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Steady-State Analysis of Adaptive Notch Filter Using Error Spectrum Shaping

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This paper presents a method to analyze the steady-state characteristics of an adaptive notch filter, when the effects of finite precision arithmetic are taken into account. The reduction of these effects by means of error spectrum shaping is also discussed. Closed-form expressions for the steady-state mean square error of variable coefficient have been obtained. Finally the results of computer simulation are shown which confirm the theoretical prediction.

§ 1. Introduction

Adaptive digital notch filters have many applications such as the detection of sinusoids in noise or eliminating sinusoidal interference from a broad-band signal. Implementations using finite impulse response (FIR) and infinite impulse response (IIR) filters have been discussed in the literatures [1]–[4]. An advantage of IIR notch filter is that it requires fewer multiplication operations per output sample, when compared to FIR filters.

Errors due to finite wordlengh can be quite large in narrow-band IIR filters. It has been shown that these errors may be considerably reduced by using a technique known as error spectrum shaping (ESS) [5]–[7].

In this paper we present an adaptive IIR notch filter with ESS. The adaptive detection of a sinusoid with additive white Gaussian noise has been investigated. Floating-point arithmetic with rounding is examined and the expressions for the steady-state mean square errors of output sinusoid and variable coefficient are found.

The outline of this paper is as follows. Some preliminaries on adaptive notch filter with ESS are presented in Section 2. In Section 3, the steady-state mean square errors of output sinusoid and variable coefficient are obtained. Some simulation results that verify the expressions derived for the steady-state analysis are presented in Section 4. Section 5 concludes the paper.

§ 2. Preliminaries

Figure 1 (a) shows the block diagram of adaptive notch filter. For simplicity we consider only the case where the received signal consists of single sinusoid and additive Gaussian noise. The input data are written as

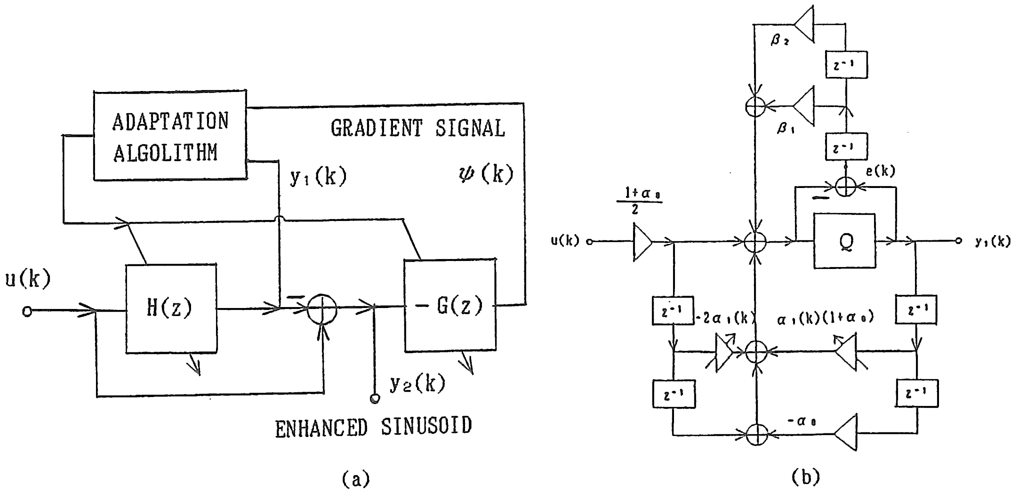

Fig. 1 Adaptive notch filter with error spectrum shaping. (a) Block diagram. (b) Variable notch filter with error spectrum shaping.

$$u(k) = A\cos((\pi/K)k + \theta) + n(k), \qquad k = 0, 1, 2, \cdots, \tag{1}$$

where k is the time index, K is an unknown constant determining the angular frequency of the input sinusoid, and $n(k)$ is a sequence of Gaussian random variables with mean zero and variance σ_n^2.

The transfer function of second-order IIR notch filter is given by

$$H(z) = \frac{1+\alpha_0}{2}\,\frac{1 - 2\alpha_1(k)z^{-1} + z^{-2}}{1 - \alpha_1(k)(1+\alpha_0)z^{-1} + \alpha_0 z^{-2}}, \tag{2}$$

where $\alpha_1(k)$ is variable coefficient controlling a notch frequency. The rejection bandwidth can be determined by α_0 [1].

For brevity we consider only the gradient algorithms which do not require any matrix inversion. The updating formula for notch filter becomes

$$\alpha_1(k+1) = \alpha_1(k) - \mu y_1(k)\psi(k), \tag{3}$$

where μ is a constant controlling the convergence rate. In (3), $y_1(k)$ is the output of notch filter and $\psi(k)$ is the defferential of $y_1(k)$ with restpect to $\alpha_1(k)$. $\psi(k)$ is generated by the circuit having the transfer function

$$G(z) = \frac{(1+\alpha_0)z^{-1}}{1 - \alpha_1(k)(1+\alpha_0)z^{-1} + \alpha_0 z^{-2}} \tag{4}$$

Figure 1 (b) shows the variable notch filter with ESS. Here we will consider the special case where a sum of products is accumulated in full precision and only the final result is quantized.

As is well known, the quantization error in the floating-point arithmetic depends on the signal to be quantized. The quantization error can be described by

$$e(k) = \varepsilon(k)w(k), \qquad \varepsilon(k) = (m_r(k) - m(k))/m(k) \tag{5}$$

where $w(k)$ is the signal to be quantized, and $\varepsilon(k)$ is the relative error. $m(k)$ and $m_r(k)$ denote the mantissa and the mantissa after rounding, respectively. For convenience in analysis, $(m_r(k) - m(k))$ is assumed to be uniformly distributed random variable in the interval, $(-(1/2)2^{-b}, (1/2)2^{-b})$, and $m(k)$ is assumed to be uniformly distributed in (0.5, 1.0), where $b+1$ is the mantissa wordlength (one bit for sign).

Assuming that $(m_r(k) - m(k))$ and $m(k)$ are white noise process, and that $(m_r(k) - m(k))$, $m(k)$ and the signal $w(k)$ are mutually independent, $e(k)$ in (5) is mean zero white noise process.

At the steady-state, the input sinusoid is rejected and the power spectral density of the signal $w(k)$ can be expressed by $|H(z)|^2$. In this case, the variance of $e(k)$ is obtained easily as [8], [9]

$$\sigma_1^2 = (2^{-2b}/12)(1 + \alpha_0)\sigma_n^2. \tag{6}$$

The transfer function from the error signal to the filter output is given by

$$N(z) = \frac{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 - \alpha_1(k)(1 + \alpha_0)z^{-1} + \alpha_0 z^{-2}}. \tag{7}$$

Table I lists the ESS structures considered in this paper. ① is the direct form filter with no ESS. ② is simple form of ESS requiring no additional multiplications. ③ requires one additional multiplication. ④ is the optimal ESS structure and requires two additional multiplications [5].

Table I Variable Notch Filters with Error Spectrum Shaping

FILTER STRUCTURE	β_1	β_2
① WITHOUT ESS	0	0
② SUBOPTIMAL ESS 1	-2	1
③ SUBOPTIMAL ESS 2	$-2\alpha_1(k)$	1
④ OPTIMAL ESS	$-\alpha_1(k)(1+\alpha_0)$	α_0

§3. Steady-State Analysis

Here, we consider the error of sinusoidal output due to the roundoff error at the notch filter output. The variance of output sinusoid, σ_{out}^2, is given by

$$\sigma_{out}^2 = \frac{\sigma_1^2}{2\pi}\int_{-\pi}^{\pi} |N(e^{j\omega})|^2 \, d\omega. \tag{8}$$

By using the β_1 and β_2 given in Table I, we get σ_{out}^2 as shown in Table II. From the results of this table, it is seen that the variance of output error, σ_{out}^2, can be reduced by means of ESS.

Table II Variances of Output Errors, σ_{out}^2

①	$\sigma_1^2 \dfrac{1}{(1-\alpha_0^2)\{1-\alpha_1(k)^2\}}$
②	$\sigma_1^2 \dfrac{2\{3-\alpha_1(k)(1+\alpha_0)-\alpha_0\}}{(1-\alpha_0^2)\{1+\alpha_1(k)\}}$
③	$\sigma_1^2 \dfrac{2}{1+\alpha_0}$
④	σ_1^2

Next, we provide a quantitative analysis of the steady-state mean square error of $\alpha_1(k)$ under the assumption that the effect of fluctuation of $\alpha_1(k)$ to filter performance can be neglected. In order to analyze the effects of quantization error, we will use the method developed in [10]. At the steady-state, the value of $\alpha_1(k)$ fluctuates around $\cos\pi/K$, causing a mean-square error of the sine wave frequency estimate. Under the assumption that fluctuation of $\alpha_1(k)$ is very small, it follows for $z = e^{j\pi/K}$

$$|H(e^{j\pi/K})| = \beta|\varDelta\alpha_1(k)| \tag{9a}$$

where

$$\beta = (1+\alpha_0)/\{(1-\alpha_0)\sin(\pi/K)\}, \quad \varDelta\alpha_1(k) = \alpha_1(k) - \cos(\pi/K). \tag{9b}$$

After the convergence the output of the notch filter is given as

$$y_1(k) = -A\beta\varDelta\alpha_1(k)\sin((\pi/K)k+\theta) + \eta_1(k), \tag{10}$$

where $\eta_1(k)$ is the noise term due to the quantization. We restrict attention to the coefficient error due to the quantization, the noise term due to the input noise $n(k)$ is omitted in (10). $\psi(k)$ will have a single sinusoid being tracked and it can be expressed as

$$\psi(k) = -A\beta \sin((\pi/K)k + \theta), \tag{11}$$

in the steady-state.

Substituting (10) and (11) in (3), we have

$$\begin{aligned} \varDelta\alpha_1(k+1) = \{1 - \mu A^2\beta^2 \sin^2((\pi/K)k + \theta)\}\, \varDelta\alpha_1(k) \\ - \mu A\beta\eta_1(k) \sin((\pi/K)k + \theta). \end{aligned} \tag{12}$$

Equation (12) is a first-order time-varying ordinary difference equation with respect to $\varDelta\alpha_1(k)$. The second term of the right-hand side is input term. By taking the expected value of coefficient of $\varDelta\alpha_1(k)$, we obtain the time-invarient difference quation

$$\varDelta\alpha_1(k+1) = \left(1 - \frac{\mu}{2}A^2\beta^2\right)\varDelta\alpha_1(k) - \mu A\beta\eta_1(k)\sin((\pi/K)k + \theta). \tag{13}$$

From (13) the steady-state variance of $\varDelta\alpha_1(k)$ can be obtained as

$$\sigma_2^2 = \frac{1}{2\pi}\int_{\pi}^{\pi} |F(e^{j\omega})|^2 P(\omega)\, d\omega, \tag{14}$$

where

$$F(e^{j\omega}) = \frac{-\mu A\beta}{1 - \left(1 - \frac{\mu}{2}A^2\beta^2\right)e^{-j\omega}}, \tag{15a}$$

$$P(\omega) = \frac{1}{4}\left\{N\left(\omega - \frac{\pi}{k}\right) + N\left(\omega + \frac{\pi}{k}\right)\right\}. \tag{15b}$$

$F(e^{j\omega})$ of (15a) is the transfer function from input $\eta_1(k)\sin((\pi/K)k + \theta)$ to $\varDelta\alpha_1(k)$ and it has low-pass characteristics having very narrow bandwidth. $N(\omega)$ indicates the power spectrum of $\eta_1(k)$. For given μ, α_0, K and input SNR, the mean-square error of $\alpha_1(k)$ can be obtained from (14), (15a) and (15b).

§4. Simulation Results

The numerical examples are presented here in order to illustrate the results in Section 3. The input sinusoid had unit magnitude and was corrupted by the white Gaussian noise with signal-to-noise ratios of 0.0 dB and 10 dB. The input was applied to a filter using infinite precision, and slso a filter with same coefficients, but using a shorter mantissa in the computation. The outputs and the values of variable coefficients of the two filters were then subtracted, squared, and averaged over 5000 samples to obtain the estimates of their variances. The values of K and μ are fixed at 5 and 0.00005, respectively.

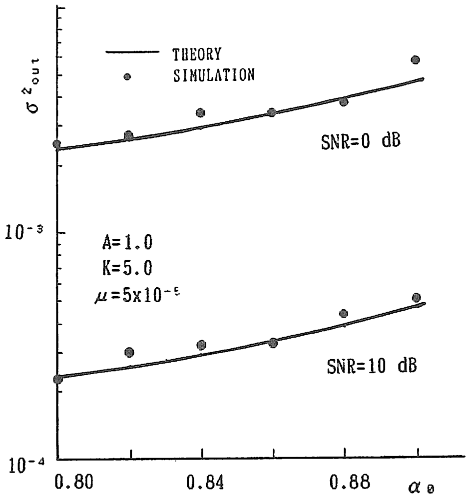

Fig. 2 Variances of output error, σ^2_{out}, for ① WITHOUT ESS ($b = 4$ bits).

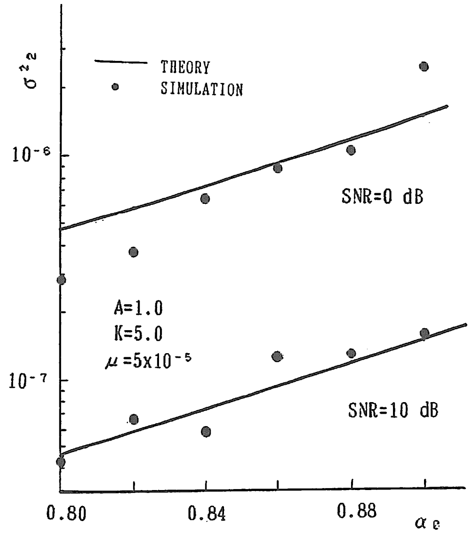

Fig. 3 Variances of coefficient error, σ_2^2, for ① WITHOUT ESS ($b = 4$ bits).

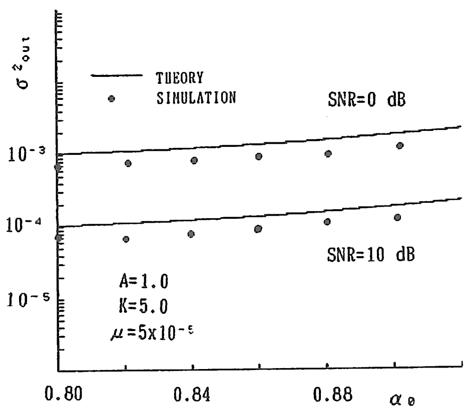

Fig. 4 Variances of output error, σ^2_{out}, for ② SUBOPTIMAL ESS 1 ($b = 4$ bits).

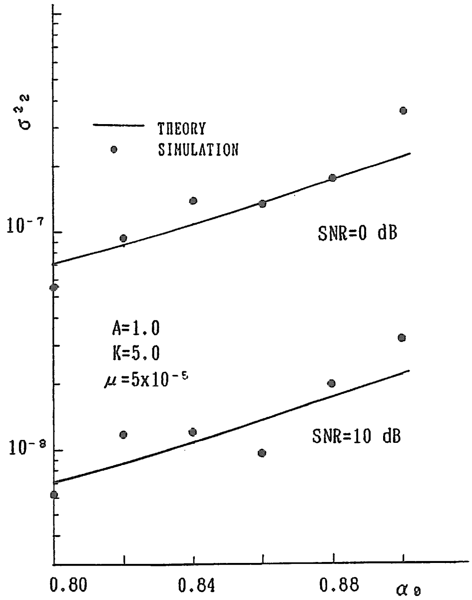

Fig. 5 Variances of coefficient error, σ_2^2, for ② SUBOPTIMAL ESS 1 ($b = 4$ bits).

Figs 2 and 3 show results for ① WITHOUT ESS. Figure 2 shows the theoretical and simulated results for the variances of output error, σ_{out}^2. As expected, the values of σ_{out}^2 for floating-point arithmetic is proportional to the power of input noise. While those for fixed-point arithmetic did not depend on the variance of input additive noise [11]. In Fig. 3 the simulated variances of $\Delta\alpha_1(k)$, σ_2^2, are plotted versus α_0, along with the theoretical curves. It should be noted that the values of σ_2^2 inclease as α_0 approaches one. Note from Figs 2 and 3 that the theoretical and simulated results agree reasonably well.

In Figs. 4 and 5, we show the results for ② SUBOPTIMAL ESS 1. Figure 4 shows the theoretical and simulated results for the variances of output error, σ_{out}^2. As expected, σ_{out}^2 for ② SUBOPTIMAL ESS 1 is much smaller than thot for ① WITHOUT ESS. In Fig. 5 the simulated variances of $\Delta\alpha_1(k)$, σ_2^2, are plotted versus α_0, along with the theoretical curves. It should be noted that the values of σ_2^2 are redused by the ESS. Note also from Figs. 4 and 5 that the theoretical and simulated results agree reasonably well.

§5. Conclusions

In this paper, we have studied the steady-state behavior of adaptive notch filters implemented with floating-point arithmetic. Approximate and simple closed-form results are derived to obtain the mean-square errors of enhanced sinusoid and coefficient fluctuation due to finite wordlength.

The resulting expression is useful for design purposes and also leads to several observations about the effects of finite precision arithmetic for adaptive IIR filters.

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