Z_2)$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. It is shown that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\, [\, i(-Z_2 + Z_3)\,],\ [\, i(-Z_2 + Z_l)\,],\ [\, i(Z_1 - Z_2)\,]\, \}$$

(see (5.4.32)). About the above *Spr*-elements, it is known that

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}}(i(-Z_2 + Z_3)) &= \mathfrak{so}(4, 2l-6) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(-Z_2 + Z_l)) &= \mathfrak{su}(2, l-2) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(Z_1 - Z_2)) &= \mathfrak{so}(2, 2l-4) \oplus \mathfrak{t}^1 \end{aligned} \tag{5.4.33}$$

(cf. Theorem 6.16 in [Bm]). Therefore since $l \geq 5$, three *Spr*-elements $i(-Z_2 + Z_3)$, $i(-Z_2 + Z_l)$ and $i(Z_1 - Z_2)$ are not mutually equivalent. Besides, by (5.4.33) and Lemma 3.1.1, we confirm that $(\mathfrak{g}, \mathfrak{so}(4, 2l-6) \oplus \mathfrak{t}^1)$, $(\mathfrak{g}, \mathfrak{su}(2, l-2) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}(2, 2l-4) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_2 + Z_3)$, $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_2 + Z_l)$ and $\rho_3 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_1 - Z_2)$, respectively. Thus, we have got the conclusion. □

5.4.6. *Case* DI $\mathfrak{so}(2j, 2l-2j) : 3 \leq j \leq l-3$. Our purpose in this paragraph is to classify *Spr*-elements of $\mathfrak{so}(2j, 2l-2j)$ (cf. Proposition 5.4.18).

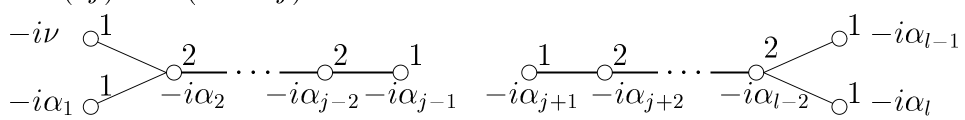
First, let us define an involutive automorphism θ_6 of $\mathfrak{g}_u$ such that $\mathfrak{so}(2j, 2l-2j)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_6. We define an inner automorphism θ_6 of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ by

$$\theta_6 := \exp \pi \operatorname{ad}_{\mathfrak{d}_l} iZ_j. \tag{5.4.34}$$

Then, it follows from $iZ_j \in \mathfrak{g}_u$ and $\theta_6|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ that (c1) $\theta_6(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_6(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_6(\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}$. The result of Murakami [Mu3, pp. 297, type DI] means that θ_6 is an involutive automorphism of $\mathfrak{d}_l$. Besides, his result also assures that $\{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^l$ is the set of simple roots in $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$, its Dynkin diagram is

[15] This φ is an outer automorphism of $\mathfrak{so}(4, 2l-4)$.

$\mathfrak{k} = \mathfrak{so}(2j) \oplus \mathfrak{so}(2l-2j)$:



and $\mathfrak{so}(2j, 2l-2j)$ is the real form of $\mathfrak{d}_l$ given by (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_6(K) = K\}$ and $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_6(P) = -P\}$. Here, $-i\nu$ denotes the lowest root $i(\alpha_1 + 2\sum_{d=2}^{l-2} \alpha_d + \alpha_{l-1} + \alpha_l) \in \triangle(\mathfrak{g}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (see Remark 2.3.1). Now, let us assume $\{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\}$ (resp. $\{-i\alpha_t\}_{t=j+1}^{l}$) to be the set of simple roots of $\mathfrak{k}_1 := \mathfrak{so}(2j)$ (resp. $\mathfrak{k}_2 := \mathfrak{so}(2l-2j)$). Then, the highest root $-i\mu_1 \in \triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2 \in \triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$(5.4.35)\qquad \begin{cases} -i\mu_1 = -i(\alpha_1 + 2\sum_{u=2}^{j-2} \alpha_u + \alpha_{j-1} + \nu), \\ -i\mu_2 = -i(\alpha_{j+1} + 2\sum_{v=j+2}^{l-2} \alpha_v + \alpha_{l-1} + \alpha_l). \end{cases}$$

Next, we will describe the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of the dual basis $\{Z_a\}_{a=1}^{l}$ of $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}$. Define an element $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$ by setting $-i\alpha_s(T_a) = \delta_{s,a}$ $(1 \leq s \leq j-1)$, $-i\nu(T_a) = \delta_{j,a}$ and $-i\alpha_t(T_a) = \delta_{t,a}$ $(j+1 \leq t \leq l)$. Then, the elements T_a $(1 \leq a \leq l)$ are as follows:

$$(5.4.36)\qquad \begin{cases} T_1 = i(Z_1 - \frac{1}{2}Z_j), \\ T_p = i(Z_p - Z_j) \quad \text{for } 2 \leq p \leq j-1, \\ T_j = -\frac{i}{2}Z_j, \\ T_q = i(-Z_j + Z_q) \quad \text{for } j+1 \leq q \leq l-2, \\ T_n = i(-\frac{1}{2}Z_j + Z_n) \quad \text{for } n = l-1,\, l, \end{cases}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$ and $-i\nu = i(\alpha_1 + 2\sum_{d=2}^{l-2} \alpha_d + \alpha_{l-1} + \alpha_l)$. This (5.4.36) is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$. By use of (5.4.36), we are going to prove Lemma 5.4.15.

Lemma 5.4.15. *With the above assumptions; an element $T \in \mathfrak{W}_{\mathfrak{k}}^6$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$ if and only if it is one of the following:*

$$\begin{array}{lll} i(Z_{j-1} - Z_j), & i(-Z_j + Z_{j+1}), & i(Z_1 - Z_j + Z_{l-1}), \\ i(Z_1 - Z_j + Z_l), & i(-Z_j + Z_{l-1}), & i(-Z_j + Z_l). \end{array}$$

Here, $\{Z_a\}_{a=1}^{l}$ is the dual basis of $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{l}$, and $\mathfrak{W}_{\mathfrak{k}}^6$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$;

$$\mathfrak{W}_{\mathfrak{k}}^6 = \left\{ T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\alpha_1(T) \geq 0, \cdots, -i\alpha_{j-1}(T) \geq 0, -i\nu(T) \geq 0, \\ -i\alpha_{j+1}(T) \geq 0, \cdots, -i\alpha_l(T) \geq 0 \end{array} \right\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^6$ is an Spr-element of $\mathfrak{g}$. Since (5.4.35) and $\{T_a\}_{a=1}^{l}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_s\}_{s=1}^{j-1} \cup \{-i\nu\} \cup \{-i\alpha_t\}_{t=j+1}^{l}$, Lemma

4.2.2 implies that the element T is one of the following fifteen elements:

$$\begin{array}{lll} (\mathrm{b}'\text{-}1.1)\ T_1, & (\mathrm{b}'\text{-}1.2)\ T_{j-1}, & (\mathrm{b}'\text{-}1.3)\ T_j, \\ (\mathrm{b}'\text{-}2.1)\ T_{j+1}, & (\mathrm{b}'\text{-}2.2)\ T_{l-1}, & (\mathrm{b}'\text{-}2.3)\ T_l, \\ (\mathrm{b}'\text{-}3.1)\ T_1+T_{j+1}, & (\mathrm{b}'\text{-}3.2)\ T_1+T_{l-1}, & (\mathrm{b}'\text{-}3.3)\ T_1+T_l, \\ (\mathrm{b}'\text{-}3.4)\ T_{j-1}+T_{j+1}, & (\mathrm{b}'\text{-}3.5)\ T_{j-1}+T_{l-1}, & (\mathrm{b}'\text{-}3.6)\ T_{j-1}+T_l, \\ (\mathrm{b}'\text{-}3.7)\ T_j+T_{j+1}, & (\mathrm{b}'\text{-}3.8)\ T_j+T_{l-1}, & (\mathrm{b}'\text{-}3.9)\ T_j+T_l. \end{array}$$

By use of (5.4.36), let us rewrite the above elements as

$$\begin{array}{lll} (\mathrm{b}\text{-}1.1)\ i(Z_1-\frac{1}{2}Z_j), & (\mathrm{b}\text{-}1.2)\ i(Z_{j-1}-Z_j), & (\mathrm{b}\text{-}1.3)\ -\frac{i}{2}Z_j, \\ (\mathrm{b}\text{-}2.1)\ i(-Z_j+Z_{j+1}), & (\mathrm{b}\text{-}2.2)\ i(-\frac{1}{2}Z_j+Z_{l-1}), & (\mathrm{b}\text{-}2.3)\ i(-\frac{1}{2}Z_j+Z_l), \\ (\mathrm{b}\text{-}3.1)\ i(Z_1-\frac{3}{2}Z_j+Z_{j+1}), & (\mathrm{b}\text{-}3.2)\ i(Z_1-Z_j+Z_{l-1}), & \\ (\mathrm{b}\text{-}3.3)\ i(Z_1-Z_j+Z_l), & & \\ (\mathrm{b}\text{-}3.4)\ i(Z_{j-1}-2Z_j+Z_{j+1}), & (\mathrm{b}\text{-}3.5)\ i(Z_{j-1}-\frac{3}{2}Z_j+Z_{l-1}), & \\ (\mathrm{b}\text{-}3.6)\ i(Z_{j-1}-\frac{3}{2}Z_j+Z_l), & & \\ (\mathrm{b}\text{-}3.7)\ i(-\frac{3}{2}Z_j+Z_{j+1}), & (\mathrm{b}\text{-}3.8)\ i(-Z_j+Z_{l-1}), & (\mathrm{b}\text{-}3.9)\ i(-Z_j+Z_l). \end{array}$$

Due to Lemma 4.1.1, the Spr-element T must satisfy $\beta(T) = \pm i$ for all roots $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{d}_l, \tilde{\mathfrak{h}})$. Therefore, it follows from (5.4.3) and $\alpha_a(Z_b) = \delta_{a,b}$ that the element T is one of the following:

$$\begin{array}{ll} (\mathrm{b}\text{-}1.2)\ i(Z_{j-1}-Z_j), & (\mathrm{b}\text{-}2.1)\ i(-Z_j+Z_{j+1}), \\ (\mathrm{b}\text{-}3.2)\ i(Z_1-Z_j+Z_{l-1}), & (\mathrm{b}\text{-}3.3)\ i(Z_1-Z_j+Z_l), \\ (\mathrm{b}\text{-}3.8)\ i(-Z_j+Z_{l-1}), & (\mathrm{b}\text{-}3.9)\ i(-Z_j+Z_l). \end{array}$$

Conversely, suppose that an element T' is one of the above elements. Then, by virtue of (5.4.3) and $\alpha_a(Z_b) = \delta_{a,b}$, one has $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{d}_l, \tilde{\mathfrak{h}})$. So, the element T' is an Spr-element of $\mathfrak{g}$ (cf. Lemma 4.1.1). For the reasons, we have completed the proof of Lemma 5.4.15. □

Lemma 5.4.15 and (4.1.2) allow us to lead the following:

$$(5.4.37)\quad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{lll} [i(Z_{j-1}-Z_j)], & [i(-Z_j+Z_{j+1})], & [i(Z_1-Z_j+Z_{l-1})], \\ [i(Z_1-Z_j+Z_l)], & [i(-Z_j+Z_{l-1})], & [i(-Z_j+Z_l)] \end{array} \right\},$$

where $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$. From now on, let us aim to show that the above Spr-element $i(Z_1-Z_j+Z_{l-1})$ (resp. $i(-Z_j+Z_{l-1})$) is equivalent to $i(Z_1-Z_j+Z_l)$ (resp. $i(-Z_j+Z_l)$). Recall that the involution θ_1 in Paragraph 5.4.1 satisfies $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$ and $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_1'$. Since $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_1'$ and (5.4.1), one obtains $\theta_1(i(Z_1-Z_j+Z_{l-1})) = i(Z_1-Z_j+Z_l)$ and $\theta_1(i(-Z_j+Z_{l-1})) = i(-Z_j+Z_l)$. Hence, our aim is accomplished, if the involution θ_1 is an automorphism of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$. By using $\theta_1|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_1'$ and (5.4.1) again, we have $\theta_1(iZ_j) = iZ_j$ (because $3 \le j \le l-3$). Thus from (5.4.34), it is obvious that $\theta_1 \circ \theta_6 = \theta_6 \circ \theta_1$. This, together with Proposition 2.2.3, shows that the involution θ_1 is an automorphism of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$.[16] For the

[16]This θ_1 becomes an outer automorphism of $\mathfrak{so}(2j, 2l-2j)$.

reasons, it follows from (5.4.37) that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{ll} [\,i(Z_{j-1} - Z_j)\,], & [\,i(-Z_j + Z_{j+1})\,], \\ [\,i(Z_1 - Z_j + Z_l)\,], & [\,i(-Z_j + Z_l)\,] \end{array} \right\}. \tag{5.4.38}$$

Let us show that the above *Spr*-element $i(Z_1 - Z_j + Z_l)$ is equivalent to $i(-Z_j + Z_l)$. The involution φ in Lemma 5.4.8 satisfies $\varphi(i(Z_1 - Z_j + Z_l)) = i(-Z_j + Z_l)$ because $3 \le j \le l-3$. Consequently, if the involution φ is an automorphism of $\mathfrak{g}$, then the *Spr*-element $i(Z_1 - Z_j + Z_l)$ is equivalent to $i(-Z_j + Z_l)$ via φ. Hence, we devote ourselves to confirming that the involution φ is an automorphism of $\mathfrak{g} = \mathfrak{so}(2j, 2l - 2j)$. Since φ satisfies $\varphi(iZ_j) = i(-2Z_1 + Z_j)$, and since (5.4.34) $\theta_6 = \exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} iZ_j$, we comprehend

$$\varphi \circ \theta_6 = \varphi \circ \exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} iZ_j = \exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} i(-2Z_1 + Z_j) \circ \varphi.$$

Since $\exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} iZ_1$ $(= \theta_3)$ is involutive (ref. Paragraph 5.4.3), one can deduce that $\exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} i(-2Z_1) = \mathrm{id}$ on $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$. Accordingly, it follows from $[Z_1, Z_j] = 0$ that $\exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} i(-2Z_1 + Z_j) = \exp \pi \,\mathrm{ad}_{\mathfrak{d}_l} iZ_j = \theta_6$, so that $\varphi \circ \theta_6 = \theta_6 \circ \varphi$. This asserts that the involution φ in Lemma 5.4.8 is an automorphism of $\mathfrak{g} = \mathfrak{so}(2j, 2l - 2j)$ (see Proposition 2.2.3).[17] By (5.4.38) and the above arguments, we have

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \left\{ \begin{array}{l} [\,i(Z_{j-1} - Z_j)\,],\ [\,i(-Z_j + Z_{j+1})\,], \\ [\,i(-Z_j + Z_l)\,] \end{array} \right\}, \tag{5.4.39}$$

where $\mathfrak{g} = \mathfrak{so}(2j, 2l - 2j)$. Henceforth, let us investigate whether the *Spr*-elements in (5.4.39) are equivalent to each other or not. In the first place, we will show Lemma 5.4.16.

Lemma 5.4.16. *In the above setting; the Spr-element* $i(-Z_j + Z_l)$ *of* $\mathfrak{g}$ *is not equivalent to* $i(Z_{j-1} - Z_j)$ *and* $i(-Z_j + Z_{j+1})$.

Proof. Suppose that the *Spr*-element $i(-Z_j + Z_l)$ of $\mathfrak{g}$ is equivalent to either $i(Z_{j-1} - Z_j)$ or $i(-Z_j + Z_{j+1})$. It is known that

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_l)) &= \mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j)) &= \mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1})) &= \mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1 \end{aligned} \tag{5.4.40}$$

(cf. Theorem 6.16 in [Bm]). Hence, the supposition makes us lead $j = 2$ and $l = 4$, which contradicts the hypothesis $3 \le j \le l-3$. Therefore, the *Spr*-element $i(-Z_j + Z_l)$ of $\mathfrak{g}$ is not equivalent to $i(Z_{j-1} - Z_j)$ and $i(-Z_j + Z_{j+1})$. Hence, we have shown Lemma 5.4.16. □

In the second place, let us verify Lemma 5.4.17.

Lemma 5.4.17. *In the setting on Paragraph* 5.4.6; *the Spr-element* $i(Z_{j-1} - Z_j)$ *of* $\mathfrak{g}$ *is equivalent to* $i(-Z_j + Z_{j+1})$ *if and only if* $2j = l$. *Here,* $\mathfrak{g} = \mathfrak{so}(2j, 2l - 2j)$.

[17] This φ becomes an outer automorphism of $\mathfrak{so}(2j, 2l - 2j)$.

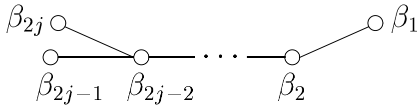
Proof. Naturally, it follows from (5.4.40) that, if the *Spr*-element $T_{j-1} = i(Z_{j-1} - Z_j)$ is equivalent to $T_{j+1} = i(-Z_j + Z_{j+1})$, then one has $2j = l$. So, it is sufficient to confirm that, in case of $2j = l$, there exists an automorphism ψ of $\mathfrak{g}$ satisfying $\psi(T_{j-1}) = T_{j+1}$. Suppose that $2j = l$ henceforth. Let us construct an automorphism ψ of $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j) = \mathfrak{so}(2j, 2j)$ such that $\psi(T_{j-1}) = T_{j+1}$. Define an involutive linear isomorphism ψ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{2j}$ by

$$(5.4.41) \qquad \begin{cases} \psi'(iZ_1) := i(Z_{2j-1} - Z_{2j}), \\ \psi'(iZ_d) := i(Z_{2j-d} - 2Z_{2j}) \quad \text{for } 2 \leq d \leq 2j-2, \\ \psi'(iZ_{2j-1}) := i(Z_1 - Z_{2j}), \\ \psi'(iZ_{2j}) := -iZ_{2j}. \end{cases}$$

Then, the linear involution ψ' satisfies

$$(5.4.42) \qquad \begin{cases} {}^t\psi'_{\mathbb{C}}(\alpha_k) = \alpha_{2j-k} \quad \text{for } 1 \leq k \leq 2j-1, \\ {}^t\psi'_{\mathbb{C}}(\alpha_{2j}) = -(\alpha_1 + 2\sum_{d=2}^{2j-2}\alpha_d + \alpha_{2j-1} + \alpha_{2j}) \end{cases}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$. Here, $\psi'_{\mathbb{C}}$ denotes the complex linear extension of ψ' to $\tilde{\mathfrak{h}}$. It comes from (5.4.42) that the Dynkin diagram of $\{{}^t\psi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2j}$ is as follows:



where $\beta_a := {}^t\psi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq 2j$. Hence, the linear involution ψ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}}$ satisfies

$$(5.4.43) \qquad {}^t\psi'_{\mathbb{C}}\big(\triangle(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}})$$

because the Dynkin diagram of $\{{}^t\psi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{2j}$ is the same as that of $\Pi_{\triangle(\mathfrak{d}_{2j},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{2j}$ (cf. Murakami [Mu3, Lemma 1, pp. 295]). For the reasons, there exists an involutive automorphism $\bar{\psi}$ of $\mathfrak{d}_{2j} = \mathfrak{so}(4j, \mathbb{C})$ such that (i) $\bar{\psi}(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (iii) $\bar{\psi}(X_{\pm\alpha_a}) = X_{\pm{}^t\bar{\psi}(\alpha_a)}$ (see Proposition 2.3.2). From now on, we are going to prove that the involution $\bar{\psi}$ satisfies the two conditions (a) and (b) in Proposition 2.3.4. The definition (5.4.34) of θ_6 means that $\theta_6 = \mathrm{id}$ on $\tilde{\mathfrak{h}}$. Thus, $\theta_6 \circ \bar{\psi} = \bar{\psi} \circ \theta_6$ on $i\tilde{\mathfrak{h}}_{\mathbb{R}}$—that is, the involution $\bar{\psi}$ of $\mathfrak{d}_{2j}$ satisfies the condition (a) in Proposition 2.3.4. We want to show that the involution $\bar{\psi}$ also satisfies the condition (b) in Proposition 2.3.4. It follows from (5.4.34) that

$$\triangle_1(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}} : \theta_6) = \left\{ \sum_{a=1}^{2j} n_a\alpha_a \in \triangle(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}}) \;\middle|\; n_j = 0 \text{ or } \pm 2 \right\}$$

(see (2.3.4) for $\triangle_1(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}} : \theta_6)$). The coefficient of any root $\alpha \in \triangle^+(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}})$ with respect to α_{2j} is either 1 or zero. Therefore, $(n_j, n_{2j}) = (0,0)$, $(0,1)$, $(2,0)$ or $(2,1)$ for each root $\beta = \sum_{a=1}^{2j} n_a\alpha_a \in \triangle_1^+(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}} : \theta_6)$. On the other hand, it follows from $\bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.4.42) that ${}^t\bar{\psi}(\alpha_k) = \alpha_{2j-k}$ $(1 \leq k \leq 2j-1)$ and ${}^t\bar{\psi}(\alpha_{2j}) = -(\alpha_1 + 2\sum_{d=2}^{2j-2}\alpha_d + \alpha_{2j-1} + \alpha_{2j})$. Consequently, the coefficient of ${}^t\bar{\psi}(\beta)$

with respect to α_j is ± 2 or zero, for every root $\beta \in \triangle_1^+(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}} : \theta_6)$. Hence since $\bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.4.43), we deduce that

$$ {}^t\bar{\psi}\big(\triangle_1(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}} : \theta_6)\big) = \triangle_1(\mathfrak{d}_{2j}, \tilde{\mathfrak{h}} : \theta_6). $$

So, $\bar{\psi}$ also satisfies the condition (b) in Proposition 2.3.4. Consequently, the involution $\bar{\psi}$ satisfies the two conditions (a) and (b). By Proposition 2.3.4, there exists an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ such that $\bar{\psi} \circ \exp \mathrm{ad}_{\mathfrak{d}_{2j}} iH \in \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{g}_u)$. Define ψ by $\psi := \bar{\psi} \circ \exp \mathrm{ad}_{\mathfrak{d}_{2j}} iH$.[18] Since $\exp \mathrm{ad}_{\mathfrak{d}_{2j}} iH = \mathrm{id}$ on $\tilde{\mathfrak{h}}$, one has $\psi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$. So, it follows from (5.4.41) that

$$ \psi(T_{j-1}) = \psi(i(Z_{j-1} - Z_j)) = i(-Z_j + Z_{j+1}) = T_{j+1}. $$

For the reasons, we have proved Lemma 5.4.17. □

Now, we are going to show Proposition 5.4.18.

Proposition 5.4.18. *Under our equivalence relation, Spr-elements of* DI: $\mathfrak{g} = \mathfrak{so}(2j, 2l-2j)$, $3 \leq j \leq l-3$, *are classified as follows*:

$$ Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \begin{cases} \{\ [i(Z_{j-1} - Z_j)],\ [i(-Z_j + Z_{j+1})],\ [i(-Z_j + Z_l)]\ \} & \text{if } 2j \neq l, \\ \{\ [i(Z_{j-1} - Z_j)],\ [i(-Z_j + Z_l)]\ \} & \text{if } 2j = l. \end{cases} $$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1)$, (2) $(\mathfrak{g}, \mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1)$ *and* (3) $(\mathfrak{g}, \mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_{j-1} - Z_j)$, $\rho_2 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(-Z_j + Z_{j+1})$ *and* $\rho_3 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(-Z_j + Z_l)$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. Two Lemmas 5.4.16 and 5.4.17, combined with (5.4.39), enable us to deduce the first half of statements on this proposition. The latter half of the statements follows from Lemma 3.1.1 and (5.4.40). So, Proposition 5.4.18 has been confirmed. □

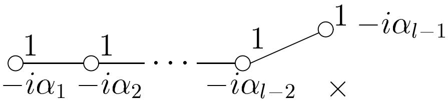
5.4.7. *Case* DIII $\mathfrak{so}^*(2l)$. This paragraph is devoted to classifying *Spr*-elements of $\mathfrak{so}^*(2l)$ (see Proposition 5.4.22).

Let us define an involutive automorphism θ_7 of $\mathfrak{g}_u$ such that $\mathfrak{so}^*(2l)$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_7. Let θ_7 be an inner automorphism of $\mathfrak{d}_l = \mathfrak{so}(2l, \mathbb{C})$ defined by

$$ \theta_7 := \exp \pi\, \mathrm{ad}_{\mathfrak{d}_l} iZ_l. \tag{5.4.44} $$

Then since $iZ_l \in \mathfrak{g}_u$ and $\theta_7|_{\tilde{\mathfrak{h}}} = \mathrm{id}$, one deduces that the automorphism θ_7 satisfies the conditions in Paragraph 2.3.2; (c1) $\theta_7(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_7(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_7(\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})}$. Due to the result of Murakami [Mu3, pp. 297, type DIII], we know that this automorphism θ_7 is involutive. Moreover, his result states that $\{-i\alpha_d\}_{d=1}^{l-1}$ is the simple root system of $\mathfrak{k}$, the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d\}_{d=1}^{l-1}$ is

[18] This ψ is an outer automorphism of $\mathfrak{so}(2j, 2j)$.

$\mathfrak{k} = \mathfrak{su}(l) \oplus \mathfrak{t}^1$: 

(cf. Remark 2.3.3) and $\mathfrak{so}^*(2l)$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{d}_l$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_7(K) = K\}$ and $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_7(P) = -P\}$. Here, we remark that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

$$-i\mu = -i(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}). \tag{5.4.45}$$

Denote by $\mathfrak{W}^7_{\mathfrak{k}}$ the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d\}_{d=1}^{l-1}$;

$$\mathfrak{W}^7_{\mathfrak{k}} = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \,|\, -i\alpha_1(T) \geq 0, -i\alpha_2(T) \geq 0, \cdots, -i\alpha_{l-1}(T) \geq 0\}.$$

With the notation, we are going to prove Lemma 5.4.19.

Lemma 5.4.19. *In the above setting; an element $T \in \mathfrak{W}^7_{\mathfrak{k}}$ is an Spr-element of $\mathfrak{g} = \mathfrak{so}^*(2l)$ if and only if it is one of the following:*

$$iZ_1, \quad i(Z_d - Z_l) \text{ for } 1 \leq d \leq l-1, \quad iZ_{l-1}, \quad \pm iZ_l.$$

Proof. Suppose that an element $T \in \mathfrak{W}^7_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. For every root $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ one obtains $-i\gamma(iZ_l) \equiv 0$, by virtue of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_d\}_{d=1}^{l-1}$ and $\alpha_a(Z_b) = \delta_{a,b}$. Thus, the element iZ_l is a central element of $\mathfrak{k}$. So, Lemma 4.2.3, combined with (5.4.45), $\alpha_a(Z_b) = \delta_{a,b}$ and $T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{l}$, implies that one of the following cases only occurs:

$$(\text{c}'\text{-1})\ T = i(Z_d + \lambda_l \cdot Z_l) \text{ for } 1 \leq d \leq l-1, \quad (\text{c}'\text{-2})\ T = i\,\lambda_l \cdot Z_l,$$

where λ_l is a real number ($\lambda_l \neq 0$ in Case (c$'$-2)). Since T is an *Spr*-element, it must satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{d}_l, \tilde{\mathfrak{h}})$ (ref. Lemma 4.1.1). Therefore by $\alpha_a(Z_b) = \delta_{a,b}$ and (5.4.3), the value of λ_l is determined as follows:

$$\text{Case (c}'\text{-1): } \lambda_l = \begin{cases} -1 \text{ or } 0 & \text{if } d = 1 \text{ or } d = l-1, \\ -1 & \text{if } 2 \leq d \leq l-2. \end{cases} \quad \text{Case (c}'\text{-2): } \lambda_l = \pm 1.$$

Accordingly, if an element $T \in \mathfrak{W}^7_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$, then one of the following cases only occurs:

$$\begin{array}{ll} (\text{c-1.1})\ T = iZ_1, & (\text{c-1.2})\ T = i(Z_d - Z_l) \text{ for } 1 \leq d \leq l-1, \\ (\text{c-1.3})\ T = iZ_{l-1}, & (\text{c-2})\ T = \pm iZ_l. \end{array}$$

Conversely, if an element T' is one of the above elements, then it satisfies $\beta(T') = \pm i$ for all roots $\beta \in \triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{d}_l, \tilde{\mathfrak{h}})$; and therefore, it follows from Lemma 4.1.1 that the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{so}^*(2l)$. Thus, we have completed the proof of Lemma 5.4.19. □

By (4.1.2) and Lemma 5.4.19, we confirm that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \left\{ \begin{array}{l} [iZ_1],\ [i(Z_d - Z_l)], \\ [iZ_{l-1}],\ [iZ_l] \end{array} \,\middle|\, 1 \leq d \leq l-1 \right\}, \tag{5.4.46}$$

where $\mathfrak{g} = \mathfrak{so}^*(2l)$. The following lemma implies that the above *Spr*-element iZ_1, $i(Z_k - Z_l)$ and iZ_{l-1} are equivalent to $i(Z_{l-1} - Z_l)$, $i(Z_{l-k} - Z_l)$ and $i(Z_1 - Z_l)$, respectively $(2 \leq k \leq l-2)$.

Lemma 5.4.20. *In the setting on Paragraph* 5.4.7; *there exists an automorphism* ϕ *of* $\mathfrak{g} = \mathfrak{so}^*(2l)$ *such that*

$$\begin{cases} \phi(iZ_1) = i(Z_{l-1} - Z_l), \\ \phi(iZ_k) = i(Z_{l-k} - 2Z_l) \quad \textit{for } 2 \leq k \leq l-2, \\ \phi(iZ_{l-1}) = i(Z_1 - Z_l), \\ \phi(iZ_l) = -iZ_l. \end{cases}$$

Proof. The construction of the automorphism ψ in the proof of Lemma 5.4.17 enables us to obtain an automorphism ϕ of $\mathfrak{d}_l$ which satisfies $\phi(\mathfrak{g}_u) \subset \mathfrak{g}_u$ and

$$\begin{cases} \phi(iZ_1) = i(Z_{l-1} - Z_l), \\ \phi(iZ_k) = i(Z_{l-k} - 2Z_l) \quad \text{for } 2 \leq k \leq l-2, \\ \phi(iZ_{l-1}) = i(Z_1 - Z_l), \\ \phi(iZ_l) = -iZ_l. \end{cases} \tag{5.4.47}$$

Hence, the rest of proof is to verify that ϕ is an automorphism of $\mathfrak{g} = \mathfrak{so}^*(2l)$. From (5.4.44), (5.4.47) and θ_7 being involutive, it follows that

$$\phi \circ \theta_7 = \exp \pi \operatorname{ad}_{\mathfrak{d}_l} \phi(iZ_l) \circ \phi = \theta_7 \circ \phi.$$

Thus, Proposition 2.2.3 assures that ϕ is an automorphism of $\mathfrak{g}$.[19] This completes the proof of Lemma 5.4.20. □

Lemma 5.4.20 and (5.4.46) mean that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \left\{ [iZ_1],\ [i(Z_c - Z_l)],\ [iZ_l] \;\middle|\; 1 \leq c \leq [l/2] \right\}, \tag{5.4.48}$$

where $\mathfrak{g} = \mathfrak{so}^*(2l)$. Now, let us investigate whether the above *Spr*-elements are equivalent to each other. Theorem 6.16 in [Bm] implies that

$$\begin{aligned} \mathfrak{c}_{\mathfrak{g}}(iZ_1) &= \mathfrak{so}^*(2l-2) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(Z_c - Z_l)) &= \mathfrak{su}(c, l-c) \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(iZ_l) &= \mathfrak{su}(l) \oplus \mathfrak{t}^1, \end{aligned} \tag{5.4.49}$$

where $1 \leq c \leq [l/2]$. This shows that

$$iZ_l \text{ is not equivalent to } iZ_1 \text{ and } i(Z_c - Z_l); \tag{5.4.50}$$

and that for any $1 \leq c, c' \leq [l/2]$

$$i(Z_c - Z_l) \text{ is equivalent to } i(Z_{c'} - Z_l) \text{ if and only if } c = c'. \tag{5.4.51}$$

In addition, it follows from (5.4.49) that for $1 \leq c \leq [l/2]$

$$\begin{aligned} &i(Z_c - Z_l) \text{ is not equivalent to } iZ_1, \\ &\qquad\qquad \text{except for the case of } l = 4 \text{ and } c = 1. \end{aligned} \tag{5.4.52}$$

[19] This ϕ is an outer automorphism of $\mathfrak{so}^*(2l)$.

We will demonstrate that the *Spr*-element $i(Z_1 - Z_l)$ is equivalent to iZ_1 in case of $l = 4$, by proving the following:

Lemma 5.4.21. *In the above setting* ($l = 4$); *there exists an automorphism* φ *of* $\mathfrak{g} = \mathfrak{so}^*(8)$ *such that* $\varphi(iZ_1) = iZ_3$, $\varphi(iZ_2) = iZ_2$, $\varphi(iZ_3) = iZ_1$ *and* $\varphi(iZ_4) = iZ_4$.

Proof. Let us construct an automorphism which satisfies condition in this lemma. Define an involutive linear isomorphism φ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^4$ by

$$\varphi'(iZ_1) := iZ_3, \quad \varphi'(iZ_2) := iZ_2, \quad \varphi'(iZ_3) := iZ_1, \quad \varphi'(iZ_4) := iZ_4. \tag{5.4.53}$$

Then, by $\alpha_a(Z_b) = \delta_{a,b}$ we have

$$^t\varphi'_{\mathbb{C}}(\alpha_1) = \alpha_3, \quad ^t\varphi'_{\mathbb{C}}(\alpha_2) = \alpha_2, \quad ^t\varphi'_{\mathbb{C}}(\alpha_3) = \alpha_1, \quad ^t\varphi'_{\mathbb{C}}(\alpha_4) = \alpha_4, \tag{5.4.54}$$

where $\varphi'_{\mathbb{C}}$ denotes the complex linear extension of φ' to $\tilde{\mathfrak{h}}$. Therefore from (5.4.3), it is natural that

$$^t\varphi'_{\mathbb{C}}\big(\triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{d}_4, \tilde{\mathfrak{h}}).$$

Thus, Proposition 2.3.2 enables us to get an involutive automorphism φ of $\mathfrak{d}_4 = \mathfrak{so}(8, \mathbb{C})$ such that (i) $\varphi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (iii) $\varphi(X_{\pm\alpha_a}) = X_{\pm{}^t\varphi(\alpha_a)}$. By virtue of $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.4.53), the rest of proof is to confirm that the involution φ of $\mathfrak{d}_4$ is an automorphism of $\mathfrak{g}$. Since $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.4.53), one obtains $\varphi(iZ_4) = iZ_4$. So, the involution φ of $\mathfrak{d}_4$ is commutative with θ_7, by the definition (5.4.44) of θ_7. Hence, Proposition 2.2.3 implies that the involution φ is an automorphism of $\mathfrak{g} = \mathfrak{so}^*(8)$.[20] For the reasons, we have proved Lemma 5.4.21. □

By using two automorphisms φ in Lemma 5.4.21 and ϕ in Lemma 5.4.20, we obtain $\phi(\varphi(iZ_1)) = \phi(iZ_3) = i(Z_1 - Z_4)$ in case of $l = 4$.[21] Thus, it is shown that

$$i(Z_c - Z_l) \text{ is equivalent to } iZ_1, \text{ in case of } l = 4 \text{ and } c = 1. \tag{5.4.55}$$

Now, let us verify Proposition 5.4.22.

Proposition 5.4.22. *Under our equivalence relation, Spr-elements of* DIII: $\mathfrak{g} = \mathfrak{so}^*(2l)$ *are classified as follows*:

$$\begin{aligned} &Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) \\ &\qquad = \begin{cases} \{\, [iZ_1],\ [i(Z_c - Z_l)],\ [iZ_l] \mid 1 \le c \le [l/2] \,\} & \textit{if } l \neq 4, \\ \{\, [i(Z_c - Z_l)],\ [iZ_l] \mid 1 \le c \le [l/2] \,\} & \textit{if } l = 4. \end{cases} \end{aligned}$$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}^*(2l-2) \oplus \mathfrak{t}^1)$, (2) $(\mathfrak{g}, \mathfrak{su}(c, l-c) \oplus \mathfrak{t}^1)$ *and* (3) $(\mathfrak{g}, \mathfrak{su}(l) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_1$, $\rho_2 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_c - Z_l)$ *and* $\rho_3 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, iZ_l$, *respectively. Here,* $\{Z_a\}_{a=1}^l$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{d}_l, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^l$.

Proof. Due to (5.4.48), (5.4.50)–(5.4.52) and (5.4.55), we conclude the first half of statements on this proposition. Lemma 3.1.1 and (5.4.49) enable us to deduce the latter half of the statements. □

[20] This φ is an outer automorphism of $\mathfrak{so}^*(8)$.

[21] This $\phi \circ \varphi$ is an inner automorphism of $\mathfrak{so}^*(8)$.

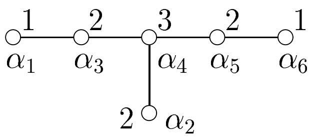
The consequences in Subsection 5.4 are as follows:

Table IV.

		DI
8-1	$\mathfrak{g}$	$\mathfrak{so}(2j+1, 2l-2j-1)$: $l \geq 4$, $j = 0$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_{j+1})\,]$ where $Z_0 = 0$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1$
8-2	$\mathfrak{g}$	$\mathfrak{so}(2j+1, 2l-2j-1)$: $l \geq 4$, $1 \leq j \leq l-3$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_1\,]$, $[\,i(-Z_j + Z_{j+1})\,]$: if $2j+1 \neq l$
		$[\,iZ_1\,]$: if $2j+1 = l$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_1)$	$\mathfrak{so}(2j-1, 2l-2j-1) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1$
9-1	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j)$: $l \geq 4$, $j = 1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1}-Z_j)\,]$, $[\,i(-Z_j+Z_{j+1})\,]$, $[\,i(-Z_j+Z_l)\,]$ where $Z_0 = 0$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j))$	$\mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_l))$	$\mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$
9-2	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j)$: $l = 4$, $j = 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_{j+1})\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
9-3	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j)$: $l \geq 5$, $j = 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1}-Z_j)\,]$, $[\,i(-Z_j+Z_{j+1})\,]$, $[\,i(-Z_j+Z_l)\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j))$	$\mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_l))$	$\mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$
9-4	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j)$: $l \geq 6$, $3 \leq j \leq l-3$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1} - Z_j)\,]$, $[\,i(-Z_j + Z_{j+1})\,]$, $[\,i(-Z_j + Z_l)\,]$: if $2j \neq l$
		$[\,i(Z_{j-1} - Z_j)\,]$, $[\,i(-Z_j + Z_l)\,]$: if $2j = l$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_{j-1} - Z_j))$	$\mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j + Z_{j+1}))$	$\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$

	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_j+Z_l))$	$\mathfrak{su}(j,l-j)\oplus\mathfrak{t}^1$
		DIII
10	$\mathfrak{g}$	$\mathfrak{so}^*(2l)$: $l\geq 4$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	$[iZ_1]$, $[i(Z_c-Z_l)]$, $[iZ_l]$, $1\leq c\leq [l/2]$: if $l\neq 4$
		$[i(Z_c-Z_l)]$, $[iZ_l]$, $1\leq c\leq [l/2]$: if $l=4$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_1)$	$\mathfrak{so}^*(2l-2)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_c-Z_l))$	$\mathfrak{su}(c,l-c)\oplus\mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_l)$	$\mathfrak{su}(l)\oplus\mathfrak{t}^1$

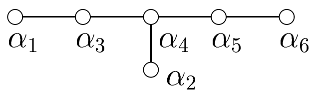
5.5. **Type** E_6**.** This subsection consists of three paragraphs. Each paragraph is devoted to classifying *Spr*-elements of each real form of the exceptional complex simple Lie algebra $\mathfrak{e}_6^{\mathbb{C}}$. First, let us introduce our setting. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{e}_6^{\mathbb{C}}$, let $\{\alpha_a\}_{a=1}^6$ be the set of simple roots in $\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})$ whose Dynkin diagram is

$\mathfrak{e}_6^{\mathbb{C}}$:


(cf. Bourbaki [Br, Plate V, pp. 275–276]) and let $\mathfrak{g}_u$ be the compact real form of $\mathfrak{e}_6^{\mathbb{C}}$ given by $\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})$ and (2.3.1). Then, we denote by $\{Z_a\}_{a=1}^6$ ($Z_a\in\tilde{\mathfrak{h}}$) the dual basis of $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}=\{\alpha_a\}_{a=1}^6$. In these setting, we are going to classify *Spr*-elements of each real form of $\mathfrak{e}_6^{\mathbb{C}}$.

Notation 5.5.1. In Subsection 5.5, we utilize the following notation:

- $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}=\{\alpha_a\}_{a=1}^6$.



- $\mathfrak{g}_u$: the compact real form of $\mathfrak{e}_6^{\mathbb{C}}$ given by $\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})$ and (2.3.1).
- $\{Z_a\}_{a=1}^6$: the dual basis of $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}=\{\alpha_a\}_{a=1}^6$.

5.5.1. *Case* EI $\mathfrak{e}_{6(6)}$. Our aim in this paragraph is to classify *Spr*-elements of $\mathfrak{e}_{6(6)}$.

In the first place, we will obtain an involutive automorphism θ_1 of $\mathfrak{e}_6^{\mathbb{C}}$ such that (I) it satisfies (c1) $\theta_1(\mathfrak{g}_u)\subset\mathfrak{g}_u$, (c2) $\theta_1(\tilde{\mathfrak{h}})\subset\tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})})=\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}$ and (II) $\mathfrak{e}_{6(6)}$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_1. In order to do so, let us construct an involutive automorphism θ_0 of $\mathfrak{e}_6^{\mathbb{C}}$ such that (c1) $\theta_0(\mathfrak{g}_u)\subset\mathfrak{g}_u$, (c2) $\theta_0(\tilde{\mathfrak{h}})\subset\tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_0(\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})})=\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}$. By use of θ_0, we will define an involutive automorphism θ_1 afterward. Let θ_0' be an involutive linear isomorphism of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{6}$ defined by

$$\text{(5.5.1)} \qquad \begin{cases} \theta'_0(iZ_1) := iZ_6, & \theta'_0(iZ_2) := iZ_2, \\ \theta'_0(iZ_3) := iZ_5, & \theta'_0(iZ_4) := iZ_4, \\ \theta'_0(iZ_5) := iZ_3, & \theta'_0(iZ_6) := iZ_1 \end{cases}$$

(see (2.3.2) for $\tilde{\mathfrak{h}}_{\mathbb{R}}$). Then, the complex linear extension $\theta'_{0\mathbb{C}}$ of θ'_0 to $\tilde{\mathfrak{h}}$ satisfies

$$\text{(5.5.2)} \qquad \begin{cases} {}^t\theta'_{0\mathbb{C}}(\alpha_1) = \alpha_6, & {}^t\theta'_{0\mathbb{C}}(\alpha_2) = \alpha_2, \\ {}^t\theta'_{0\mathbb{C}}(\alpha_3) = \alpha_5, & {}^t\theta'_{0\mathbb{C}}(\alpha_4) = \alpha_4, \\ {}^t\theta'_{0\mathbb{C}}(\alpha_5) = \alpha_3, & {}^t\theta'_{0\mathbb{C}}(\alpha_6) = \alpha_1, \end{cases}$$

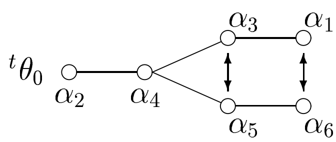
since $\alpha_a(Z_b) = \delta_{a,b}$. This shows that

$${}^t\theta'_{0\mathbb{C}}\big(\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$$

because $\triangle^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$ is as follows:

$$\text{(5.5.3)} \quad \triangle^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) = \left\{ \begin{array}{l} \pm\epsilon_i + \epsilon_j \ (1 \leq i < j \leq 5), \\ \dfrac{1}{2}\big(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}\epsilon_i\big) \text{ with } \sum_{i=1}^{5}\nu(i) \text{ even} \end{array} \right\},$$

where $\alpha_1 = (1/2)\cdot(\epsilon_1 + \epsilon_8 - \sum_{p=2}^{7}\epsilon_p)$, $\alpha_2 = \epsilon_1 + \epsilon_2$ and $\alpha_q = \epsilon_{q-1} - \epsilon_{q-2}$ $(3 \leq q \leq 6)$ (cf. Bourbaki [Br, Plate V, pp. 275]). Accordingly, Proposition 2.3.2 means that there exists an involutive automorphism θ_0 of $\mathfrak{e}_6^{\mathbb{C}}$ satisfying (i) $\theta_0(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\theta_0|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_0$ and (iii) $\theta_0(X_{\pm\alpha_a}) = X_{\pm{}^t\theta_0(\alpha_a)}$. It follows from $\theta_0|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta'_0$ and (5.5.2) that the involution satisfies the condition (c3) ${}^t\theta_0(\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})}$. So, we have constructed an involution θ_0 of $\mathfrak{e}_6^{\mathbb{C}}$ which satisfies the three conditions (c1), (c2) and (c3). Notice that the involution θ_0 is the same as θ_ρ in Murakami [Mu3, pp. 305, type EIV].[22]

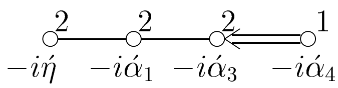


Now, define an automorphism θ_1 of $\mathfrak{e}_6^{\mathbb{C}}$ as follows:

$$\text{(5.5.4)} \qquad \theta_1 := \theta_0 \circ \exp \pi \operatorname{ad}_{\mathfrak{e}_6^{\mathbb{C}}} iZ_2.$$

Since $iZ_2 \in \mathfrak{g}_u$ and $\theta_1|_{\tilde{\mathfrak{h}}} = \theta_0|_{\tilde{\mathfrak{h}}}$, it satisfies (c1) $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_1(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})}$. In addition, it is the same as the involution θ_1 in Murakami [Mu3, pp. 305, type EI]. Therefore, Murakami's result [Mu3] enables us to deduce that $\{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_1|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_3|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_4|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$ is the set of simple roots in $\triangle(\mathfrak{k}, \mathfrak{k} \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$, where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_1(K) = K\}$ and $\eta := \alpha_2 + \alpha_3 + \alpha_4$, and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{sp}(4)$:



[22] Erratum: pp. 305, the last line on [Mu3], read "EI (resp. EIV)" instead of "EIV (resp. EI)".

Here $-i\acute{\alpha}_c := -i\alpha_c|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$ $(c = 1, 3, 4)$ and $-i\acute{\eta} := -i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}$. Besides, his result also implies that the highest root $-i\mu \in \triangle(\mathfrak{k}, \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is

$$-i\mu = -i(2\eta + 2\alpha_1 + 2\alpha_3 + \alpha_4)|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, \tag{5.5.5}$$

and that $\mathfrak{e}_{6(6)}$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k}\oplus i\mathfrak{p}$ of $\mathfrak{e}_6^{\mathbb{C}}$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_1(P) = -P\}$. In the second place, let us describe the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})}$ in terms of the dual basis $\{Z_a\}_{a=1}^6$ of $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^6$. This description will be needed in the third place. Since $\theta_1|_{\tilde{\mathfrak{h}}} = \theta_0|_{\tilde{\mathfrak{h}}}$, and since $\theta_0|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \theta_0'$ and (5.5.1), we perceive that

$$\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{i(Z_1 + Z_6), iZ_2, i(Z_3 + Z_5), iZ_4\}. \tag{5.5.6}$$

Now, let $\{T_d\}_{d=1}^4$, $T_d \in \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}$, denote the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_1|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_3|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_4|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$. Taking $\eta = \alpha_2 + \alpha_3 + \alpha_4$, (5.5.6) and $\alpha_a(Z_b) = \delta_{a,b}$ into consideration, we can obtain

$$\begin{cases} T_1 = iZ_2, \\ T_2 = i(Z_1 + Z_6), \\ T_3 = i(-Z_2 + Z_3 + Z_5), \\ T_4 = i(-Z_2 + Z_4). \end{cases} \tag{5.5.7}$$

In the third place, let us search a Weyl chamber $\mathfrak{W}_{\mathfrak{k}}^1$ of $\mathfrak{k}$ for *Spr*-elements of $\mathfrak{g} = \mathfrak{e}_{6(6)}$. Here, $\mathfrak{W}_{\mathfrak{k}}^1$ is given by

$$\mathfrak{W}_{\mathfrak{k}}^1 := \{T \in \mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\eta(T) \geq 0, -i\alpha_1(T) \geq 0, -i\alpha_3(T) \geq 0, -i\alpha_4(T) \geq 0\}.$$

Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^1$ is an *Spr*-element of $\mathfrak{g}$. Then, Lemma 4.2.4 and (5.5.5) enable us to have $T = T_4$ because $\{T_d\}_{d=1}^4$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\eta|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_1|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_3|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}, -i\alpha_4|_{\mathfrak{k}\cap i\tilde{\mathfrak{h}}_{\mathbb{R}}}\}$. Hence, $T = i(-Z_2 + Z_4)$ comes from (5.5.7). Consequently, this "$T = i(-Z_2 + Z_4)$" is a necessary condition for an element $T \in \mathfrak{W}_{\mathfrak{k}}^1$ to be an *Spr*-element of $\mathfrak{g} = \mathfrak{e}_{6(6)}$. However, it is not the sufficient condition. Indeed, there exists a root $\beta = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \in \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$ (see (5.5.3)), and it follows from $\alpha_a(Z_b) = \delta_{a,b}$ that $\beta(i(-Z_2 + Z_4)) = 2i \neq \pm i$, so that the element $T = i(-Z_2 + Z_4)$ can not be an *Spr*-element (cf. Lemma 4.1.1). For the reasons, the set of *Spr*-elements which belong to $\mathfrak{W}_{\mathfrak{k}}^1$ is an empty set. Accordingly, by (4.1.2) we assert the following:

Proposition 5.5.2. *The set of Spr-elements of* EI: $\mathfrak{e}_{6(6)}$ *is an empty set.*

5.5.2. *Case* EII $\mathfrak{e}_{6(2)}$. In this paragraph, we will classify *Spr*-elements of $\mathfrak{e}_{6(2)}$ (cf. Proposition 5.5.5).

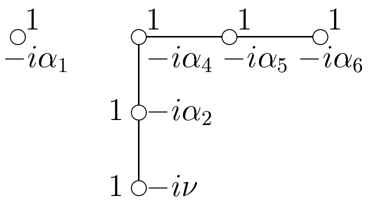
Let us define an involutive automorphism θ_2 of $\mathfrak{g}_u$ such that $\mathfrak{e}_{6(2)}$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_2. Define an inner automorphism θ_2 of $\mathfrak{e}_6^{\mathbb{C}}$ by

$$\theta_2 := \exp \pi \operatorname{ad}_{\mathfrak{e}_6^{\mathbb{C}}} iZ_3 \tag{5.5.8}$$

(see Notation 5.5.1 for Z_3). Then, this automorphism θ_2 is involutive and satisfies the three conditions in Paragraph 2.3.2; (c1) $\theta_2(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_2(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3)

${}^t\theta_2(\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})}$ (ref. Murakami [Mu3]). The result of Murakami [Mu3, pp. 297, type EII] states that $\{-i\alpha_1, -i\nu, -i\alpha_2, -i\alpha_q\}_{q=4}^6$ is the simple root system of $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_2(K) = K\}$ and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(6)$:



where $-i\nu := i(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)$. Moreover from his result, one knows that $\mathfrak{e}_{6(2)}$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{e}_6^{\mathbb{C}}$ (where $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_2(P) = -P\}$). Now, let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ denote $\mathfrak{su}(2)$ and $\mathfrak{su}(6)$, respectively. We assume $\{-i\alpha_1\}$ (resp. $\{-i\nu, -i\alpha_2, -i\alpha_q\}_{q=4}^6$) to be the set of simple roots in $\triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (resp. $\triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$). In this case, the highest root $-i\mu_1 \in \triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2 \in \triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$\text{(5.5.9)} \qquad \begin{cases} -i\mu_1 = -i\alpha_1, \\ -i\mu_2 = -i(\nu + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6). \end{cases}$$

From now on, we are going to describe the dual basis $\{T_a\}_{a=1}^6$ of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_2, -i\alpha_q\}_{q=4}^6$ in terms of $\{Z_a\}_{a=1}^6$. The description will be utilized in the proof of Lemma 5.5.3. Define an element $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}}$ by setting $-i\alpha_1(T_a) = \delta_{1,a}$, $-i\nu(T_a) = \delta_{2,a}$, $-i\alpha_2(T_a) = \delta_{3,a}$ and $-i\alpha_q(T_a) = \delta_{q,a}$ $(4 \leq q \leq 6)$. Then, since $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^6$ and $\alpha_a(Z_b) = \delta_{a,b}$, and since $-i\nu = i(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$, one perceives that

$$\text{(5.5.10)} \qquad \begin{cases} T_1 = i(Z_1 - \frac{1}{2}Z_3), & T_2 = -\frac{i}{2}Z_3, & T_3 = i(Z_2 - Z_3), \\ T_4 = i(-\frac{3}{2}Z_3 + Z_4), & T_5 = i(-Z_3 + Z_5), & T_6 = i(-\frac{1}{2}Z_3 + Z_6). \end{cases}$$

Let us show the following:

Lemma 5.5.3. *In the above setting; an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an Spr-element of $\mathfrak{g} = \mathfrak{e}_{6(2)}$ if and only if it is one of the following:*

$$i(Z_2 - Z_3), \quad i(-Z_3 + Z_5), \quad i(Z_1 - Z_3), \quad i(Z_1 - Z_3 + Z_6).$$

Here, $\mathfrak{W}_{\mathfrak{k}}^2$ is a Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_2, -i\alpha_q\}_{q=4}^6$;

$$\mathfrak{W}_{\mathfrak{k}}^2 = \left\{ T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\alpha_1(T) \geq 0, -i\nu(T) \geq 0, \;\; -i\alpha_2 \geq 0, \\ -i\alpha_4(T) \geq 0, -i\alpha_5(T) \geq 0, -i\alpha_6(T) \geq 0 \end{array} \right\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an *Spr*-element of $\mathfrak{g}$. Then, Lemma 4.2.2, together with (5.5.9), means that the element T is one of the following eleven elements:

$$\text{(b}'\text{-1) } T_1, \quad \text{(b}'\text{-2) } T_p \text{ for } 2 \leq p \leq 6, \quad \text{(b}'\text{-3) } T_1 + T_p \text{ for } 2 \leq p \leq 6$$

because $\{T_a\}_{a=1}^6$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_1, -i\nu, -i\alpha_2, -i\alpha_q\}_{q=4}^6$. In terms of (5.5.10), we can rewrite the above elements as

(b-1) $i(Z_1 - \frac{1}{2}Z_3)$,
(b-2.1) $-\frac{i}{2}Z_3$, (b-2.2) $i(Z_2 - Z_3)$, (b-2.3) $i(-\frac{3}{2}Z_3 + Z_4)$,
(b-2.4) $i(-Z_3 + Z_5)$, (b-2.5) $i(-\frac{1}{2}Z_3 + Z_6)$,
(b-3.1) $i(Z_1 - Z_3)$, (b-3.2) $i(Z_1 + Z_2 - \frac{3}{2}Z_3)$, (b-3.3) $i(Z_1 - 2Z_3 + Z_4)$,
(b-3.4) $i(Z_1 - \frac{3}{2}Z_3 + Z_5)$, (b-3.5) $i(Z_1 - Z_3 + Z_6)$.

Due to the supposition and Lemma 4.1.1, the element T has to satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$. For the reason, the element T is one of the following:

(b-2.2) $i(Z_2 - Z_3)$, (b-2.4) $i(-Z_3 + Z_5)$,
(b-3.1) $i(Z_1 - Z_3)$, (b-3.5) $i(Z_1 - Z_3 + Z_6)$

since $\alpha_a(Z_b) = \delta_{a,b}$ and (5.5.3). Conversely, suppose that an element T' is one of the above elements. Then $T' \in \mathfrak{W}_{\mathfrak{k}}^2$, and it follows from $\alpha_a(Z_b) = \delta_{a,b}$ and (5.5.3) that it satisfies $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$, so that the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{e}_{6(2)}$ (cf. Lemma 4.1.1). Therefore, we have proved Lemma 5.5.3. □

By (4.1.2) and Lemma 5.5.3, we deduce that

(5.5.11) $$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\, [i(Z_2 - Z_3)],\ [i(-Z_3 + Z_5)],\ [i(Z_1 - Z_3)],\ [i(Z_1 - Z_3 + Z_6)]\,\},$$

where $\mathfrak{g} = \mathfrak{e}_{6(2)}$. The following lemma implies that the above *Spr*-element $i(Z_2 - Z_3)$ (resp. $i(Z_1 - Z_3)$) is equivalent to $i(-Z_3 + Z_5)$ (resp. $i(Z_1 - Z_3 + Z_6)$).

Lemma 5.5.4. *In the setting on Paragraph* 5.5.2; *there exists an automorphism* ψ *of* $\mathfrak{g} = \mathfrak{e}_{6(2)}$ *such that*

$$\begin{cases} \psi(iZ_1) = i(Z_1 - Z_6), & \psi(iZ_2) = i(Z_5 - 2Z_6), \\ \psi(iZ_3) = i(Z_3 - 2Z_6), & \psi(iZ_4) = i(Z_4 - 3Z_6), \\ \psi(iZ_5) = i(Z_2 - 2Z_6), & \psi(iZ_6) = -iZ_6. \end{cases}$$

Here, $\{Z_a\}_{a=1}^6$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^6$.

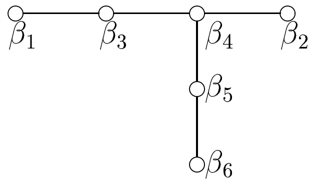
Proof. Define an involutive linear isomorphism ψ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^6$ by

(5.5.12) $$\begin{cases} \psi'(iZ_1) := i(Z_1 - Z_6), & \psi'(iZ_2) := i(Z_5 - 2Z_6), \\ \psi'(iZ_3) := i(Z_3 - 2Z_6), & \psi'(iZ_4) := i(Z_4 - 3Z_6), \\ \psi'(iZ_5) := i(Z_2 - 2Z_6), & \psi'(iZ_6) := -iZ_6. \end{cases}$$

Then by $\alpha_a(Z_b) = \delta_{a,b}$, the complex linear extension $\psi'_{\mathbb{C}}$ of ψ' to $\tilde{\mathfrak{h}}$ satisfies

(5.5.13) $$\begin{cases} {}^t\psi'_{\mathbb{C}}(\alpha_1) = \alpha_1, & {}^t\psi'_{\mathbb{C}}(\alpha_2) = \alpha_5, \\ {}^t\psi'_{\mathbb{C}}(\alpha_3) = \alpha_3, & {}^t\psi'_{\mathbb{C}}(\alpha_4) = \alpha_4, \\ {}^t\psi'_{\mathbb{C}}(\alpha_5) = \alpha_2, & {}^t\psi'_{\mathbb{C}}(\alpha_6) = \nu. \end{cases}$$

Here, $\nu = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$. In terms of (5.5.13), the Dynkin diagram of $\{{}^t\psi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^6$ is as follows:



where $\beta_a := {}^t\psi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq 6$. Therefore, we see that

$$(5.5.14) \qquad {}^t\psi'_{\mathbb{C}}\big(\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$$

(ref. Murakami [Mu3, Lemma 1, pp. 295]). So, Proposition 2.3.2 assures that there exists an involutive automorphism $\bar{\psi}$ of $\mathfrak{e}_6^{\mathbb{C}}$ satisfying (i) $\bar{\psi}(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (iii) $\bar{\psi}(X_{\pm\alpha_a}) = X_{\pm{}^t\bar{\psi}(\alpha_a)}$. Now, let us prove that the involution $\bar{\psi}$ satisfies the two conditions (a) and (b) in Proposition 2.3.4. From the definition (5.5.8) of θ_2, it is obvious that $\theta_2 = \mathrm{id}$ on $\tilde{\mathfrak{h}}$. Thus $\theta_2 \circ \bar{\psi} = \bar{\psi} \circ \theta_2$ on $i\tilde{\mathfrak{h}}_{\mathbb{R}}$. Accordingly, $\bar{\psi}$ satisfies the condition (a) in Proposition 2.3.4. Let us show that the involution $\bar{\psi}$ also satisfies the condition (b) in Proposition 2.3.4. By (5.5.8), one obtains

$$\triangle_1^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_2) = \left\{ \sum_{a=1}^{6} n_a\alpha_a \in \triangle^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) \;\middle|\; n_3 = 0 \text{ or } 2 \right\}$$

(see (2.3.4) for $\triangle_1^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_2)$). Since $\bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.5.13), we have ${}^t\bar{\psi}(\alpha_1) = \alpha_1$, ${}^t\bar{\psi}(\alpha_2) = \alpha_5$, ${}^t\bar{\psi}(\alpha_3) = \alpha_3$, ${}^t\bar{\psi}(\alpha_4) = \alpha_4$, ${}^t\bar{\psi}(\alpha_5) = \alpha_2$, ${}^t\bar{\psi}(\alpha_6) = \nu = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$. The coefficient of each root $\alpha \in \triangle^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$ with respect to α_6 is either 1 or zero. Therefore, it follows that $(n_3, n_6) = (0, 0)$, $(0, 1)$, $(2, 0)$ or $(2, 1)$ for any root $\beta = \sum_{a=1}^{6} n_a\alpha_a \in \triangle_1^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_2)$; and hence the coefficient of ${}^t\bar{\psi}(\beta)$ with respect to α_3 is ± 2 or zero, for any root $\beta \in \triangle_1^+(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_2)$. Accordingly, since $\bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.5.14), we conclude that

$${}^t\bar{\psi}\big(\triangle_1(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_2)\big) = \triangle_1(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_2),$$

namely the involution $\bar{\psi}$ of $\mathfrak{e}_6^{\mathbb{C}}$ also satisfies the condition (b) in Proposition 2.3.4. Therefore, $\bar{\psi}$ satisfies the two conditions (a) and (b). Proposition 2.3.4 enables us to have an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ such that $\bar{\psi} \circ \exp \mathrm{ad}_{\mathfrak{e}_6^{\mathbb{C}}} iH$ is an automorphism of $\mathfrak{g} = \mathfrak{e}_{6(2)}$. Define an automorphism ψ of $\mathfrak{g}$ by $\psi := \bar{\psi} \circ \exp \mathrm{ad}_{\mathfrak{e}_6^{\mathbb{C}}} iH$.[23] By virtue of $\psi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \bar{\psi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \psi'$ and (5.5.12), we conclude Lemma 5.5.4. □

Now, let us demonstrate Proposition 5.5.5.

Proposition 5.5.5. *Under our equivalence relation, Spr-elements of* EII: $\mathfrak{g} = \mathfrak{e}_{6(2)}$ *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(Z_2 - Z_3)],\ [i(Z_1 - Z_3)]\ \}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}(6,4) \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{so}^*(10) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_2 - Z_3)$ *and* $\rho_2 = \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(Z_1 - Z_3)$, *respectively. Here,* $\{Z_a\}_{a=1}^6$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^6$.

[23]This ψ is an outer automorphism of $\mathfrak{e}_{6(2)}$.

Proof. Lemma 5.5.4 and (5.5.11) imply that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ \, [i(Z_2 - Z_3)], \ [i(Z_1 - Z_3)] \, \}.$$

It is known that

$$\mathfrak{c}_{\mathfrak{g}}(i(Z_2 - Z_3)) = \mathfrak{so}(6,4) \oplus \mathfrak{t}^1,$$
$$\mathfrak{c}_{\mathfrak{g}}(i(Z_1 - Z_3)) = \mathfrak{so}^*(10) \oplus \mathfrak{t}^1$$

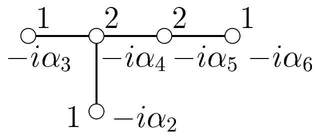
(cf. Theorem 6.16 in [Bm]). This shows that the *Spr*-element $i(Z_2 - Z_3)$ is not equivalent to $i(Z_1 - Z_3)$; and that $(\mathfrak{g}, \mathfrak{so}(6,4) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}^*(10) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(Z_2 - Z_3)$ and $\rho_2 := \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(Z_1 - Z_3)$, respectively (cf. Lemma 3.1.1). Therefore, we have got the conclusion. □

5.5.3. *Case* EIII $\mathfrak{e}_{6(-14)}$. In this paragraph, we devote ourselves to classifying *Spr*-elements of $\mathfrak{e}_{6(-14)}$ (see Proposition 5.5.8).

Let us define an involutive automorphism θ_3 of $\mathfrak{g}_u$ such that $\mathfrak{e}_{6(-14)}$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_3. Define an inner automorphism θ_3 of $\mathfrak{e}_6^{\mathbb{C}}$ by

(5.5.15) $$\theta_3 := \exp \pi \, \mathrm{ad}_{\mathfrak{e}_6^{\mathbb{C}}} \, iZ_1.$$

Then, since $iZ_1 \in \mathfrak{g}_u$ and $\theta_3|_{\tilde{\mathfrak{h}}} = \mathrm{id}$, one has (c1) $\theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_3(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_3(\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})}$. Murakami's result [Mu3, pp. 297, type EIII] states that the automorphism θ_3 is involutive, the simple root system of $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_3(K) = K\}$ is $\{-i\alpha_c\}_{c=2}^6$, and the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^6$ is as follows:

$\mathfrak{k} = \mathfrak{so}(10) \oplus \mathfrak{t}^1$: ×


Moreover, it follows from his result that $\mathfrak{e}_{6(-14)}$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{e}_6^{\mathbb{C}}$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \,|\, \theta_3(P) = -P\}$, and that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is as follows:

(5.5.16) $$-i\mu = -i(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6).$$

Now, denote by $\mathfrak{W}_{\mathfrak{k}}^3$ a Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^6$;

$$\mathfrak{W}_{\mathfrak{k}}^3 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \,|\, -i\alpha_2(T) \geq 0, -i\alpha_3(T) \geq 0, \cdots, -i\alpha_6(T) \geq 0\}.$$

We are going to search this Weyl chamber $\mathfrak{W}_{\mathfrak{k}}^3$ of $\mathfrak{k}$ for *Spr*-elements of $\mathfrak{g} = \mathfrak{e}_{6(-14)}$.

Lemma 5.5.6. *With the above notation; an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ is an Spr-element of $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ if and only if it is one of the following:*

$$i(-Z_1 + Z_2), \quad i(-Z_1 + Z_3), \quad i(-Z_1 + Z_6),$$
$$iZ_6, \qquad \pm iZ_1.$$

Here, $\{Z_a\}_{a=1}^6$ is the dual basis of $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^6$.

Proof. Suppose that an element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. It comes from $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=2}^6$ and $\alpha_a(Z_b) = \delta_{a,b}$ that $-i\gamma(iZ_1) \equiv 0$ for all roots $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$. So, iZ_1 is a central element of $\mathfrak{k}$. Then, Lemma 4.2.3 and (5.5.16) allow us to deduce that one of the following cases only occurs:

$$\begin{array}{ll} (\text{c}'\text{-1.1})\ T = i(\lambda \cdot Z_1 + Z_2), & (\text{c}'\text{-1.2})\ T = i(\lambda \cdot Z_1 + Z_3), \\ (\text{c}'\text{-1.3})\ T = i(\lambda \cdot Z_1 + Z_6), & (\text{c}'\text{-2})\ T = i\,\lambda \cdot Z_1, \end{array}$$

because of $\alpha_a(Z_b) = \delta_{a,b}$ and $T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^6$. Here, λ is a real number ($\lambda \neq 0$ in Case (c$'$-2)). Since $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is an *Spr*-element, Lemma 4.1.1 implies that it must satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$. Therefore by $\alpha_a(Z_b) = \delta_{a,b}$ and (5.5.3), the value of λ is determined as follows: $\lambda = -1$ in two Cases (c$'$-1.1) and (c$'$-1.2), $\lambda = -1$ or 0 in Case (c$'$-1.3), and $\lambda = \pm 1$ in Case (c$'$-2). Accordingly, the *Spr*-element $T \in \mathfrak{W}^3_{\mathfrak{k}}$ is one of the following:

$$\begin{array}{ll} (\text{c-1.1})\ i(-Z_1 + Z_2), & (\text{c-1.2})\ i(-Z_1 + Z_3), \\ (\text{c-1.3})\ i(-Z_1 + Z_6),\ iZ_6, & (\text{c-2})\ \pm iZ_1. \end{array}$$

Conversely, if an element T' is one of the above elements, then $T' \in \mathfrak{W}^3_{\mathfrak{k}}$ and it satisfies $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})$; and hence it is an *Spr*-element of $\mathfrak{g}$ (see Lemma 4.1.1). Thus, Lemma 5.5.6 has been proved. □

Lemma 5.5.6, combined with (4.1.2), enables us to lead the following:

$$(5.5.17)\quad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \operatorname{Aut}(\mathfrak{g})) = \left\{ \begin{array}{lll} [\,i(-Z_1 + Z_2)\,], & [\,i(-Z_1 + Z_3)\,], & [\,i(-Z_1 + Z_6)\,], \\ [\,iZ_6\,], & [\,iZ_1\,] & \end{array} \right\},$$

where $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. The following lemma means that the above *Spr*-element $i(-Z_1 + Z_2)$ and $i(-Z_1 + Z_6)$ are equivalent to $i(-Z_1 + Z_3)$ and iZ_6, respectively.

Lemma 5.5.7. *In the setting on Paragraph* 5.5.3; *there exists an involutive automorphism* ϕ *of* $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ *satisfying*

$$\begin{array}{lll} \phi(iZ_1) = -iZ_1, & \phi(iZ_2) = i(-2Z_1 + Z_3), & \phi(iZ_3) = i(-2Z_1 + Z_2), \\ \phi(iZ_4) = i(-3Z_1 + Z_4), & \phi(iZ_5) = i(-2Z_1 + Z_5), & \phi(iZ_6) = i(-Z_1 + Z_6). \end{array}$$

Here, $\{Z_a\}_{a=1}^6$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^6$.

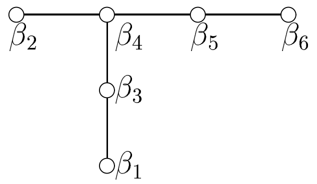
Proof. Let us define an involutive, linear isomorphism ϕ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^6$ by

$$(5.5.18)\qquad \left\{ \begin{array}{ll} \phi'(iZ_1) := -iZ_1, & \phi'(iZ_2) := i(-2Z_1 + Z_3), \\ \phi'(iZ_3) := i(-2Z_1 + Z_2), & \phi'(iZ_4) := i(-3Z_1 + Z_4), \\ \phi'(iZ_5) := i(-2Z_1 + Z_5), & \phi'(iZ_6) := i(-Z_1 + Z_6). \end{array} \right.$$

Then since $\alpha_a(Z_b) = \delta_{a,b}$, the complex linear extension $\phi'_{\mathbb{C}}$ of ϕ' to $\tilde{\mathfrak{h}}$ satisfies

$$(5.5.19)\qquad \left\{ \begin{array}{ll} {}^t\phi'_{\mathbb{C}}(\alpha_1) = \nu, & {}^t\phi'_{\mathbb{C}}(\alpha_2) = \alpha_3, \\ {}^t\phi'_{\mathbb{C}}(\alpha_3) = \alpha_2, & {}^t\phi'_{\mathbb{C}}(\alpha_4) = \alpha_4, \\ {}^t\phi'_{\mathbb{C}}(\alpha_5) = \alpha_5, & {}^t\phi'_{\mathbb{C}}(\alpha_6) = \alpha_6. \end{array} \right.$$

Here, $\nu = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)$. Therefore, the Dynkin diagram of $\{{}^t\phi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^6$ is as follows:



where $\beta_a := {}^t\phi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq 6$. This shows that

$$ {}^t\phi'_{\mathbb{C}}\big(\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}}) $$

(cf. Murakami [Mu3, Lemma 1, pp. 295]). For the reasons, Proposition 2.3.2 enables us to get an involutive automorphism ϕ of $\mathfrak{e}_6^{\mathbb{C}}$ such that (i) $\phi(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (iii) $\phi(X_{\pm\alpha_a}) = X_{\pm{}^t\phi(\alpha_a)}$. By virtue of $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.5.18), the rest of proof is to demonstrate that the involution ϕ of $\mathfrak{e}_6^{\mathbb{C}}$ is an automorphism of $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. From $\phi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \phi'$ and (5.5.18), it is easy to see that $\phi(iZ_1) = -iZ_1$. Thus, since the involution θ_3 is defined by (5.5.15), the involution ϕ is commutative with θ_3. Therefore, the involution ϕ is an automorphism of $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ (see Proposition 2.2.3).[24] Accordingly, we have shown Lemma 5.5.7. □

Now, let us prove the following:

Proposition 5.5.8. *Under our equivalence relation, Spr-elements of* EIII: $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ *are classified as follows*:

$$ Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_3)],\ [iZ_6],\ [iZ_1]\ \}. $$

Besides, (1) $(\mathfrak{g}, \mathfrak{so}^*(10) \oplus \mathfrak{t}^1)$, (2) $(\mathfrak{g}, \mathfrak{so}(8,2) \oplus \mathfrak{t}^1)$ *and* (3) $(\mathfrak{g}, \mathfrak{so}(10) \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_3)$, $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_6$ *and* $\rho_3 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$, *respectively. Here,* $\{Z_a\}_{a=1}^6$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_6^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^6$.

Proof. It is immediate from (5.5.17) and Lemma 5.5.7 that

$$ Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_3)],\ [iZ_6],\ [iZ_1]\ \}. $$

The centralizers of the above Spr-elements in $\mathfrak{g} = \mathfrak{e}_{6(-14)}$ are as follows:

$$ \begin{aligned} &\mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_3)) = \mathfrak{so}^*(10) \oplus \mathfrak{t}^1, \\ &\mathfrak{c}_{\mathfrak{g}}(iZ_6) = \mathfrak{so}(8,2) \oplus \mathfrak{t}^1, \\ &\mathfrak{c}_{\mathfrak{g}}(iZ_1) = \mathfrak{so}(10) \oplus \mathfrak{t}^1 \end{aligned} \tag{5.5.20} $$

(ref. Theorem 6.16 in [Bm]). Therefore, Lemma 3.1.1 implies that $(\mathfrak{g}, \mathfrak{so}^*(10) \oplus \mathfrak{t}^1)$, $(\mathfrak{g}, \mathfrak{so}(8,2) \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{so}(10) \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_3)$, $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_6$ and $\rho_3 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$, respectively. Besides, it follows from (5.5.20) that the three Spr-elements $i(-Z_1 + Z_3)$, iZ_6 and iZ_1 are not mutually equivalent. Thus, we have completed the proof of Proposition 5.5.8. □

[24]This ϕ is an outer automorphism of $\mathfrak{e}_{6(-14)}$.

By Proposition 4.2.5, and by Propositions in Subsection 5.5, we have the following:

Table V.

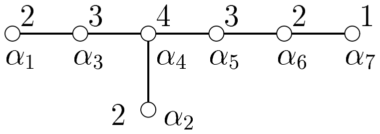
		EI
11	$\mathfrak{g}$	$\mathfrak{e}_{6(6)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	None

		EII
12	$\mathfrak{g}$	$\mathfrak{e}_{6(2)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_2 - Z_3)\,]$, $[\,i(Z_1 - Z_3)\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_2 - Z_3))$	$\mathfrak{so}(6,4) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_1 - Z_3))$	$\mathfrak{so}^*(10) \oplus \mathfrak{t}^1$

		EIII
13	$\mathfrak{g}$	$\mathfrak{e}_{6(-14)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_1 + Z_3)\,]$, $[\,iZ_6\,]$, $[\,iZ_1\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_3))$	$\mathfrak{so}^*(10) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_6)$	$\mathfrak{so}(8,2) \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_1)$	$\mathfrak{so}(10) \oplus \mathfrak{t}^1$

		EIV
14	$\mathfrak{g}$	$\mathfrak{e}_{6(-26)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	None

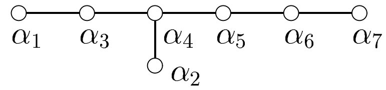
5.6. **Type** E_7**.** This subsection is devoted to classifying *Spr*-elements of each real form of the exceptional, complex simple Lie algebra $\mathfrak{e}_7^{\mathbb{C}}$. Let us introduce our setting. Let $\tilde{\mathfrak{h}}$ be a Cartan subalgebra of $\mathfrak{e}_7^{\mathbb{C}}$, let $\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ be the set of non-zero roots of $\mathfrak{e}_7^{\mathbb{C}}$ with respect to $\tilde{\mathfrak{h}}$, and let $\{\alpha_a\}_{a=1}^7$ be the set of simple roots in $\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ whose Dynkin diagram is as follows:

$\mathfrak{e}_7^{\mathbb{C}}$:


(cf. Plate VI in Bourbaki [Br, pp. 279–280]). We denote by $\mathfrak{g}_u$ the compact real form of $\mathfrak{e}_7^{\mathbb{C}}$ which is given by $\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ and (2.3.1), and we denote by $\{Z_a\}_{a=1}^7$ ($Z_a \in \tilde{\mathfrak{h}}$) the dual basis of $\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^7$. In the setting, we will classify *Spr*-elements of each real form of $\mathfrak{e}_7^{\mathbb{C}}$.

Notation 5.6.1. In Subsection 5.6, we utilize the following notation:

- $\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^7$.



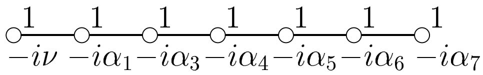
- $\mathfrak{g}_u$: the compact real form of $\mathfrak{e}_7^{\mathbb{C}}$ given by $\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})$ and (2.3.1).
- $\{Z_a\}_{a=1}^7$: the dual basis of $\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^7$.

5.6.1. *Case* EV $\mathfrak{e}_{7(7)}$. Our aim in this paragraph is to classify *Spr*-elements of $\mathfrak{e}_{7(7)}$ (see Proposition 5.6.4).

In the first place, let us give an involutive automorphism θ_1 of $\mathfrak{g}_u$ such that $\mathfrak{e}_{7(7)}$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_1. Define an inner automorphism θ_1 of $\mathfrak{e}_7^{\mathbb{C}}$ by

$$\theta_1 := \exp \pi \operatorname{ad}_{\mathfrak{e}_7^{\mathbb{C}}} iZ_2. \tag{5.6.1}$$

Then, one has (c1) $\theta_1(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_1(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_1(\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})}$ since $iZ_2 \in \mathfrak{g}_u$ and $\theta_1|_{\tilde{\mathfrak{h}}} = \mathrm{id}$. It is shown by Murakami [Mu3, pp. 297, type EV] that the automorphism θ_1 is involutive, the simple root system of $\mathfrak{k} := \{K \in \mathfrak{g}_u \mid \theta_1(K) = K\}$ is $\{-i\nu, -i\alpha_1, -i\alpha_c\}_{c=3}^7$, and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{su}(8)$: 

Here, $-i\nu$ denotes the lowest root $i(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$ in $\triangle(\mathfrak{g}_u, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ (ref. Remark 2.3.1). Besides, it is also shown that $\mathfrak{e}_{7(7)}$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{e}_7^{\mathbb{C}}$, where $\mathfrak{p}$ denotes the -1-eigenspace of θ_1 in $\mathfrak{g}_u$. Remark that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is

$$-i\mu = -i(\nu + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7). \tag{5.6.2}$$

Now, we will describe the dual basis $\{T_a\}_{a=1}^7$ of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\nu, -i\alpha_1, -i\alpha_c\}_{c=3}^7$ in terms of $\{Z_a\}_{a=1}^7$, for the sake of the second place. Let $T_a \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^7$ be an element defined by $-i\nu(T_a) = \delta_{1,a}$, $-i\alpha_1(T_a) = \delta_{2,a}$ and $-i\alpha_c(T_a) = \delta_{c,a}$ $(3 \le c \le 7)$. Then, we obtain

$$\begin{cases} T_1 = -\frac{i}{2}Z_2, & T_2 = i(Z_1 - Z_2), \quad T_3 = i(-\frac{3}{2}Z_2 + Z_3), \\ T_4 = i(-2Z_2 + Z_4), & T_5 = i(-\frac{3}{2}Z_2 + Z_5), \quad T_6 = i(-Z_2 + Z_6), \\ T_7 = i(-\frac{1}{2}Z_2 + Z_7) & \end{cases} \tag{5.6.3}$$

because of $-i\nu = i(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$ and $\alpha_a(Z_b) = \delta_{a,b}$. This description (5.6.3) will be useful immediately. In the second place, let us prove the following:

Lemma 5.6.2. *In the above setting; an element $T \in \mathfrak{W}_{\mathfrak{k}}^1$ is an Spr-element of $\mathfrak{g} = \mathfrak{e}_{7(7)}$ if and only if it is either $i(Z_1 - Z_2)$ or $i(-Z_2 + Z_6)$. Here, $\mathfrak{W}_{\mathfrak{k}}^1$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\nu, -i\alpha_1, -i\alpha_c\}_{c=3}^7$;*

$$\mathfrak{W}_{\mathfrak{k}}^1 = \left\{ T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \;\middle|\; \begin{array}{l} -i\nu(T) \ge 0, \ -i\alpha_1(T) \ge 0, -i\alpha_3(T) \ge 0, -i\alpha_4(T) \ge 0, \\ -i\alpha_5(T) \ge 0, -i\alpha_6(T) \ge 0, -i\alpha_7(T) \ge 0 \end{array} \right\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an *Spr*-element of $\mathfrak{g}$. Then, due to (5.6.2) and Lemma 4.2.4, there exists an integer $a \in \{1, 2, \ldots, 7\}$ satisfying $T = T_a$ because $\{T_a\}_{a=1}^7$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\nu, -i\alpha_1, -i\alpha_c\}_{c=3}^7$. Therefore, it follows from (5.6.3) that the element T is one of the following elements:

$$\begin{cases} -\frac{i}{2}Z_2, & i(Z_1 - Z_2), & i(-\frac{3}{2}Z_2 + Z_3), \\ i(-2Z_2 + Z_4), & i(-\frac{3}{2}Z_2 + Z_5), & i(-Z_2 + Z_6), \\ i(-\frac{1}{2}Z_2 + Z_7). \end{cases}$$

Since $T \in \mathfrak{W}^1_{\mathfrak{k}}$ is an *Spr*-element, Lemma 4.1.1 implies that it must satisfy $\beta(T) = \pm i$ for every root $\beta \in \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \backslash \triangle_T(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$. Accordingly, the *Spr*-element T is either $i(Z_1 - Z_2)$ or $i(-Z_2 + Z_6)$, because $\alpha_a(Z_b) = \delta_{a,b}$ and $\triangle^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ is as follows:

$$(5.6.4) \qquad \triangle^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) = \left\{ \begin{array}{l} \pm\epsilon_i + \epsilon_j \ (1 \le i < j \le 6), \quad -\epsilon_7 + \epsilon_8, \\ \dfrac{1}{2}\big(-\epsilon_7 + \epsilon_8 + \sum_{i=1}^6 (-1)^{\nu(i)}\epsilon_i\big) \text{ with } \sum_{i=1}^6 \nu(i) \text{ odd} \end{array} \right\},$$

where $\alpha_1 = (1/2)\cdot(\epsilon_1 + \epsilon_8 - \sum_{p=2}^7 \epsilon_p)$, $\alpha_2 = \epsilon_1 + \epsilon_2$ and $\alpha_c = \epsilon_{c-1} - \epsilon_{c-2}$ $(3 \le c \le 7)$ (cf. Bourbaki [Br, pp. 279]). Conversely, suppose that an element T' is either $i(Z_1 - Z_2)$ or $i(-Z_2 + Z_6)$. Then, the element T' satisfies $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ because $\alpha_a(Z_b) = \delta_{a,b}$ and (5.6.4). Consequently, Lemma 4.1.1 allows us to deduce that the element T' is an *Spr*-element of $\mathfrak{g} = \mathfrak{e}_{7(7)}$. Hence, we have shown Lemma 5.6.2. □

By (4.1.2) and Lemma 5.6.2, one sees that

$$(5.6.5) \qquad Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(Z_1 - Z_2)],\ [i(-Z_2 + Z_6)]\ \},$$

where $\mathfrak{g} = \mathfrak{e}_{7(7)}$. In the third place, we will verify that the above *Spr*-element $i(Z_1 - Z_2)$ is equivalent to $i(-Z_2 + Z_6)$.

Lemma 5.6.3. *With the above assumptions; there exists an automorphism φ of $\mathfrak{g}_u$ such that it stabilizes $\mathfrak{g} = \mathfrak{e}_{7(7)}$ and satisfies*

$$\begin{cases} \varphi(iZ_1) = i(Z_6 - 2Z_7), & \varphi(iZ_2) = i(Z_2 - 2Z_7), & \varphi(iZ_3) = i(Z_5 - 3Z_7), \\ \varphi(iZ_4) = i(Z_4 - 4Z_7), & \varphi(iZ_5) = i(Z_3 - 3Z_7), & \varphi(iZ_6) = i(Z_1 - 2Z_7), \\ \varphi(iZ_7) = -iZ_7. \end{cases}$$

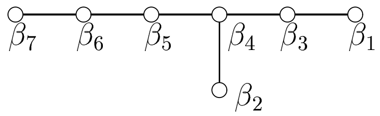
Proof. Let us construct such an automorphism φ in this lemma. Define an involutive, linear isomorphism φ' of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^7$ by setting

$$(5.6.6) \quad \begin{cases} \varphi'(iZ_1) := i(Z_6 - 2Z_7), \varphi'(iZ_2) := i(Z_2 - 2Z_7), \varphi'(iZ_3) := i(Z_5 - 3Z_7), \\ \varphi'(iZ_4) := i(Z_4 - 4Z_7), \varphi'(iZ_5) := i(Z_3 - 3Z_7), \varphi'(iZ_6) := i(Z_1 - 2Z_7), \\ \varphi'(iZ_7) := -iZ_7. \end{cases}$$

This, together with $\alpha_a(Z_b) = \delta_{a,b}$, yields that

$$(5.6.7) \qquad \begin{cases} {}^t\varphi'_{\mathbb{C}}(\alpha_1) = \alpha_6, & {}^t\varphi'_{\mathbb{C}}(\alpha_2) = \alpha_2, & {}^t\varphi'_{\mathbb{C}}(\alpha_3) = \alpha_5, \\ {}^t\varphi'_{\mathbb{C}}(\alpha_4) = \alpha_4, & {}^t\varphi'_{\mathbb{C}}(\alpha_5) = \alpha_3, & {}^t\varphi'_{\mathbb{C}}(\alpha_6) = \alpha_1, \\ {}^t\varphi'_{\mathbb{C}}(\alpha_7) = \nu, \end{cases}$$

where $\varphi'_{\mathbb{C}}$ denotes the complex linear extension of φ' to $\tilde{\mathfrak{h}}$. Here, $\nu = -(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$. Therefore, the Dynkin diagram of $\{{}^t\varphi'_{\mathbb{C}}(\alpha_a)\}_{a=1}^{7}$ is as follows:



where $\beta_a := {}^t\varphi'_{\mathbb{C}}(\alpha_a)$ for $1 \leq a \leq 7$. So, φ' satisfies

$$ {}^t\varphi'_{\mathbb{C}}\big(\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})\big) = \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \tag{5.6.8} $$

(ref. Murakami [Mu3, Lemma 1, pp. 295]). Consequently, there exists an involutive automorphism $\bar{\varphi}$ of $\mathfrak{e}_7^{\mathbb{C}}$ such that (i) $\bar{\varphi}(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (ii) $\bar{\varphi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (iii) $\bar{\varphi}(X_{\pm\alpha_a}) = X_{\pm {}^t\bar{\varphi}(\alpha_a)}$ (see Proposition 2.3.2). From now on, let us show that this involution $\bar{\varphi}$ satisfies the two conditions (a) and (b) in Proposition 2.3.4. Naturally, $\theta_1|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ follows from (5.6.1). Hence, one deduces that $\theta_1 \circ \bar{\varphi} = \bar{\varphi} \circ \theta_1$ on $i\tilde{\mathfrak{h}}_{\mathbb{R}}$, and so the involution $\bar{\varphi}$ satisfies the condition (a) in Proposition 2.3.4. We want to verify that the involution $\bar{\varphi}$ also satisfies the condition (b) in Proposition 2.3.4. By virtue of (5.6.1), we have

$$ \triangle_1^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_1) = \left\{ \sum_{a=1}^{7} n_a\alpha_a \in \triangle^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \;\middle|\; n_2 = 0 \text{ or } 2 \right\} $$

(refer to (2.3.4) for $\triangle_1^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_1)$). Since $\bar{\varphi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.6.7), the involution $\bar{\varphi}$ of $\mathfrak{e}_7^{\mathbb{C}}$ satisfies ${}^t\bar{\varphi}(\alpha_1) = \alpha_6$, ${}^t\bar{\varphi}(\alpha_2) = \alpha_2$, ${}^t\bar{\varphi}(\alpha_3) = \alpha_5$, ${}^t\bar{\varphi}(\alpha_4) = \alpha_4$, ${}^t\bar{\varphi}(\alpha_5) = \alpha_3$, ${}^t\bar{\varphi}(\alpha_6) = \alpha_1$ and ${}^t\bar{\varphi}(\alpha_7) = \nu = -(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$. The coefficient of any root $\alpha \in \triangle^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ with respect to α_7 is either 1 or zero. So, one comprehends $(n_2, n_7) = (0,0), (0,1), (2,0)$ or $(2,1)$, for any root $\beta = \sum_{a=1}^{7} n_a\alpha_a \in \triangle_1^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_1)$. Consequently, for any root $\beta \in \triangle_1^+(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_1)$, the coefficient of ${}^t\bar{\varphi}(\beta)$ with respect to α_2 is ± 2 or zero. Thus, it follows from (5.6.8) that

$$ {}^t\bar{\varphi}\big(\triangle_1(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_1)\big) = \triangle_1(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}} : \theta_1). $$

Hence, the involution $\bar{\varphi}$ also satisfies the condition (b) in Proposition 2.3.4. Therefore, $\bar{\varphi}$ satisfies the two conditions (a) and (b). By Proposition 2.3.4, one has an element $H \in \tilde{\mathfrak{h}}_{\mathbb{R}}$ such that $\bar{\varphi} \circ \exp \mathrm{ad}_{\mathfrak{e}_7^{\mathbb{C}}} iH \in \mathrm{Aut}(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{g}_u)$. Define φ by $\varphi := \bar{\varphi} \circ \exp \mathrm{ad}_{\mathfrak{e}_7^{\mathbb{C}}} iH$.[25] Since $\varphi|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \bar{\varphi}|_{i\tilde{\mathfrak{h}}_{\mathbb{R}}} = \varphi'$ and (5.6.6), we can get the conclusion. □

Lemma 5.6.3 and (5.6.5) imply that

$$ Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_2 + Z_6)]\ \}, $$

where $\mathfrak{g} = \mathfrak{e}_{7(7)}$. Lemma 3.1.1 assures that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(i(-Z_2 + Z_6)))$ is the pseudo-Hermitian symmetric Lie algebra by an involution $\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{g}}\, i(-Z_2 + Z_6)$. Theorem 6.16 in [Bm] means that $\mathfrak{c}_{\mathfrak{g}}(i(-Z_2 + Z_6))) = \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$. For the reasons, we assert the following:

[25]This φ is an outer automorphism of $\mathfrak{e}_{7(7)}$.

Proposition 5.6.4. *Under our equivalence relation, Spr-elements of* EV: $\mathfrak{g} = \mathfrak{e}_{7(7)}$ *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ \, [\, i(-Z_2 + Z_6) \,] \, \}.$$

Besides, $(\mathfrak{g}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1)$ *is the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho = \exp \pi \, \mathrm{ad}_{\mathfrak{g}} \, i(-Z_2 + Z_6)$*. Here,* $\{Z_a\}_{a=1}^{7}$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^{7}$.

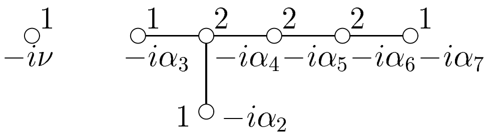
5.6.2. *Case* EVI $\mathfrak{e}_{7(-5)}$. In this paragraph, we classify *Spr*-elements of $\mathfrak{e}_{7(-5)}$ (see Proposition 5.6.6).

Let us define an involutive automorphism θ_2 of $\mathfrak{g}_u$ such that $\mathfrak{e}_{7(-5)}$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_2. We define an inner automorphism θ_2 of $\mathfrak{e}_7^{\mathbb{C}}$ by

$$\theta_2 := \exp \pi \, \mathrm{ad}_{\mathfrak{e}_7^{\mathbb{C}}} \, iZ_1.$$

This θ_2 satisfies (c1) $\theta_2(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_2(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_2(\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})}$ because of $iZ_1 \in \mathfrak{g}_u$ and $\theta_2|_{\tilde{\mathfrak{h}}} = \mathrm{id}$. Murakami's result [Mu3, pp. 297, type EVI] states that the automorphism θ_2 is involutive, $\{-i\nu, -i\alpha_p\}_{p=2}^{7}$ is the set of simple roots in $\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and its Dynkin diagram is as follows:

$\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{so}(12)$:



where $\mathfrak{k} := \{K \in \mathfrak{g}_u \,|\, \theta_2(K) = K\}$ and $-i\nu := i(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$. Moreover, it follows that $\mathfrak{e}_{7(-5)}$ is the real form of $\mathfrak{e}_7^{\mathbb{C}}$ given by (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$, where $\mathfrak{p}$ denotes the -1-eigenspace of θ_2 in $\mathfrak{g}_u$. Now, $\mathfrak{k}$ is the direct sum of two simple ideals $\mathfrak{k}_1 := \mathfrak{su}(2)$ and $\mathfrak{k}_2 := \mathfrak{so}(12)$. We assume that $\{-i\nu\}$ and $\{-i\alpha_p\}_{p=2}^{7}$ are the simple root system of $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. Then, the highest root $-i\mu_1 \in \triangle(\mathfrak{k}_1, \mathfrak{k}_1 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ and $-i\mu_2 \in \triangle(\mathfrak{k}_2, \mathfrak{k}_2 \cap i\tilde{\mathfrak{h}}_{\mathbb{R}})$ are as follows:

$$(5.6.9) \qquad \begin{cases} -i\mu_1 = -i\nu, \\ -i\mu_2 = -i(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7). \end{cases}$$

Let T_a be an element of $i\tilde{\mathfrak{h}}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^{7}$ defined by $-i\nu(T_a) = \delta_{1,a}$ and $-i\alpha_p(T_a) = \delta_{p,a}$ $(2 \leq p \leq 7)$. Since $\alpha_a(Z_a) = \delta_{a,b}$ and $-i\nu = i(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$, we have

$$(5.6.10) \qquad \begin{cases} T_1 = -\frac{i}{2} Z_1, & T_2 = i(-Z_1 + Z_2), \quad T_3 = i(-\frac{3}{2} Z_1 + Z_3), \\ T_4 = i(-2Z_1 + Z_4), & T_5 = i(-\frac{3}{2} Z_1 + Z_5), \quad T_6 = (-Z_1 + Z_6), \\ T_7 = i(-\frac{1}{2} Z_1 + Z_7). \end{cases}$$

This $\{T_a\}_{a=1}^{7}$ is the dual basis of $\Pi_{\triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\nu, -i\alpha_p\}_{p=2}^{7}$. By use of (5.6.10), we will prove the following:

Lemma 5.6.5. *With the above notation*; *an element* $T \in \mathfrak{W}_{\mathfrak{k}}^2$ *is an Spr-element of* $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ *if and only if it is either* $i(-Z_1 + Z_2)$ *or* $i(-Z_1 + Z_7)$*. Here,* $\mathfrak{W}_{\mathfrak{k}}^2$ *is the* *positive Weyl chamber with respect to* $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\nu, -i\alpha_p\}_{p=2}^7$;

$$\mathfrak{W}_{\mathfrak{k}}^2 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\nu(T) \geq 0, -i\alpha_2(T) \geq 0, \cdots, -i\alpha_7(T) \geq 0\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an *Spr*-element of $\mathfrak{g}$. Then, Lemma 4.2.2 and (5.6.9) enable us to see that the element T is one of the seven elements T_1, T_2, T_3, T_7, $T_1 + T_2$, $T_1 + T_3$ and $T_1 + T_7$, because $\{T_a\}_{a=1}^7$ is the dual basis of $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\nu, -i\alpha_p\}_{p=2}^7$. Hence, it follows from (5.6.10) that one of the following seven cases only occurs:

$$\begin{array}{lll} T = -\frac{i}{2}Z_1, & & \\ T = i(-Z_1 + Z_2), & T = i(-\frac{3}{2}Z_1 + Z_3), & T = i(-\frac{1}{2}Z_1 + Z_7), \\ T = i(-\frac{3}{2}Z_1 + Z_2), & T = i(-2Z_1 + Z_3), & T = i(-Z_1 + Z_7). \end{array}$$

Since $T \in \mathfrak{W}_{\mathfrak{k}}^2$ is an *Spr*-element, it must satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ (cf. Lemma 4.1.1). Therefore, since $\alpha_a(Z_b) = \delta_{a,b}$ and (5.6.4), one of the following two cases only occurs:

$$T = i(-Z_1 + Z_2), \quad T = i(-Z_1 + Z_7).$$

Conversely, suppose that an element T' is either $i(-Z_1+Z_2)$ or $i(-Z_1+Z_7)$. Then, it belongs to $\mathfrak{W}_{\mathfrak{k}}^2$, and satisfies $\beta(T') = \pm i$ for any root $\beta \in \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$, because of $\alpha_a(Z_b) = \delta_{a,b}$ and (5.6.4). So, Lemma 4.1.1 implies that the element T' is an *Spr*-element of $\mathfrak{g}$. Accordingly, we have completed the proof of Lemma 5.6.5. □

Let us demonstrate Proposition 5.6.6.

Proposition 5.6.6. *Under our equivalence relation, Spr-elements of* EVI: $\mathfrak{g} = \mathfrak{e}_{7(-5)}$ *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_2)],\ [i(-Z_1 + Z_7)]\ \}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_2)$ *and* $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_7)$, *respectively. Here,* $\{Z_a\}_{a=1}^7$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^7$.

Proof. By virtue of (4.1.2) and Lemma 5.6.5, we deduce that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{\ [i(-Z_1 + Z_2)],\ [i(-Z_1 + Z_7)]\ \}.$$

About the above *Spr*-elements, it is known that

$$(5.6.11) \qquad \begin{cases} \mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_2)) = \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1, \\ \mathfrak{c}_{\mathfrak{g}}(i(-Z_1 + Z_7)) = \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1 \end{cases}$$

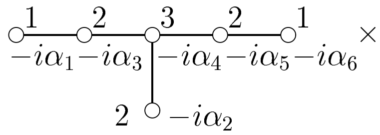
(cf. Theorem 6.16 in [Bm]). This (5.6.11) shows that the *Spr*-elements $i(-Z_1+Z_2)$ and $i(-Z_1 + Z_7)$ are not equivalent to each other. Besides, Lemma 3.1.1 and (5.6.11) mean that $(\mathfrak{g}, \mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_2)$ and $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_7)$, respectively. For the reasons, we have proved Proposition 5.6.6. □

5.6.3. *Case* EVII $\mathfrak{e}_{7(-25)}$. This paragraph deals with the classification of *Spr*-elements of $\mathfrak{e}_{7(-25)}$ (cf. Proposition 5.6.9).

First, we give an involutive automorphism θ_3 of $\mathfrak{g}_u$ such that $\mathfrak{e}_{7(-25)}$ is related to $\mathfrak{g}_u$ as in the formulae (2.2.1), (2.2.2) and (2.2.3) by means of θ_3. Define an inner automorphism θ_3 of $\mathfrak{e}_7^{\mathbb{C}}$ as follows:

$$\theta_3 := \exp \pi \operatorname{ad}_{\mathfrak{e}_7^{\mathbb{C}}} iZ_7. \tag{5.6.12}$$

In this case, it follows from $iZ_7 \in \mathfrak{g}_u$ and $\theta_3|_{\tilde{\mathfrak{h}}} = \mathrm{id}$ that (c1) $\theta_3(\mathfrak{g}_u) \subset \mathfrak{g}_u$, (c2) $\theta_3(\tilde{\mathfrak{h}}) \subset \tilde{\mathfrak{h}}$ and (c3) ${}^t\theta_3(\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})}) = \Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}},\tilde{\mathfrak{h}})}$. Due to the result of Murakami [Mu3, pp. 297, type EVII], one knows that the automorphism θ_3 is involutive, the simple root system of $\mathfrak{k}$ is $\{-i\alpha_c\}_{c=1}^6$, and the Dynkin diagram of $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=1}^6$ is as follows:

$\mathfrak{k} = \mathfrak{e}_6 \oplus \mathfrak{t}^1$: 

Here, $\mathfrak{k}$ denotes the $+1$-eigenspace of θ_3 in $\mathfrak{g}_u$. Moreover, his result implies that $\mathfrak{e}_{7(-25)}$ is the real form (2.2.3) $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{e}_7^{\mathbb{C}}$, where $\mathfrak{p} := \{P \in \mathfrak{g}_u \mid \theta_3(P) = -P\}$. Remark that the highest root $-i\mu \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$ is

$$-i\mu = -i(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6). \tag{5.6.13}$$

Next, let us show Lemma 5.6.7.

Lemma 5.6.7. *In the above setting; an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ is an Spr-element of $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ if and only if it is one of the following:*

$$i(Z_1 - Z_7), \quad i(Z_6 - Z_7), \quad \pm iZ_7.$$

Here, $\mathfrak{W}_{\mathfrak{k}}^3$ is the positive Weyl chamber with respect to $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=1}^6$;

$$\mathfrak{W}_{\mathfrak{k}}^3 = \{T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} \mid -i\alpha_1(T) \geq 0, -i\alpha_2(T) \geq 0, \cdots, -i\alpha_6(T) \geq 0\}.$$

Proof. Suppose that an element $T \in \mathfrak{W}_{\mathfrak{k}}^3$ is an *Spr*-element of $\mathfrak{g}$. From $\alpha_a(Z_b) = \delta_{a,b}$ and $\Pi_{\triangle(\mathfrak{k},i\tilde{\mathfrak{h}}_{\mathbb{R}})} = \{-i\alpha_c\}_{c=1}^6$, one obtains $-i\gamma(iZ_7) \equiv 0$ for any root $-i\gamma \in \triangle(\mathfrak{k}, i\tilde{\mathfrak{h}}_{\mathbb{R}})$. So, the element iZ_7 is a central element of $\mathfrak{k}$. Hence, the supposition and Lemma 4.2.3, combined with (5.6.13), assure that one of the following cases only occurs:

$$\text{(c}'\text{-1.1) } T = i(Z_1 + \lambda \cdot Z_7), \qquad \text{(c}'\text{-1.2) } T = i(Z_6 + \lambda \cdot Z_7), \qquad \text{(c}'\text{-2) } T = i\,\lambda \cdot Z_7$$

because $T \in i\tilde{\mathfrak{h}}_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{iZ_a\}_{a=1}^7$ and $\alpha_a(Z_b) = \delta_{a,b}$. Here, λ is a real number ($\lambda \neq 0$ in Case (c′-2)). Since T is an *Spr*-element, it must satisfy $\beta(T) = \pm i$ for any root $\beta \in \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_T(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ (see Lemma 4.1.1). Therefore, by virtue of $\alpha_a(Z_b) = \delta_{a,b}$ and (5.6.4), the value of λ can be determined as follows: $\lambda = -1$ in two Cases (c′-1.1) and (c′-1.2), and $\lambda = \pm 1$ in Case (c′-2). For the reasons, the element T is one of the following:

$$\text{(c-1.1) } i(Z_1 - Z_7), \qquad \text{(c-1.2) } i(Z_6 - Z_7), \qquad \text{(c-2) } \pm iZ_7.$$

Conversely, if an element T' is one of the above elements, then it satisfies $\beta(T') = \pm i$ for all roots $\beta \in \triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}}) \setminus \triangle_{T'}(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})$ because of $\alpha_a(Z_b) = \delta_{a,b}$ and (5.6.4). So, the element T' is an Spr-element of $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ (see Lemma 4.1.1 again). Consequently, we have shown Lemma 5.6.7. □

Lemma 5.6.7 and (4.1.2) enable us to deduce that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ [i(Z_1 - Z_7)], [i(Z_6 - Z_7)], [iZ_7] \}, \tag{5.6.14}$$

where $\mathfrak{g} = \mathfrak{e}_{7(-25)}$. The following lemma means that the above Spr-element $i(Z_1 - Z_7)$ is equivalent to $i(Z_6 - Z_7)$.

Lemma 5.6.8. *In the setting on Paragraph* 5.6.3; *there exists an automorphism* φ *of* $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ *such that*

$$\begin{cases} \varphi(iZ_1) = i(Z_6 - 2Z_7), & \varphi(iZ_2) = i(Z_2 - 2Z_7), \quad \varphi(iZ_3) = i(Z_5 - 3Z_7), \\ \varphi(iZ_4) = i(Z_4 - 4Z_7), & \varphi(iZ_5) = i(Z_3 - 3Z_7), \quad \varphi(iZ_6) = i(Z_1 - 2Z_7), \\ \varphi(iZ_7) = -iZ_7. \end{cases}$$

Proof. The automorphism φ in Lemma 5.6.3 satisfies

$$\begin{cases} \varphi(iZ_1) = i(Z_6 - 2Z_7), & \varphi(iZ_2) = i(Z_2 - 2Z_7), \quad \varphi(iZ_3) = i(Z_5 - 3Z_7), \\ \varphi(iZ_4) = i(Z_4 - 4Z_7), & \varphi(iZ_5) = i(Z_3 - 3Z_7), \quad \varphi(iZ_6) = i(Z_1 - 2Z_7), \\ \varphi(iZ_7) = -iZ_7. \end{cases}$$

Therefore, it suffices to confirm that the automorphism φ of $\mathfrak{g}_u$ is an automorphism of $\mathfrak{g} = \mathfrak{e}_{7(-25)}$. Since $\varphi(iZ_7) = -iZ_7$, and since θ_3 is involutive and (5.6.12), φ is commutative with θ_3. Accordingly, Proposition 2.2.3 implies that φ is an automorphism of $\mathfrak{g} = \mathfrak{e}_{7(-25)}$.[26] Hence, Lemma 5.6.8 has been proved. □

Now, let us demonstrate Proposition 5.6.9.

Proposition 5.6.9. *Under our equivalence relation, Spr-elements of* EVII: $\mathfrak{g} = \mathfrak{e}_{7(-25)}$ *are classified as follows*:

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ [i(Z_6 - Z_7)], [iZ_7] \}.$$

Besides, (1) $(\mathfrak{g}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1)$ *and* (2) $(\mathfrak{g}, \mathfrak{e}_6 \oplus \mathfrak{t}^1)$ *are the pseudo-Hermitian symmetric Lie algebra by an involution* $\rho_1 = \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_6 - Z_7)$ *and* $\rho_2 = \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_7$, *respectively. Here,* $\{Z_a\}_{a=1}^7$ *is the dual basis of* $\Pi_{\triangle(\mathfrak{e}_7^{\mathbb{C}}, \tilde{\mathfrak{h}})} = \{\alpha_a\}_{a=1}^7$.

Proof. It follows from (5.6.14) and Lemma 5.6.8 that

$$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) = \{ [i(Z_6 - Z_7)], [iZ_7] \}.$$

It is known that

$$\begin{aligned} &\mathfrak{c}_{\mathfrak{g}}(i(Z_6 - Z_7)) = \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1, \\ &\mathfrak{c}_{\mathfrak{g}}(iZ_7) = \mathfrak{e}_6 \oplus \mathfrak{t}^1 \end{aligned} \tag{5.6.15}$$

(ref. Theorem 6.16 in [Bm]). By (5.6.15), we perceive that the Spr-elements $i(Z_6 - Z_7)$ and iZ_7 are not equivalent to each other. Moreover, from (5.6.15) and Lemma 3.1.1 we deduce that $(\mathfrak{g}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1)$ and $(\mathfrak{g}, \mathfrak{e}_6 \oplus \mathfrak{t}^1)$ are the pseudo-Hermitian symmetric Lie algebra by an involution $\rho_1 := \exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_6 - Z_7)$ and $\rho_2 := \exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_7$, respectively. Hence, we have got the conclusion. □

[26] This φ is an outer automorphism of $\mathfrak{e}_{7(-25)}$.

The results obtained so far are as follows:

Table VI.

		EV
15	$\mathfrak{g}$	$\mathfrak{e}_{7(7)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_2+Z_6)\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_2+Z_6))$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$
		EVI
16	$\mathfrak{g}$	$\mathfrak{e}_{7(-5)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_1+Z_2)\,]$, $[\,i(-Z_1+Z_7)\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_1+Z_2))$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(i(-Z_1+Z_7))$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1$
		EVII
17	$\mathfrak{g}$	$\mathfrak{e}_{7(-25)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_6-Z_7)\,]$, $[\,iZ_7\,]$
	$\mathfrak{c}_{\mathfrak{g}}(i(Z_6-Z_7))$	$\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1$
	$\mathfrak{c}_{\mathfrak{g}}(iZ_7)$	$\mathfrak{e}_6 \oplus \mathfrak{t}^1$

By Proposition 4.1.2, and by collecting Tables I through VI, we assert the following theorem:

Theorem 5.6.10. *Under our equivalence relation, Spr-elements of* $\mathfrak{g}$ *are classified as follows*:

		AI
1	$\mathfrak{g}$	$\mathfrak{sl}(2k+1,\mathbb{R}) : k \geq 1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	*None*
2	$\mathfrak{g}$	$\mathfrak{sl}(2k,\mathbb{R}) : k \geq 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_k\,]$
		AII
3	$\mathfrak{g}$	$\mathfrak{su}^*(2k) : k \geq 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_k\,]$
		AIII

4-1	$\mathfrak{g}$	$\mathfrak{su}(j, l+1-j) : l \geq 1,\ j = 1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_{j+b})\,],\ [\,iZ_j\,],\ j \leq b \leq l-j$
4-2	$\mathfrak{g}$	$\mathfrak{su}(j, l+1-j) : l \geq 3,\ 2 \leq j \leq l-1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_a - Z_j)\,],\ [\,i(-Z_j + Z_{j+b})\,],$ $[\,i(Z_c - Z_j + Z_{j+d})\,],\ [\,iZ_j\,],$ $1 \leq a \leq j-1,\ 1 \leq b \leq l-j,$ $1 \leq c \leq [(j-2)/2]+1,$ $1 \leq d \leq l-j :$ *if* $l+1 \neq 2j$
		$[\,i(Z_a - Z_j)\,],\ [\,i(Z_c - Z_j + Z_{j+d})\,],\ [\,iZ_j\,],$ $1 \leq a \leq j-1,$ $1 \leq c \leq [(j-2)/2]+1,$ $c \leq d \leq j-c :$ *if* $l+1 = 2j$
		BI
5	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j+1) : l \geq 2,\ 1 \leq j \leq l$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1} - Z_j)\,],\ [\,i(-Z_j + Z_{j+1})\,]$ *where* $Z_0 = 0$: *if* $1 \leq j \leq l-1$
		$[\,i(Z_{j-1} - Z_j)\,]$: *if* $j = l$
		CI
6	$\mathfrak{g}$	$\mathfrak{sp}(l, \mathbb{R}) : l \geq 3$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_d - Z_l)\,],\ [\,iZ_l\,],\ 1 \leq d \leq [l/2]$
		CII
7	$\mathfrak{g}$	$\mathfrak{sp}(j, l-j) : l \geq 3,\ 1 \leq j \leq l-1$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_l)\,]$
		DI
8-1	$\mathfrak{g}$	$\mathfrak{so}(2j+1, 2l-2j-1) : l \geq 4,\ j = 0$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_{j+1})\,]$ *where* $Z_0 = 0$
8-2	$\mathfrak{g}$	$\mathfrak{so}(2j+1, 2l-2j-1) : l \geq 4,\ 1 \leq j \leq l-3$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_1\,],\ [\,i(-Z_j + Z_{j+1})\,]$: *if* $2j+1 \neq l$
		$[\,iZ_1\,]$: *if* $2j+1 = l$
9-1	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l \geq 4,\ j = 1$

	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1}-Z_j)\,]$, $[\,i(-Z_j+Z_{j+1})\,]$, $[\,i(-Z_j+Z_l)\,]$ *where* $Z_0 = 0$
9-2	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l = 4,\ j = 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_j + Z_{j+1})\,]$
9-3	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l \geq 5,\ j = 2$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1}-Z_j)\,]$, $[\,i(-Z_j+Z_{j+1})\,]$, $[\,i(-Z_j+Z_l)\,]$
9-4	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l \geq 6,\ 3 \leq j \leq l-3$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_{j-1} - Z_j)\,]$, $[\,i(-Z_j + Z_{j+1})\,]$, $[\,i(-Z_j + Z_l)\,]$: *if* $2j \neq l$
		$[\,i(Z_{j-1} - Z_j)\,]$, $[\,i(-Z_j + Z_l)\,]$: *if* $2j = l$
		DIII
10	$\mathfrak{g}$	$\mathfrak{so}^*(2l) : l \geq 4$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,iZ_1\,]$, $[\,i(Z_c - Z_l)\,]$, $[\,iZ_l\,]$, $1 \leq c \leq [l/2]$: *if* $l \neq 4$
		$[\,i(Z_c - Z_l)\,]$, $[\,iZ_l\,]$, $1 \leq c \leq [l/2]$: *if* $l = 4$
		EI
11	$\mathfrak{g}$	$\mathfrak{e}_{6(6)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	*None*
		EII
12	$\mathfrak{g}$	$\mathfrak{e}_{6(2)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_2 - Z_3)\,]$, $[\,i(Z_1 - Z_3)\,]$
		EIII
13	$\mathfrak{g}$	$\mathfrak{e}_{6(-14)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_1 + Z_3)\,]$, $[\,iZ_6\,]$, $[\,iZ_1\,]$
		EIV
14	$\mathfrak{g}$	$\mathfrak{e}_{6(-26)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$	*None*
		EV
15	$\mathfrak{g}$	$\mathfrak{e}_{7(7)}$

	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_2+Z_6)\,]$
		EVI
16	$\mathfrak{g}$	$\mathfrak{e}_{7(-5)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(-Z_1+Z_2)\,]$, $[\,i(-Z_1+Z_7)\,]$
		EVII
17	$\mathfrak{g}$	$\mathfrak{e}_{7(-25)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	$[\,i(Z_6-Z_7)\,]$, $[\,iZ_7\,]$
		EVIII
18	$\mathfrak{g}$	$\mathfrak{e}_{8(8)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	*None*
		EIX
19	$\mathfrak{g}$	$\mathfrak{e}_{8(-24)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	*None*
		FI
20	$\mathfrak{g}$	$\mathfrak{f}_{4(4)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	*None*
		FII
21	$\mathfrak{g}$	$\mathfrak{f}_{4(-20)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	*None*
		G
22	$\mathfrak{g}$	$\mathfrak{g}_{2(2)}$
	$Spr_{\mathfrak{g}}/(\{\pm 1\}\times \mathrm{Aut}(\mathfrak{g}))$	*None*

Two Theorems 3.2.1 and 5.6.10 allow us to lead the following:

Corollary 5.6.11. *Under Berger's equivalence relation, simple irreducible pseudo-Hermitian symmetric Lie algebras* $(\mathfrak{g},\rho)$ *are classified as follows* (*where* $\mathfrak{r}$ *denotes the* $+1$*-eigenspace of* ρ *in* $\mathfrak{g}$) :

		AI
1	$\mathfrak{g}$	$\mathfrak{sl}(2k+1,\mathbb{R}) : k\geq 1$

	ρ	*None*
2	$\mathfrak{g}$	$\mathfrak{sl}(2k,\mathbb{R}) : k \geq 2$
	ρ	$\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_k$
	$\mathfrak{r}$	$\mathfrak{sl}(k,\mathbb{C}) \oplus \mathfrak{t}^1$
		AII
3	$\mathfrak{g}$	$\mathfrak{su}^*(2k) : k \geq 2$
	ρ	$\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_k$
	$\mathfrak{r}$	$\mathfrak{sl}(k,\mathbb{C}) \oplus \mathfrak{t}^1$
		AIII
4-1	$\mathfrak{g}$	$\mathfrak{su}(j, l+1-j) : l \geq 1,\ j = 1$
	ρ	(2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+b}),\ j \leq b \leq l-j;$ (4) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_j$
	$\mathfrak{r}$	(2) $\mathfrak{su}(b) \oplus \mathfrak{su}(l+1-j-b, j) \oplus \mathfrak{t}^1$
		(4) $\mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1$
4-2	$\mathfrak{g}$	$\mathfrak{su}(j, l+1-j) : l \geq 3,\ 2 \leq j \leq l-1$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_a - Z_j),\ 1 \leq a \leq j-1;$ (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+b}),\ 1 \leq b \leq l-j;$ (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_c - Z_j + Z_{j+d}),$ $1 \leq c \leq [(j-2)/2]+1,$ $1 \leq d \leq l-j;$ (4) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_j :$ *if* $l+1 \neq 2j$
		(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_a - Z_j),\ 1 \leq a \leq j-1;$ (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_c - Z_j + Z_{j+d}),$ $1 \leq c \leq [(j-2)/2]+1,$ $c \leq d \leq j-c;$ (4) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_j :$ *if* $l+1 = 2j$
	$\mathfrak{r}$	(1) $\mathfrak{su}(j-a) \oplus \mathfrak{su}(l+1-j, a) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{su}(b) \oplus \mathfrak{su}(l+1-j-b, j) \oplus \mathfrak{t}^1$
		(3) $\mathfrak{su}(c,d) \oplus \mathfrak{su}(j-c, l+1-j-d) \oplus \mathfrak{t}^1$
		(4) $\mathfrak{su}(j) \oplus \mathfrak{su}(l+1-j) \oplus \mathfrak{t}^1$
		BI

5	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j+1) : l \geq 2,\ 1 \leq j \leq l$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_{j-1} - Z_j),$ *where* $Z_0 = 0$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$: *if* $1 \leq j \leq l-1$
		(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_{j-1} - Z_j)$: *if* $j = l$
	$\mathfrak{r}$	(1) $\mathfrak{so}(2j-2, 2l-2j+1) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}(2j, 2l-2j-1) \oplus \mathfrak{t}^1$
		CI
6	$\mathfrak{g}$	$\mathfrak{sp}(l, \mathbb{R}) : l \geq 3$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_d - Z_l),\ 1 \leq d \leq [l/2]$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_l$
	$\mathfrak{r}$	(1) $\mathfrak{su}(d, l-d) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{su}(l) \oplus \mathfrak{t}^1$
		CII
7	$\mathfrak{g}$	$\mathfrak{sp}(j, l-j) : l \geq 3,\ 1 \leq j \leq l-1$
	ρ	$\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$
	$\mathfrak{r}$	$\mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$
		DI
8-1	$\mathfrak{g}$	$\mathfrak{so}(2j+1, 2l-2j-1) : l \geq 4,\ j = 0$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1}),$ *where* $Z_0 = 0$
	$\mathfrak{r}$	(1) $\mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1$
8-2	$\mathfrak{g}$	$\mathfrak{so}(2j+1, 2l-2j-1) : l \geq 4,\ 1 \leq j \leq l-3$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$: *if* $2j+1 \neq l$
		(2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$: *if* $2j+1 = l$
	$\mathfrak{r}$	(1) $\mathfrak{so}(2j+1, 2l-2j-3) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}(2j-1, 2l-2j-1) \oplus \mathfrak{t}^1$
9-1	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l \geq 4,\ j = 1$

	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_{j-1} - Z_j)$, *where* $Z_0 = 0$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$
	$\mathfrak{r}$	(1) $\mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
		(3) $\mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$
9-2	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l = 4,\ j = 2$
	ρ	(2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$
	$\mathfrak{r}$	(2) $\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
9-3	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l \geq 5,\ j = 2$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_{j-1} - Z_j)$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$
	$\mathfrak{r}$	(1) $\mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
		(3) $\mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$
9-4	$\mathfrak{g}$	$\mathfrak{so}(2j, 2l-2j) : l \geq 6,\ 3 \leq j \leq l-3$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_{j-1} - Z_j)$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_{j+1})$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$: *if* $2j \neq l$
		(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_{j-1} - Z_j)$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_j + Z_l)$: *if* $2j = l$
	$\mathfrak{r}$	(1) $\mathfrak{so}(2j-2, 2l-2j) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}(2j, 2l-2j-2) \oplus \mathfrak{t}^1$
		(3) $\mathfrak{su}(j, l-j) \oplus \mathfrak{t}^1$

		DIII
10	$\mathfrak{g}$	$\mathfrak{so}^*(2l) : l \geq 4$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_c - Z_l)$, $1 \leq c \leq [l/2]$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_l$: *if* $l \neq 4$

		(2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_c - Z_l),\ 1 \leq c \leq [l/2]$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_l$: *if* $l = 4$
	$\mathfrak{r}$	(1) $\mathfrak{so}^*(2l-2) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{su}(c, l-c) \oplus \mathfrak{t}^1$
		(3) $\mathfrak{su}(l) \oplus \mathfrak{t}^1$
		EI
11	$\mathfrak{g}$	$\mathfrak{e}_{6(6)}$
	ρ	*None*
		EII
12	$\mathfrak{g}$	$\mathfrak{e}_{6(2)}$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_2 - Z_3)$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_1 - Z_3)$
	$\mathfrak{r}$	(1) $\mathfrak{so}(6,4) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}^*(10) \oplus \mathfrak{t}^1$
		EIII
13	$\mathfrak{g}$	$\mathfrak{e}_{6(-14)}$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_3)$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_6$; (3) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_1$
	$\mathfrak{r}$	(1) $\mathfrak{so}^*(10) \oplus \mathfrak{t}^1$
		(2) $\mathfrak{so}(8,2) \oplus \mathfrak{t}^1$
		(3) $\mathfrak{so}(10) \oplus \mathfrak{t}^1$
		EIV
14	$\mathfrak{g}$	$\mathfrak{e}_{6(-26)}$
	ρ	*None*
		EV
15	$\mathfrak{g}$	$\mathfrak{e}_{7(7)}$
	ρ	$\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_2 + Z_6)$
	$\mathfrak{r}$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$

		EVI
16	$\mathfrak{g}$	$\mathfrak{e}_{7(-5)}$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_2)$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(-Z_1 + Z_7)$
	$\mathfrak{r}$	(1) $\mathfrak{e}_{6(2)} \oplus \mathfrak{t}^1$
		(2) $\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1$
		EVII
17	$\mathfrak{g}$	$\mathfrak{e}_{7(-25)}$
	ρ	(1) $\exp \pi \operatorname{ad}_{\mathfrak{g}} i(Z_6 - Z_7)$; (2) $\exp \pi \operatorname{ad}_{\mathfrak{g}} iZ_7$
	$\mathfrak{r}$	(1) $\mathfrak{e}_{6(-14)} \oplus \mathfrak{t}^1$
		(2) $\mathfrak{e}_6 \oplus \mathfrak{t}^1$
		EVIII
18	$\mathfrak{g}$	$\mathfrak{e}_{8(8)}$
	ρ	*None*
		EIX
19	$\mathfrak{g}$	$\mathfrak{e}_{8(-24)}$
	ρ	*None*
		FI
20	$\mathfrak{g}$	$\mathfrak{f}_{4(4)}$
	ρ	*None*
		FII
21	$\mathfrak{g}$	$\mathfrak{f}_{4(-20)}$
	ρ	*None*
		G
22	$\mathfrak{g}$	$\mathfrak{g}_{2(2)}$
	ρ	*None*

Remark 5.6.12. At the table in Corollary 5.6.11, each element S satisfying $\rho = \exp \pi \operatorname{ad}_{\mathfrak{g}} S$ is the canonical central element of $\mathfrak{r}$ relative to $(\mathfrak{g}, \mathfrak{r})$.

6. A classification of simple irreducible pseudo-Hermitian symmetric spaces

In this section, we define an equivalence relation on the set of simple irreducible pseudo-Hermitian symmetric spaces $(G/R, \Sigma, J, \mathsf{g})$, and we give a correspondence between the equivalence relation on the set of $(G/R, \Sigma, J, \mathsf{g})$ and that on the set of *Spr*-elements of $\mathfrak{g} = \mathrm{Lie}(G)$ (see Theorem 6.2.1).

Remark 6.0.13 (Shapiro [Sh, pp. 532]). A simple pseudo-Hermitian symmetric space G/R is irreducible if and only if $\mathfrak{g} = \mathrm{Lie}(G)$ is a real form of a complex simple Lie algebra.

6.1. **An Equivalence relation.** Let us fix a connected Lie group G whose Lie algebra $\mathfrak{g}$ is a real form of a complex simple Lie algebra, and let us define an equivalence relation on the set of simple irreducible pseudo-Hermitian symmetric spaces. Let $(G/R_p, \Sigma_p, J_p, \mathsf{g}_p)$, $p = 1, 2$, be a pseudo-Hermitian symmetric space endowed with an invariant complex structure J_p and an invariant pseudo-Hermitian metric g_p (with respect to J_p), where Σ_p is an involutive automorphism of G such that $(G_{\Sigma_p})_0 \subset R_p \subset G_{\Sigma_p}$. Here, $(G_{\Sigma_p})_0$ denotes the identity component of $G_{\Sigma_p} := \{g \in G \,|\, \Sigma_p(g) = g\}$. In the setting, we say that $(G/R_1, \Sigma_1, J_1, \mathsf{g}_1)$ is *equivalent* to $(G/R_2, \Sigma_2, J_2, \mathsf{g}_2)$, if there exists an automorphism Φ of G satisfying the following four conditions:

- (C.1) $\Phi \circ \Sigma_1 \circ \Phi^{-1} = \Sigma_2$.
- (C.2) $\Phi(R_1) = R_2$.
- (C.3) A G-equivariant diffeomorphism $\bar{\Phi}$ of G/R_1 onto G/R_2, defined by $\bar{\Phi}(gR_1) := \Phi(g)R_2$ for $gR_1 \in G/R_1$, is holomorphic or anti-holomorphic—that is, $d\bar{\Phi} \circ J_1 = \pm J_2 \circ d\bar{\Phi}$.
- (C.4) There exists a non-zero number $\lambda \in \mathbb{R}$ such that $\bar{\Phi}^* \mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$.

This is an equivalence relation on the set of simple irreducible pseudo-Hermitian symmetric spaces $(G/R, \Sigma, J, \mathsf{g})$. In the next subsection, we will demonstrate that this equivalence relation corresponds to the equivalence relation on the set of *Spr*-elements of $\mathfrak{g} = \mathrm{Lie}(G)$. For the goal, let us verify the following:

Proposition 6.1.1. *Let $(G/R_1, \Sigma_1, J_1, \mathsf{g}_1)$ and $(G/R_2, \Sigma_2, J_2, \mathsf{g}_2)$ be two simple irreducible pseudo-Hermitian symmetric spaces. Then, $(G/R_1, \Sigma_1, J_1, \mathsf{g}_1)$ is equivalent to $(G/R_2, \Sigma_2, J_2, \mathsf{g}_2)$ if and only if there exists an automorphism Φ of G such that $\Phi \circ \Sigma_1 \circ \Phi^{-1} = \Sigma_2$.*

Proof. The necessary condition is obvious. So, we devote ourselves to proving the sufficient condition. Suppose that an automorphism Φ of G satisfies $\Phi \circ \Sigma_1 \circ \Phi^{-1} = \Sigma_2$. For $p = 1, 2$, there exists an *Spr*-element $S_p \in \mathfrak{g} = \mathrm{Lie}(G)$ such that

$$\text{(a) } R_p = C_G(S_p), \quad \text{(b) } J_p = J_{s_p}, \quad \text{(c) } \Sigma_p = \mathrm{A}_{\exp \pi S_p}$$

(cf. Remark 2.1.7). Since $R_p = C_G(S_p)$ are connected (ref. Shapiro [Sh, pp. 533]), the supposition "$\Phi \circ \Sigma_1 \circ \Phi^{-1} = \Sigma_2$" enables us to deduce that the automorphism Φ of G satisfies the condition (C.2) $\Phi(R_1) = R_2$. From now on, let us confirm that a diffeomorphism $\bar{\Phi}$ of G/R_1 onto G/R_2, $gR_1 \mapsto \Phi(g)R_2$, satisfies the condition

(C.3) $d\bar{\Phi} \circ J_1 = \pm J_2 \circ d\bar{\Phi}$. In order to do so, it is sufficient to confirm that $d\bar{\Phi} \circ J_1 = \pm J_2 \circ d\bar{\Phi}$ at the origin $o \in G/R_1$ because $\bar{\Phi}$ is G-equivariant and J_p are invariant. Identify the tangent space $T_o(G/R_1)$ (resp. $T_{\bar{\Phi}(o)}(G/R_2)$) with the -1-eigenspace $\mathfrak{q}_1 := [S_1, \mathfrak{g}]$ of $\exp \pi \operatorname{ad}_{\mathfrak{g}} S_1$ (resp. $\mathfrak{q}_2 := [S_2, \mathfrak{g}]$ of $\exp \pi \operatorname{ad}_{\mathfrak{g}} S_2$) in $\mathfrak{g}$, henceforth (see Lemma 3.1.1-(1) for $\mathfrak{q}_p = [S_p, \mathfrak{g}]$). Here, it follows from (c) that the differential homomorphism $(\Sigma_p)_*$ of Σ_p accords with $\exp \pi \operatorname{ad}_{\mathfrak{g}} S_p$. This identification allows us to assume that $(J_1)_o = \operatorname{ad}_{\mathfrak{g}} S_1|_{\mathfrak{q}_1}$ on $\mathfrak{q}_1 = T_o(G/R_1)$ and $(J_2)_{\bar{\Phi}(o)} = \operatorname{ad}_{\mathfrak{g}} S_2|_{\mathfrak{q}_2}$ on $\mathfrak{q}_2 = T_{\bar{\Phi}(o)}(G/R_2)$ (since (b)), and also allows us to assume that $(d\bar{\Phi})_o = \phi|_{\mathfrak{q}_1}$, where we denote by ϕ the differential homomorphism of Φ. Accordingly, it follows from $\phi(\mathfrak{q}_1) = \mathfrak{q}_2$ that, if $\phi \circ \operatorname{ad}_{\mathfrak{g}} S_1 = \pm \operatorname{ad}_{\mathfrak{g}} S_2 \circ \phi$, then the diffeomorphism $\bar{\Phi}$ of G/R_1 onto G/R_2 satisfies the condition (C.3). So, we want to show that $\phi \circ \operatorname{ad}_{\mathfrak{g}} S_1 = \pm \operatorname{ad}_{\mathfrak{g}} S_2 \circ \phi$. By the supposition and $(\Sigma_p)_* = \exp \pi \operatorname{ad}_{\mathfrak{g}} S_p$, it is clear that $\phi \circ \exp \pi \operatorname{ad}_{\mathfrak{g}} S_1 \circ \phi^{-1} = \exp \pi \operatorname{ad}_{\mathfrak{g}} S_2$. Thus, the proof of Theorem 3.2.1 enables us to have $\phi(S_1) = \pm S_2$. This shows that $\phi \circ \operatorname{ad}_{\mathfrak{g}} S_1 = \pm \operatorname{ad}_{\mathfrak{g}} S_2 \circ \phi$. For the reasons, the diffeomorphism $\bar{\Phi}$ satisfies the condition (C.3) $d\bar{\Phi} \circ J_1 = \pm J_2 \circ d\bar{\Phi}$. Consequently, the rest of proof is to verify that the condition (C.4) holds. So, let us verify that. Proposition 2.1 in Shapiro [Sh, pp. 530] states that $\mathsf{g}_1(J_1(u), v)$ $(u, v \in T(G/R_1))$ is an invariant symplectic form on G/R_1. Thus, Theorem 1 in Matsushima [Ma, pp. 54] and its proof imply that there exists an element $W_1 \in \mathfrak{g}$ satisfying

$$\begin{cases} \operatorname{Lie}(R_1) = \mathfrak{c}_{\mathfrak{g}}(W_1), \\ \mathsf{g}_1(J_1(X), Y)_o = B_{\mathfrak{g}}([W_1, X], Y) \quad \text{for any } X, Y \in \mathfrak{q}_1 = T_o(G/R_1) \end{cases}$$

(recall Notation (n4) in Subsection 2.4, for $B_{\mathfrak{g}}$). Therefore, since $\operatorname{Lie}(R_1) = \mathfrak{c}_{\mathfrak{g}}(S_1)$ and $\dim_{\mathbb{R}} \mathfrak{c}_{\mathfrak{g}}(W_1)_{\mathfrak{z}} = \dim_{\mathbb{R}} \mathfrak{c}_{\mathfrak{g}}(S_1)_{\mathfrak{z}} = 1$ (cf. Shapiro [Sh, pp. 532]), there exists a non-zero number $\lambda_1 \in \mathbb{R}$ such that $W_1 = \lambda_1 \cdot S_1$; and hence $\mathsf{g}_1(J_1(X), Y)_o = \lambda_1 \cdot B_{\mathfrak{g}}([S_1, X], Y)$ for any $X, Y \in \mathfrak{q}_1$. This, together with $(J_1)_o = \operatorname{ad}_{\mathfrak{g}} S_1|_{\mathfrak{q}_1}$, deduces that for all vectors $X, Y \in \mathfrak{q}_1$

$$\begin{aligned} \mathsf{g}_1(X, Y)_o &= \mathsf{g}_1(J_1(X), J_1(Y))_o \\ &= \lambda_1 \cdot B_{\mathfrak{g}}([S_1, X], [S_1, Y]) \\ &= \lambda_1 \cdot B_{\mathfrak{g}}(X, Y) \end{aligned}$$

because $(\operatorname{ad}_{\mathfrak{g}} S_1)^2(Q) = -Q$ for any $Q \in \mathfrak{q}_1 = [S_1, \mathfrak{g}]$ and g_1 is a pseudo-Hermitian metric with respect to J_1. Similarly one sees that there exists a non-zero number $\lambda_2 \in \mathbb{R}$ satisfying

$$\mathsf{g}_2(X', Y')_{\bar{\Phi}(o)} = \lambda_2 \cdot B_{\mathfrak{g}}(X', Y') \quad \text{for any } X', Y' \in \mathfrak{q}_2 = T_{\bar{\Phi}(o)}(G/R_2).$$

Hence, it follows from $(d\bar{\Phi})_o = \phi|_{\mathfrak{q}_1}$ and $\phi \in \operatorname{Aut}(\mathfrak{g})$ that

$$\begin{aligned} \mathsf{g}_2\big(d\bar{\Phi}(X), d\bar{\Phi}(Y)\big)_{\bar{\Phi}(o)} &= \lambda_2 \cdot B_{\mathfrak{g}}\big(\phi(X), \phi(Y)\big) \\ &= \lambda_2 \cdot B_{\mathfrak{g}}\big(X, Y\big) \\ &= \frac{\lambda_2}{\lambda_1} \cdot \mathsf{g}_1(X, Y)_o \end{aligned}$$

for all vectors $X, Y \in \mathfrak{q}_1 = T_o(G/R_1)$. Hence, $\lambda := \lambda_2/\lambda_1$ is a non-zero real number such that $\bar{\Phi}^* \mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$ at the origin $o \in G/R_1$. Since $\bar{\Phi}$ is G-equivariant and g_p are invariant, we conclude that $\bar{\Phi}^* \mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$ on G/R_1. Thus, the condition (C.4) holds. Therefore, we have completed the proof of Proposition 6.1.1. □

The proof of Proposition 6.1.1 allows us to assert the following:

Lemma 6.1.2. *For every simple irreducible pseudo-Hermitian symmetric space* $(G/R, \Sigma, J, \mathsf{g})$*, there exist an Spr-element* $S \in \mathfrak{g} = \mathrm{Lie}(G)$ *and a non-zero number* $\lambda \in \mathbb{R}$ *such that*

$$(G/R, \Sigma, J, \mathsf{g}) = (G/C_G(S), \mathrm{A}_{\exp \pi S}, J_s, \lambda \cdot \mathsf{g}_{B_\mathfrak{g}}).$$

Here, J_s *is given in Remark* 2.1.7*, and* $\mathsf{g}_{B_\mathfrak{g}}$ *is given in Lemma* 3.1.1.

6.2. **A Correspondence.** Fix a connected Lie group G whose Lie algebra $\mathfrak{g}$ is a real form of a complex simple Lie algebra, and denote by PHSP_G the set of pseudo-Hermitian symmetric spaces $(G/R, \Sigma, J, \mathsf{g})$. Let $\mathrm{PHSP}_G / \mathrm{Aut}(G)$ be the quotient set of PHSP_G by the equivalence relation defined in Subsection 6.1. With the notation, we will demonstrate the following:

Theorem 6.2.1. *Suppose that all automorphisms of* $\mathfrak{g}$ *can be lifted to* G*. Then, the following mapping* F_2 *is a bijection of* $Spr_\mathfrak{g}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g}))$ *onto* $\mathrm{PHSP}_G / \mathrm{Aut}(G)$ *:*

$$\begin{array}{rcll} F_2 : Spr_\mathfrak{g}/(\{\pm 1\} \times \mathrm{Aut}(\mathfrak{g})) & \longrightarrow & \mathrm{PHSP}_G / \mathrm{Aut}(G) & \text{(bijective)} \\ [\,S\,] & \mapsto & [\,(G/C_G(S), \mathrm{A}_{\exp \pi S}, J_s, \mathsf{g}_{B_\mathfrak{g}})\,]. & \end{array}$$

Here, J_s *is given in Remark* 2.1.7*, and* $\mathsf{g}_{B_\mathfrak{g}}$ *is given in Lemma* 3.1.1.

Remark 6.2.2. If G is simply connected or if G is the adjoint group of $\mathfrak{g}$, then the hypothesis in Theorem 6.2.1 is always satisfied (ref. the proof of Lemma in Oshima and Sekiguchi [Os-Se, pp. 436–437]).

Proof of Theorem 6.2.1. Note that for every *Spr*-element $S \in \mathfrak{g} = \mathrm{Lie}(G)$, the quartet $(G/C_G(S), \mathrm{A}_{\exp \pi S}, J_s, \mathsf{g}_{B_\mathfrak{g}})$ is a pseudo-Hermitian symmetric space (recall Lemma 3.1.1-(3)). First, let us show that the mapping F_2 is well-defined and is injective. Let S_1 and S_2 be two *Spr*-elements of $\mathfrak{g}$. By the hypothesis, Proposition 6.1.1 and Theorem 3.2.1, we comprehend that the following four conditions are mutually equivalent:

(1) S_1 is equivalent to S_2.
(2) There exists an automorphism ϕ of $\mathfrak{g}$ such that $\phi \circ \exp \pi \operatorname{ad}_\mathfrak{g} S_1 \circ \phi^{-1} = \exp \pi \operatorname{ad}_\mathfrak{g} S_2$.
(3) There exists an automorphism Φ of G such that $\Phi \circ \mathrm{A}_{\exp \pi S_1} \circ \Phi^{-1} = \mathrm{A}_{\exp \pi S_2}$.
(4) $(G/C_G(S_1), \mathrm{A}_{\exp \pi S_1}, J_{s_1}, \mathsf{g}_{B_\mathfrak{g}})$ is equivalent to $(G/C_G(S_2), \mathrm{A}_{\exp \pi S_2}, J_{s_2}, \mathsf{g}_{B_\mathfrak{g}})$.

Therefore, the mapping F_2 is well-defined and is injective. Lemma 6.1.2 implies that the mapping F_2 is surjective. For the reasons, the mapping F_2 is bijective. So, we have got the conclusion. □

6.3. **About the equivalence relation defined in Subsection 6.1.** Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra, and let G be the adjoint group of $\mathfrak{g}$. In this case, we can restate the equivalence relation defined in Subsection 6.1 as follows:

Proposition 6.3.1. *With the above assumption; let $(G/R_p, \Sigma_p, J_p, \mathrm{g}_p)$ be two pseudo-Hermitian symmetric spaces $(p = 1, 2)$. Then, $(G/R_1, \Sigma_1, J_1, \mathrm{g}_1)$ is equivalent to $(G/R_2, \Sigma_2, J_2, \mathrm{g}_2)$ if and only if there exist a non-zero number $\lambda \in \mathbb{R}$ and a diffeomorphism f of G/R_1 onto G/R_2 which satisfy $f^*\mathrm{g}_2 = \lambda \cdot \mathrm{g}_1$.*

In order to prove Proposition 6.3.1, we first demonstrate Lemma 6.3.2.

Lemma 6.3.2. *Let $(L_p/H_p, \Sigma_p)$ be an irreducible symmetric space defined by an involution Σ_p of L_p, where L_p is a connected, semisimple Lie group and it is effective on L_p/H_p $(p = 1, 2)$, and let f be a diffeomorphism of L_1/H_2 onto L_2/H_2 which satisfies $f(o_1) = o_2$ and*

$$df({\nabla^1}_X Y) = {\nabla^2}_{df(X)} df(Y) \quad \text{for all vector fields } X, Y \text{ on } L_1/H_1,$$

where o_p denotes the origin of L_p/H_p and ∇^p denotes the canonical affine connection on L_p/H_p $(p = 1, 2)$. Then, there exists a Lie group isomorphism Φ of L_1 onto L_2 satisfying three conditions

(i) $\Phi \circ \Sigma_1 = \Sigma_2 \circ \Phi$,
(ii) $\Phi(H_1) = H_2$,
(iii) $f = \bar{\Phi}$, *where $\bar{\Phi}$ is an L_1-equivariant diffeomorphism of L_1/H_1 onto L_2/H_2 defined by $\bar{\Phi}(a_1 H_1) := \Phi(a_1) H_2$ for $a_1 H_1 \in L_1/H_1$.*

Proof. Denote by $\bar{\Sigma}_p$ an involutive, affine transformation of $(L_p/H_p, \nabla^p)$ defined by $\bar{\Sigma}_p(aH_p) := \Sigma_p(a)H_p$ for $aH_p \in L_p/H_p$ $(p = 1, 2)$. Then, it follows from $\bar{\Sigma}_p(o_p) = o_p$ and $f(o_1) = o_2$ that

$$(f \circ \bar{\Sigma}_1 \circ f^{-1})(o_2) = o_2 = \bar{\Sigma}_2(o_2).$$

Besides, since $d\bar{\Sigma}_p = -\operatorname{id}$ on $T_{o_p}(L_p/H_p)$, one deduces that $(d(f \circ \bar{\Sigma}_1 \circ f^{-1}))_{o_2} = (d\bar{\Sigma}_2)_{o_2}$. Therefore, the uniqueness of affine transformation (cf. Nomizu [No1, Lemma 6, pp. 820]) assures that

$$f \circ \bar{\Sigma}_1 \circ f^{-1} = \bar{\Sigma}_2. \tag{6.3.1}$$

Take an element $l \in L_p$, and denote by $\tau_p(l)$ a transformation of L_p/H_p defined by $\tau_p(l)\big(aH_p\big) := laH_p$ for $aH_p \in L_p/H_p$ $(p = 1, 2)$. Then, for any $l \in L_p$ and $aH_p \in L_p/H_p$, one obtains $\tau_p(\Sigma_p(l))\big(aH_p\big) = \Sigma_p(l)aH_p = \bar{\Sigma}_p(\tau_p(l)\big(\bar{\Sigma}_p(aH_p)\big))$ by virtue of Σ_p being an involutive automorphism of L_p; and thus

$$\tau_p(\Sigma_p(l)) = \bar{\Sigma}_p \circ \tau_p(l) \circ \bar{\Sigma}_p \quad \text{for all } l \in L_p \ (p = 1, 2). \tag{6.3.2}$$

Let $\operatorname{Aut}(L_p/H_p, \nabla^p)$ denote the group of affine transformations of $(L_p/H_p, \nabla^p)$ $(p = 1, 2)$. For each $\phi_1 \in \operatorname{Aut}(L_1/H_1, \nabla^1)$, it is clear that $f \circ \phi_1 \circ f^{-1} \in \operatorname{Aut}(L_2/H_2, \nabla^2)$. Hence, one can define a bijection Φ' of $\operatorname{Aut}(L_1/H_1, \nabla^1)$ onto $\operatorname{Aut}(L_2/H_2, \nabla^2)$ by

$$\begin{aligned} \Phi' : \operatorname{Aut}(L_1/H_1, \nabla^1) &\to \operatorname{Aut}(L_2/H_2, \nabla^2) \\ \phi_1 &\mapsto f \circ \phi_1 \circ f^{-1}. \end{aligned}$$

This mapping Φ' is a homeomorphic homomorphism of $\mathrm{Aut}(L_1/H_1, \nabla^1)$ onto $\mathrm{Aut}(L_2/H_2, \nabla^2)$, where $\mathrm{Aut}(L_p/H_p, \nabla^p)$ has the compact-open topology. Now, the hypothesis in this lemma, and Theorem 16.1 in Nomizu [No2] enable us to perceive that

$$\tau_p : l \mapsto \tau_p(l), \text{ is a Lie group isomorphism of } L_p \text{ onto } \mathrm{Aut}_0(L_p/H_p, \nabla^p).$$

Here, $\mathrm{Aut}_0(L_p/H_p, \nabla^p)$ denotes the identity component of $\mathrm{Aut}(L_p/H_p, \nabla^p)$. Accordingly, we can conclude that

(6.3.3) $\Phi' : \tau_1(l_1) \mapsto f \circ \tau_1(l_1) \circ f^{-1}$, is a Lie group isomorphism of $\tau_1(L_1) = \mathrm{Aut}_0(L_1/H_1, \nabla^1)$ onto $\tau_2(L_2) = \mathrm{Aut}_0(L_2/H_2, \nabla^2)$.

Let us define a Lie group isomorphism Φ of L_1 onto L_2 by setting

$$\Phi := \tau_2^{-1} \circ \Phi' \circ \tau_1.$$

First, we will prove that this isomorphism satisfies the condition (i) $\Phi \circ \Sigma_1 = \Sigma_2 \circ \Phi$. For any element $l_1 \in L_1$, there exists an element $l_2 \in L_2$ such that

$$f \circ \tau_1(l_1) \circ f^{-1} = \tau_2(l_2)$$

because of (6.3.3). Then, it follows from (6.3.1) and (6.3.2) that

$$\begin{aligned} f \circ \tau_1(\Sigma_1(l_1)) \circ f^{-1} &= f \circ (\bar{\Sigma}_1 \circ \tau_1(l_1) \circ \bar{\Sigma}_1) \circ f^{-1} \\ &= \bar{\Sigma}_2 \circ f \circ \tau_1(l_1) \circ f^{-1} \circ \bar{\Sigma}_2 \\ &= \bar{\Sigma}_2 \circ \tau_2(l_2) \circ \bar{\Sigma}_2 \\ &= \tau_2(\Sigma_2(l_2)). \end{aligned}$$

From this, one obtains the following:

$$\begin{aligned} \Phi(\Sigma_1(l_1)) &= \tau_2^{-1}\big(f \circ \tau_1(\Sigma_1(l_1)) \circ f^{-1}\big) = \tau_2^{-1}\big(\tau_2(\Sigma_2(l_2))\big) = \Sigma_2(l_2); \\ \Sigma_2(\Phi(l_1)) &= \Sigma_2\big(\tau_2^{-1}(f \circ \tau_1(l_1) \circ f^{-1})\big) = \Sigma_2\big(\tau_2^{-1}(\tau_2(l_2))\big) = \Sigma_2(l_2). \end{aligned}$$

Consequently, we have shown that Φ satisfies (i) $\Phi \circ \Sigma_1 = \Sigma_2 \circ \Phi$. Here, it has been also shown that $\Phi(l_1) = l_2$ for all elements l_1, l_2 which satisfy $f \circ \tau_1(l_1) \circ f^{-1} = \tau_2(l_2)$. Next, let us verify that

$$\pi_2 \circ \Phi = f \circ \pi_1, \tag{6.3.4}$$

where π_p denotes the projection of L_p onto L_p/H_p $(p = 1, 2)$. For any element $l_1 \in L_1$, there exists an element $l_2 \in L_2$ such that $f \circ \tau_1(l_1) \circ f^{-1} = \tau_2(l_2)$. Then, one has $\Phi(l_1) = l_2$; and therefore

$$\begin{aligned} \pi_2(\Phi(l_1)) &= \pi_2(l_2) = l_2 H_2 = \tau_2(l_2)\big(o_2\big) \\ &= (f \circ \tau_1(l_1) \circ f^{-1})(o_2) = f(\tau_1(l_1)\big(o_1\big)) = f(\pi_1(l_1)). \end{aligned}$$

Hence, we have verified (6.3.4). This (6.3.4) leads the condition (iii) $f = \bar{\Phi}$. The rest of proof is to show that Φ satisfies the condition (ii) $\Phi(H_1) = H_2$. For each element $h_1 \in H_1$, it follows from $\pi_1(h_1) = o_1$ and (6.3.4) that $\pi_2(\Phi(h_1)) = f(\pi_1(h_1)) = f(o_1) = o_2$. Thus, $\Phi(h_1) \in \pi_2^{-1}(o_2) = H_2$, and one has $\Phi(H_1) \subset H_2$. On the other hand, for any element $h_2 \in H_2$ we have $\pi_2(h_2) = o_2$, and so $\pi_1(\Phi^{-1}(h_2)) = f^{-1}(\pi_2(h_2)) = f^{-1}(o_2) = o_1$. Therefore, it is deduced that $\Phi^{-1}(h_2) \in \pi_1^{-1}(o_1) = H_1$, so that $H_2 \subset \Phi(H_1)$. For the reasons, we see that $\Phi(H_1) = H_2$. Consequently, we have demonstrated Lemma 6.3.2. □

Now, we are in a position to prove Proposition 6.3.1.

Poof of Proposition 6.3.1. We identify $\mathfrak{g}$ with the Lie algebra of G. The necessary condition is obvious (recall the condition (C.4) for $(G/R_1, \Sigma_1, J_1, \mathsf{g}_1)$ to be equivalent to $(G/R_2, \Sigma_2, J_2, \mathsf{g}_2)$). So, we will only prove the sufficient condition. Suppose that there exist a non-zero number $\lambda \in \mathbb{R}$ and a diffeomorphism f of G/R_1 onto G/R_2 which satisfy $f^*\mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$. Then, Theorem 15.5 in Nomizu [No2], together with $f^*\mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$, implies that f is an affine diffeomorphism of $(G/R_1, \nabla^1)$ onto $(G/R_2, \nabla^2)$. Here, ∇^p denotes the canonical affine connection on G/R_p $(p = 1, 2)$. Note that for $p = 1, 2$, G/R_p is an irreducible symmetric space and G acts on G/R_p effectively (because $\mathfrak{g}$ is a real form of a complex simple Lie algebra and G is its adjoint group). There exists an element $g \in G$ such that

$$(\tau_2(g) \circ f)(o_1) = o_2,$$

where o_p denotes the origin of G/R_p $(p = 1, 2)$, and $\tau_2(g)$ denotes a transformation of G/R_2 defined by $\tau_2(g)\big(aR_2\big) := gaR_2$ for $aR_2 \in G/R_2$. Then, $\tau_2(g) \circ f$ is an affine diffeomorphism of $(G/R_1, \nabla^1)$ onto $(G/R_2, \nabla^2)$ satisfying $(\tau_2(g) \circ f)(o_1) = o_2$. Lemma 6.3.2 means that there exists an automorphism Φ of G which satisfies three conditions

(i) $\Phi \circ \Sigma_1 = \Sigma_2 \circ \Phi$,
(ii) $\Phi(R_1) = R_2$,
(iii) $\tau_2(g) \circ f = \bar{\Phi}$, where $\bar{\Phi}$ is a G-equivariant diffeomorphism of G/R_1 onto G/R_2 defined by $\bar{\Phi}(aR_1) := \Phi(a)R_2$ for $aR_1 \in G/R_1$.

By existence of Φ satisfying $\Phi \circ \Sigma_1 = \Sigma_2 \circ \Phi$, and by Proposition 6.1.1, we conclude that $(G/R_1, \Sigma_1, J_1, \mathsf{g}_1)$ is equivalent to $(G/R_2, \Sigma_2, J_2, \mathsf{g}_2)$. Accordingly, if there exist a non-zero real number λ and a diffeomorphism f of G/R_1 onto G/R_2 which satisfy $f^*\mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$, then $(G/R_1, \Sigma_1, J_1, \mathsf{g}_1)$ is equivalent to $(G/R_2, \Sigma_2, J_2, \mathsf{g}_2)$. So, we have got the conclusion. □

Remark 6.3.3. By the above proof, one can deduce that in the same setting on Proposition 6.3.1,

if a diffeomorphism f of G/R_1 onto G/R_2 satisfies $f^*\mathsf{g}_2 = \lambda \cdot \mathsf{g}_1$,
then it is holomorphic or anti-holomorphic, namely $df \circ J_1 = \pm J_2 \circ df$.

Indeed, in the proof of Proposition 6.3.1, it has been proved that there exist an element $g \in G$ and an automorphism Φ of G which satisfy

(i) $\Phi \circ \Sigma_1 = \Sigma_2 \circ \Phi$,
(ii) $\Phi(R_1) = R_2$,
(iii) $\tau_2(g) \circ f = \bar{\Phi}$, where $\bar{\Phi}$ is a G-equivariant diffeomorphism of G/R_1 onto G/R_2 defined by $\bar{\Phi}(aR_1) := \Phi(a)R_2$ for $aR_1 \in G/R_1$.

The proof of Proposition 6.1.1 implies that $\bar{\Phi}$ satisfies the condition (C.3) $d\bar{\Phi} \circ J_1 = \pm J_2 \circ d\bar{\Phi}$. Hence from $\bar{\Phi} = \tau_2(g) \circ f$, it follows that f is holomorphic or anti-holomorphic, where we note that $\tau_2(g)$ is a holomorphic transformation of $(G/R_2, J_2)$.

6.4. **Appendix (Conjecture).** In 1984, Takeuchi [Ta] has classified totally real, totally geodesic submanifolds M of each Hermitian symmetric space $\bar{M}$ with $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} \bar{M}$. By arguments in this paper, we conjecture that in the Lie algebra level, totally real totally geodesic submanifolds M of each simple irreducible pseudo-Hermitian symmetric space G/R with $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} G/R$ would be classified as follows, in the case where the induced metric on M is non-degenerate.

		AI
1	G/R	$SL(2k,\mathbb{R})/(SL(k,\mathbb{C}) \times SO(2))$
	M	$SO_0(k,k)/SO(k,\mathbb{C})$
		$\big((SL(k,\mathbb{R}) \times SL(k,\mathbb{R}))/SL(k,\mathbb{R})\big) \times \mathbb{R}$
2	G/R	$SL(4m,\mathbb{R})/(SL(2m,\mathbb{C}) \times SO(2))$
	M	$Sp(2m,\mathbb{R})/Sp(m,\mathbb{C})$
		$(SL(2m,\mathbb{C})/SU^*(2m)) \times SO(2)$

		AII
3	G/R	$SU^*(2k)/(SL(k,\mathbb{C}) \times U(1))$
	M	$SO^*(2k)/SO(k,\mathbb{C})$
		$(SL(k,\mathbb{C})/SL(k,\mathbb{R})) \times U(1)$
4	G/R	$SU^*(4m)/(SL(2m,\mathbb{C}) \times U(1))$
	M	$Sp(m,m)/Sp(m,\mathbb{C})$
		$\big((SU^*(2m) \times SU^*(2m))/SU^*(2m)\big) \times \mathbb{R}$

		AIII
5	G/R	$SU(i,n-i)/S(U(k,h) \times U(i-k,n-i-h))$
	M	$SO_0(i,n-i)/(SO_0(k,h) \times SO_0(i-k,n-i-h))$
6	G/R	$SU(2a+2b,2s+2t)/S(U(2a,2s) \times U(2b,2t))$
	M	$Sp(a+b,s+t)/(Sp(a,s) \times Sp(b,t))$
7	G/R	$SU(2k,2h)/S(U(k,h) \times U(k,h))$
	M	$\big((SU(k,h) \times SU(k,h))/SU(k,h)\big) \times U(1)$

		BDI
8	G/R	$SO_0(i,n-i)/(SO_0(i-2,n-i)\times SO(2))$
	M	$(SO_0(a+1,k-a)/SO_0(a,k-a))$ $\times\ (SO_0(i-a-1,n-k-i+a)/SO_0(i-a-2,n-k-i+a))$ for $0\le k\le[(n-2)/2]$ and $0\le a\le[(i-2)/2]$
9	G/R	$SO_0(i,n-i)/(SO_0(i,n-i-2)\times SO(2))$
	M	$(SO_0(b+1,k-b)/SO_0(b,k-b))$ $\times\ (SO_0(n-i-b-1,i-k+b)/SO_0(n-i-b-2,i-k+b))$ for $0\le k\le[(n-2)/2]$ and $0\le b\le[(n-i-2)/2]$
		CI
10	G/R	$Sp(n,\mathbb{R})/U(i,n-i)$
	M	$(SL(n,\mathbb{R})/SO_0(i,n-i))\ \times\ \mathbb{R}$
11	G/R	$Sp(2a+2b,\mathbb{R})/U(2a,2b)$
	M	$Sp(a+b,\mathbb{C})/Sp(a,b)$
12	G/R	$Sp(2m,\mathbb{R})/U(m,m)$
	M	$(Sp(m,\mathbb{R})\times Sp(m,\mathbb{R}))/Sp(m,\mathbb{R})$
		$(SU(m,m)/SO^*(2m))\ \times\ U(1)$
		CII
13	G/R	$Sp(i,n-i)/U(i,n-i)$
	M	$(SU(i,n-i)/SO_0(i,n-i))\ \times\ U(1)$
14	G/R	$Sp(2a,2b)/U(2a,2b)$
	M	$(Sp(a,b)\times Sp(a,b))/Sp(a,b)$
		DI
15	G/R	$SO_0(2i,2n-2i)/U(i,n-i)$
	M	$(SO_0(i,n-i)\times SO_0(i,n-i))/SO_0(i,n-i)$
16	G/R	$SO_0(4s,4t)/U(2s,2t)$
	M	$(SU(2s,2t)/Sp(s,t))\ \times\ SO(2)$
		DIII
17	G/R	$SO^*(2n)/U(i,n-i)$

	M	$SO(n,\mathbb{C})/SO_0(i,n-i)$
18	G/R	$SO^*(4m)/U(m,m)$
	M	$(SO^*(2m)\times SO^*(2m))/SO^*(2m)$
		$(SU(m,m)/Sp(m,\mathbb{R}))\times U(1)$
19	G/R	$SO^*(2n)/(SO^*(2n-2)\times SO^*(2))$
	M	$SO(n,\mathbb{C})/SO(n-1,\mathbb{C})$
20	G/R	$SO^*(2n)/U(n)$
	M	$SO(n,\mathbb{C})/SO(n)$
21	G/R	$SO^*(4m)/U(2m)$
	M	$(SU^*(2m)/Sp(m))\times\mathbb{R}$

		EII
22	G/R	$E_{6(2)}/(SO_0(6,4)\times SO(2))$
	M	$F_{4(4)}/SO_0(5,4)$
		$Sp(4,\mathbb{R})/(Sp(2,\mathbb{R})\times Sp(2,\mathbb{R}))$
		$Sp(3,1)/(Sp(2)\times Sp(1,1))$
23	G/R	$E_{6(2)}/(SO^*(10)\times SO^*(2))$
	M	$Sp(4,\mathbb{R})/Sp(2,\mathbb{C})$

		EIII
24	G/R	$E_{6(-14)}/(SO^*(10)\times SO^*(2))$
	M	$Sp(2,2)/Sp(2,\mathbb{C})$
25	G/R	$E_{6(-14)}/(SO_0(8,2)\times SO(2))$
	M	$F_{4(-20)}/SO_0(8,1)$
		$Sp(2,2)/(Sp(1,1)\times Sp(1,1))$
26	G/R	$E_{6(-14)}/(SO(10)\times SO(2))$
	M	$F_{4(-20)}/SO(9)$
		$Sp(2,2)/(Sp(2)\times Sp(2))$

		EV
27	G/R	$E_{7(7)}/(E_{6(2)}\times SO(2))$
	M	$SL(8,\mathbb{R})/Sp(4,\mathbb{R})$

		$SU^*(8)/Sp(3,1)$
		$(E_{6(6)}/F_{4(4)}) \times \mathbb{R}$
		EVI
28	G/R	$E_{7(-5)}/(E_{6(2)} \times SO(2))$
	M	$SU(4,4)/Sp(4,\mathbb{R})$
		$SU(6,2)/Sp(3,1)$
		$(E_{6(2)}/F_{4(4)}) \times SO(2)$
29	G/R	$E_{7(-5)}/(E_{6(-14)} \times SO(2))$
	M	$SU(4,4)/Sp(2,2)$
		$(E_{6(-14)}/F_{4(-20)}) \times SO(2)$
		EVII
30	G/R	$E_{7(-25)}/(E_{6(-14)} \times SO(2))$
	M	$SU^*(8)/Sp(2,2)$
		$(E_{6(-26)}/F_{4(-20)}) \times \mathbb{R}$
31	G/R	$E_{7(-25)}/(E_6 \times SO(2))$
	M	$SU^*(8)/Sp(4)$
		$(E_{6(-26)}/F_4) \times \mathbb{R}$

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