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Spaces Characterized by Pair Bases and Pair Networks

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In this paper, we introduce the notions of semi-elastic and semi-stratonormal spaces, and show that a space is elastic (resp. stratonormal) if and only if it is semi-elastic (resp. semi-stratonormal) and monotonically normal. Furthermore, we show that semi-elastic and semi-stractonormal spaces are characteraized by some pair networks.

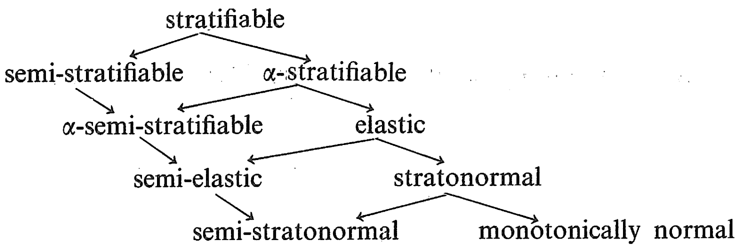
§1. Introduction

Ceder's M_3-space [5] (i.e. space with a σ-cushioned pair base) was studied and renamed "stratifiable space" by Borges [1]. By noting the properties [1, Lemma 2.1] of stratifiable spaces, Zenor introduced the notion of monotone normality in [14] and the class of monotonically normal spaces was studied in [8], [2], [3] etc. Thereafter, the intermediate classes between the class of stratifiable spaces and one of monotonically normal spaces were introduced and studied. For instance, linearly stratifiable spaces were studied in Vaughan [12] and Yasui [13], elastic spaces in Tamano and Vaughan [11], Gruenhage [7], Borges [3] and Pope [10]. Furthermore, the stratonormal space was defined by Borges [2]. An excellent survey of these classes except stratonormal spaces was given by Burke and Lutzer [4]. Note that these classes except monotonically normal spaces are characterized by some pair bases.

On the other hand, Creede [6] introduced the notion of semi-stratifiable spaces which is defined by semi-stratification. Semi-stratifiable spaces are characterized by some pair networks (Pareek [9]); that is, a space is semi-stratifiable if and only if it has a σ-cushioned pair network.

In this paper, we introduce the notions of semi-elastic and semi-stratonormal spaces. We show that a space is elastic (resp. stratonormal) if and only if it is semi-elastic (resp. semi-stratonormal) and monotonically normal. Furthermore, we show that semi-elastic, semi-stratonormal and α-semi-stratifiable ([13]) spaces are characterized by some pair networks.

These classes mentioned above are located as follows:



Throughout this paper, all spaces are assumed to be regular T_1. Cl denotes the closure operator in a space and 2^X the power set of X. Let $\leq$ be a relation on a set A. For $a, b \in A$, $a \| b$ means that neither $a \leq b$ nor $b \leq a$.

§2. Preliminaries

In this section, we state some definitions and some results, which are used to prove some theorems in section 3.

For definition and some properties of monotonically normal spaces, see [14], [8], [2] and [3]. In this paper, we assume that each monotone normality operator G is exclusively defined for each pair of separated subsets, and G satisfies $G(A, B) \cap G(B, A) = \phi$ for any pair (A, B) of separated subsets (i.e. $B \cap \mathrm{Cl}\, A = A \cap \mathrm{Cl}\, B = \phi$).

For definitions of stratifiable and semi-stratifiable spaces, see Borges [1] and Creede [6], respectively. As a connection of these spaces, there is the following theorem.

THEOREM 2.1([8, Theorem 2.5]). *A space X is stratifiable if and only if X is semi-stratifiable and monotonically normal.*

Pareek [9] introduced the notion of pair networks; i.e. a pair network $\mathscr{W}$ is a collection of pairs $W = (W_1, W_2)$ of subsets of a space such that $W_1 \subset W_2$ and, for each point x and each neighborhood U of x, there is $(W, W_2) \in \mathscr{W}$ with $x \in W_1 \subset W_2 \subset U$. By using the notion of pair network, semi-stratifiable spaces are characterized as follows.

THEOREM 2.2([9, Theorem 2.1]). *A space X is semi-stratifiable if and only if X has a σ-cushioned pair network.*

For each initial ordinal α (in this paper, α means exclusively an infinite initial ordinal number), an α-stratifiable (resp. α-semi-stratifiable space) was defined by Vaughan [12] (resp. Yasui [13]). For these spaces, there is the following theorem.

THEOREM 2.3([13, Theorem 3]). *A sapce X is α-stratifiable if and only if X is α-semi-stratifiable and monotonically normal.*

The following definitions are some generalizations of cushiond pair collection in Ceder [5].

DEFINITION 2.4([11] and [2]). Let X be a space.

(a) Let $\mathscr{U}$ be any collection of subsets of X and let R be a relation on $\mathscr{U}$ (i.e. $R \subset \mathscr{U} \times \mathscr{U}$). We shall often write $U \leq V$ instead of $(U, V) \in R$. The relation R is said to be a *framing* of $\mathscr{U}$ provided that, for every $U, V \in \mathscr{U}$ with $U \cap V \neq \phi$, either $U \leq V$ or $V \leq U$.

(b) A collection $\mathscr{U}$ is said to be *framed in a collection* $\mathscr{V}$ *with framed function* $f: \mathscr{U} \to \mathscr{V}$ provided that there exists a framing R of $\mathscr{U}$ such that for every $\mathscr{U}' \subset \mathscr{U}$ which has an R-upper bound we get that $\mathrm{Cl}(\cup \mathscr{U}') \subset \cup f(\mathscr{U}')$.

(c) If $\mathscr{U}$ is framed in $\mathscr{V}$ and R is a transitive relation, then $\mathscr{U}$ is said to be *elastic in* $\mathscr{V}$.

(d) A pair base $\mathscr{P}$ for X (i.e. $\mathscr{P}$ is a collection of pairs $P = (P_1, P_2)$ of subsets of X such that P_1 is open, $P_1 \subset P_2$ and, for each $x \in X$ and each neighborhood U of x, there exists $(P_1, P_2) \in \mathscr{P}$ with $x \in P_1 \subset P_2 \subset U$) is said to be an *elastic* (resp, *framed*) *base* if there is a framing of $\mathscr{P}_1 = \{P_1 : (P_1, P_2) \in \mathscr{P}\}$ such that $\mathscr{P}_1$ is elastic (resp. framed in $\mathscr{P}_2 = \{P_2 : (P_1, P_2) \in \mathscr{P}\}$ with respect to the function $f: \mathscr{P}_1 \to \mathscr{P}_2$ defined by $f(P_1) = P_2$ for $(P_1, P_2) \in \mathscr{P}$.

(e) A space with an elastic (resp. framed) base is called an *elastic* (resp. *stratonormal*) *space.*

For elastic spaces, the following theorem was obtained by Pope [10].

THEOREM 2.5([10, Theorem 2.8]). *A space X with a topology $\mathscr{T}$ is elastic if and only if there exist a partially ordered set $(P, \leq)$ and a function $S: P \times \mathscr{T} \to \mathscr{T}$ such that*

(a) *for each $U \in \mathscr{T}$ and $p_0 \in P$,*

$$\mathrm{Cl}(\cup \{S(p, U): p \in P, p \leq p_0\}) \subset U;$$

(b) $\cup \{S(p, U): p \in P\} = U$ *for each* $U \in \mathscr{T}$;

(c) *if* $U, V \in \mathscr{T}$ *and* $U \subset V$, $S(p, U) \subset S(p, V)$ *for all* $p \in P$;

(d) *if* $S(p, U) \cap S(p', V) \neq \phi$ *for* $p, p' \in \mathscr{P}$ *and* $U, V \in \mathscr{T}$, *then either* $p \leq p'$ *or* $p' \leq p$.

In stratonormal spaces, we can prove an analogous theorem of Theorem 2.5. Note that for a relation $\leq$ on a non-empty set A, we call $\leq$ a *preorder* if $\leq$ is reflexive and antisymmetric.

THEOREM 2.6. *A space X with a topology $\mathscr{T}$ is stratonormal if and only if there exist a preordered set $(P, \leq)$ and a function $S: P \times \mathscr{T} \to \mathscr{T}$ such that the conditions* (a)-(d) *of Theorem* 2.5 *are satisfied.*

§3. Semi-elastic and semi-stratonormal spaces

First, we introduce the notions of semi-elastic and semi-stratonormal spaces.

DEFINITION 3.1. A space X with a topology $\mathscr{T}$ is said to be *semi-elastic* (resp. *semi-stratonormal*) if there exist a partially ordered (resp. preordered) set $(P, \leq)$ and a function $S: P \times \mathscr{T} \to 2^X$ such that

(a) for each $U \in \mathscr{T}$ and each $p_0 \in P$,

$$\mathrm{Cl}(\cup\{S(p, U): p \in P,\ p \leq p_0\}) \subset U;$$

(b) $\cup\{S(p, U): p \in P\} = U$ for each $U \in \mathscr{T}$;

(c) if $U, V \in \mathscr{T}$ and $U \subset V$, then $S(p, U) \subset S(p, V)$ for all $p \in P$;

(d) for each $U \in \mathscr{T}$ and each $p \in P$, $S(p, U)$ and $\cup\{S(p', V): p' \in P,\ p \| p',\ V \in \mathscr{T}\}$ are the separated subsets of X.

The following theorem shows that the notions of semi-elastic and semi-stratonormal spaces are natural (cf. Theorems 2.1 and 2.3).

THEOREM 3.2. *A space X is elastic* (*resp. stratonormal*) *if and only if X is semi-elastic* (*resp. semi-stratonormal*) *and monotonically normal.*

PROOF. Necessity is clear by using Theorems 2.5 and 2.6.

Sufficiency: Let $(P, \leq)$ be a partially ordered (resp. preordered) set and $S: P \times \mathscr{T} \to 2^X$ a function satisfying (a)-(d) of Definition 3.1, where $\mathscr{T}$ is a topology X. Define a function $T: P \times \mathscr{T} \to \mathscr{T}$ by

$$T(p, U) = G(S(p, U), (X - U) \cup (\cup\{S(p', V): p \| p',\ p' \in P,\ V \in \mathscr{T}\}))$$

where G is a monotone normality operator. Then it can be easily that $(P, \leq)$ and T satisfy the conditions (a)-(d) of Theorem 2.5 (resp. 2.6). For instance, (a) is proved by the fact that, for each $U \in \mathscr{T}$ and each $p_0 \in P$,

$$\begin{aligned} &\mathrm{Cl}(\cup\{T(p, U): p \leq p_0,\ p \in P\}) \\ &\quad \subset \mathrm{Cl}(G(\mathrm{Cl}(\cup\{S(p, U): p \leq p_0,\ p \in P\}), X - U)) \\ &\quad \subset U \end{aligned}$$

and (d) is proved by the facts that, for $p, p' \in P$ with $p \| p'$,

$$S(p, U) \subset (X - V) \cup (\cup\{S(q, W): p' \| q,\ q \in P,\ W \in \mathscr{T}\})(= K),$$

$$T(p, U) \subset G(K, S(p', V)),$$

$$T(p', V) = G(S(p', V), K),$$

$$G(K, S(p', V)) \cap G(S(p', V), K) = \phi,$$

therefore $T(p, U) \cap T(p', V) = \phi$. Thus the proof is completed.

Finally, we give characterizations of semi-elastic, semi-stratonormal or α-semi-stratifiable spaces by some pair networks.

THEOREM 3.3. *A space X is semi-elastic (resp. semi-stratonormal) if and only if X has a pair network $\mathcal{W} = \{(W_1, W_2)\}$ such that $\mathcal{W}_1 = \{W_1 : (W_1, W_2) \in \mathcal{W}\}$ is elastic (resp. framed) in $\mathcal{W}_2 = \{W_2 : (W_1, W_2) \in \mathcal{W}\}$ with respect to the function $f: \mathcal{W}_1 \to \mathcal{W}_2$ defined by $f(W_1) = W_2$ and, for each $(W_1, W_2) \in \mathcal{W}$, W_1 and $\cup\{W_1' \in \mathcal{W}_1 : W_1 \parallel W_1'\}$ are the separated subsets.*

PROOF. Necessity: Let $\mathcal{T}$ be a topology of X and give a well order to $\mathcal{T}$. Now suppose that $(P, \leq)$ and a function $S: P \times \mathcal{T} \to 2^X$ satisfies the conditions of Definition 3.1. If we define the lexicographic order in $P \times \mathcal{T}$ and the set $\{S(p, U) : (p, U) \in P \times \mathcal{T}\}$ is equipped with the same order as that of $P \times \mathcal{T}$ (i.e. $S(p, U) \leq S(p', V)$ if and only if $(p, U) \leq (p', V)$), then $\mathcal{W} = \{(S(p, U), U) : (p, U) \in P \times \mathcal{T}\}$ satisfies the conditions of this theorem as follows: First, it is clear that $\mathcal{W}$ is a pair network. Secondly, if $S(p, U) \cap S(q, V) \neq \phi$ for $S(p, U)$, $S(q, V) \in \mathcal{W}_1$, it is clear from (d) of Definition 3.1 that $p \leq q$ or $q \leq p$, therefore $(p, U) \leq (q, V)$ or $(q, V) \leq (p, U)$. Thus, the relation $\leq$ on $\mathcal{W}_1$ is framing of $\mathcal{W}_1$. Thirdly, for each $(p_0, U_0) \in P \times \mathcal{T}$, let

$$W = \cup\{U : (p, U) \leq (p_0, U_0), (p, U) \in P \times \mathcal{T}\}.$$

Then

$$\begin{aligned} &\mathrm{Cl}(\cup\{S(p, U) : (p, U) \leq (p_0, U_0), (p, U) \in P \times \mathcal{T}\} \\ &\quad \subset \mathrm{Cl}(\cup\{S(p, W) : p \leq p_0, p \in P\}) \\ &\quad \subset W. \end{aligned}$$

Thus, for $\mathcal{W}_1' \subset \mathcal{W}_1$ which has $\leq$-upper bound, we get $\mathrm{Cl}(\cup \mathcal{W}_1') \subset \cup f(\mathcal{W}_1')$. Therefore, $\mathcal{W}_1$ is elastic (resp. framed) in $\mathcal{W}_2$ with respect to the function $f: \mathcal{W}_1 \to \mathcal{W}_2$ defined by $f(W_1) = W_2$. Finally, for each $S(p, U) \in \mathcal{W}_1$, $S(p, U)$ and $\cup\{S(p', V) : (p, U) \parallel (p', V), (p', V) \in P \times \mathcal{T}\}$ are the separated subsets by the fact

$$\begin{aligned} &\cup\{S(p', V) : (p, U) \parallel (p', V), (p', V) \in P \times \mathcal{T}\} \\ &\quad = \cup\{S(p', V) : p \parallel p', (p', V) \in P \times \mathcal{T}\}. \end{aligned}$$

Sufficiency: Suppose that $\mathcal{W} = \{(W_1, W_2)\}$ be a pair network satisfying the conditions of this theorem. Then without loss of generality, we may assume that $\mathcal{W}_1 = \{W_1 : (W_1, W_2) \in \mathcal{W}\}$ is a partially ordered (resp. preodered) set (see Pope [10, Lemma 2.7]). Define a function

$S: \mathscr{W}_1 \times \mathscr{T} \to 2^X$ by

$$S(W_1, U) = \begin{cases} W_1 & \text{if } W_2 \subset U \\ \phi & \text{otherwise.} \end{cases}$$

Then S satisfies the conditions (a)-(d) of Definition 3.1 as follows: (b), (c) and (d) are trivial. To prove (a), for each $U \in \mathscr{T}$ and each $W_1^0 \in \mathscr{W}_1$,

$$\begin{aligned} &\mathrm{Cl}(\cup\{S(W_1, U): W_1 \leq W_1^0\}) \\ &\quad \subset \mathrm{Cl}(\cup\{W_1: W_1 \leq W_1^0, W_2 \subset U\}) \\ &\quad \subset \cup\{W_2: W_1 \leq W_1^0, W_2 \subset U\} \\ &\quad \subset U. \end{aligned}$$

Thus, the proof is completed.

The α-semi-stratifiable spaces are also characterized by some pair networks as follows. The proof is easily verified, so omitted. For the definition of a linearly cushioned collection of pairs, see [12, Definition 2.5].

THEOREM 3.4. *A space X is α-semi-stratifiable if and only if X has a linearly cushioned pair network $\mathscr{P}$ and α is cofinal with $\mathscr{P}$.*

§4. Problems

The following problems naturally arise.

PROBLEM 4.1. (1) Is it true that stratonormal spaces are elastic ?

(2) Is the closed image of an elastic (resp. a stratonormal) space elastic (resp. stratonormal) ? (See [2].)

(3) Is it true that the adjunction space of two elastic (resp. stratonormal) spaces is elastic (resp. stratonormal) ?

By Theorem 3.2, these problems are reduced to the cases of semi-elastic or semi-stratonormal spaces, because the closed image of a monotonically normal space and the adjunction space of two monotonically normal spaces are also monotonically normal. In particular, the affirmative answer of Problem 4.1 (2) (elastic case) is Tamano's conjecture ([11]).

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