Mem. Fac. Sci. Shimane Univ.,
25, pp. 7–14 Dec. 25, 1991

A Maximin Cut Problem on an Infinite Network

Dedicated to Professor Masanori Kishi on his 60th birthday

Maretsugu YAMASAKI

(Received September 4, 1991)

Department of Mathematics, Shimane University, Matsue, Japan

As a dual problem of a max-flow problem on an infinite network, a maximin cut problem is considered with the aid of exceptional sets of cuts in the sense of the extremal width of the network. The penalty method in the theory of mathematical programming plays an important role in our study.

§1. Introduction

The study of duality relations between the max-flow problems and the min-cut problems seems to be one of the most important themes in the theory of networks. On a finite network, the celebrated max-flow min-cut theorem due to Ford and Fulkerson [2] has been the unique result for this direction before the work of Strang [6]. On an infinite network, Yamasaki [7] and Nakamura and Yamasaki [4] gave several max-flow min-cut theorems related to several kinds of flows and cuts. In this paper, we shall introduce a notion of an exceptional set of cuts with respect to the extremal width and consider a maximin cut problem. It will be shown by using the penalty method that the value of this maximin problem is equal to the value of a max-flow problem.

For notation and terminology, we mainly follow [3] and [5].

§2. Flows and cuts

Let X and Y be countable sets of nodes and arcs respectively and K be the node-arc incidence function. We assume that the graph $G = \{X, Y, K\}$ is connected and has no self-loop. For a strictly positive function r on Y, we call the pair $N = \{G, r\}$ an infinite network. Denote by $L(X)$ and $L(Y)$ the sets of all real functions on X and Y respectively, by $L^+(Y)$ the set of all nonnegative functions on Y and by $L_0(Y)$ the set of $w \in L(Y)$ such that the support $\{y \in Y; w(y) \neq 0\}$ of w is a finite set. Let p and q be numbers such that

$$p > 1 \quad \text{and} \quad 1/p + 1/q = 1$$

and $H_p(w)$ be the energy of $w \in L(Y)$ of order p, i.e.,

$$H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p.$$

Denote by $L_p(Y; r)$ the set of all $w \in L(Y)$ with finite energy of order p. Note that $L_p(Y; r)$ is a reflexive Banach space with the norm $[H_p(w)]^{1/p}$. We always assume that the condition

$(\mathrm{ALF})_q \quad \sum_{y \in Y} |K(x, y)| r(y)^{1-q} < \infty$

holds for all $x \in X$, i.e., N is q-almost locally finite.

REMARK 2.1. By Hölder's inequality, the inequality

(2.1) $\sum_{y \in Y} |K(x, y) w(y)| \leq [\sum_{y \in Y} |K(x, y)| r(y)^{1-q}]^{1/q} [H_p(w)]^{1/p}$

holds for $w \in L(Y)$.

For $w \in L(Y)$ and $x \in X$, we define $I(w; x)$ by

$$I(w; x) = \sum_{y \in Y} K(x, y) w(y)$$

if the sum is well-defined. By (2.1), $I(w; x)$ is well-defined for every $w \in L_p(Y; r)$.

Let A and B be mutually disjoint nonempty finite subsets of X. We say that $w \in L(Y)$ is a flow from A to B if it satisfies the following conditions:

$$\sum_{y \in Y} |K(x, y) w(y)| < \infty \quad \text{for all} \quad x \in X;$$

$$I(w; x) = 0 \quad \text{for all} \quad x \in X - A - B;$$

$$\sum_{x \in A \cup B} I(w; x) = 0.$$

Denote by $F(A, B)$ the set of all flows from A to B. We define the strength $I(w)$ of $w \in F(A, B)$ by

$$I(w) = -\sum_{x \in A} I(w; x) = \sum_{x \in B} I(w; x).$$

Put $F_0(A, B) = F(A, B) \cap L_0(Y)$ and denote by $F_p(A, B)$ the closure of $F_0(A, B)$ in the Banach space $L_p(Y; r)$. Note that $F_p(A, B)$ is a subset of $F(A, B)$ by Remark 2.1. If $w \in F_p(A, B)$ and if $\{w_n\}$ is a sequence in $F_0(A, B)$ such that $H_p(w_n - w) \to 0$ as $n \to \infty$, then $\{w_n(y)\}$ converges to $w(y)$ for each $y \in Y$ and $\{I(w_n)\}$ converges to $I(w)$ by (2.1).

REMARK 2.2. For every $x \in X$, $K(x, \cdot) r^{-1} \in L_q(Y; r)$. In fact,

$$\sum_{y \in Y} r(y) |K(x, y) r(y)^{-1}|^q = \sum_{y \in Y} |K(x, y)| r(y)^{1-q} < \infty$$

by condition $(\mathrm{ALF})_q$. Therefore, for every sequence $\{w_n\}$ in $F_p(A, B)$ which converges weakly to $w \in F_p(A, B)$, we see that $\{I(w_n; x)\}$ converges to $I(w; x)$ as $n \to \infty$ for every x, so that $I(w_n) \to I(w)$ as $n \to \infty$.

First we shall introduce a general max-flow problem. Given a (capacity) function $W \in L^+(Y)$ and a nonempty subset $\mathscr{F}$ of flows, the max-flow problem related to W and $\mathscr{F}$ is formulated as follows:

(M$\mathscr{F}$) Find $M(W; \mathscr{F}) = \sup\{I(w); w \in \mathscr{F}$ and $|w(y)| \leq W(y)$ on $Y\}$.

By the above observation, we have

$$M(W; F_0(A, B)) \leq M(W; F_p(A, B)) \leq M(W; F(A, B)).$$

To state min-cut problems, we recall some notation. For mutually disjoint nonempty subsets X_1 and X_2 of X, denote by $X_1 \ominus X_2$ the set of all arcs which connects X_1 and X_2 directly. We say that a subset Q of Y is a cut if $Q = X' \ominus (X - X')$ for some nonempty proper subset X' of X. We say that Q is a cut between A and B if there exists a subset X' of X such that $Q = X' \ominus (X - X')$, $X' \supset A$ and $X - X' \supset B$. Since the pair $\{X', X - X'\}$ is uniquely determined by Q, we put $X' = Q(A)$ and $X - X' = Q(B)$ for simplicity. Denote by $Q_{A,B}$ the set of all cuts between A and B and by $Q_{A,B}^{(f)}$ the set of all $Q \in Q_{A,B}$ such that Q is a finite subset of Y.

Given a (capacity) function $W \in L^+(Y)$ and a subset $\mathscr{C}$ of cuts, the min-cut problem related to W and $\mathscr{C}$ is formulated as follows:

(M*$\mathscr{C}$) Find $M^*(W; \mathscr{C}) = \inf\{\sum_{y \in Q} W(y); Q \in \mathscr{C}\}$.

Here we use the convention that the infimum on the empty set is equal to ∞.

The characteristic function $u_Q \in L(X)$ of $Q \in Q_{A,B}$ is defined by $u_Q(x) = 1$ on $Q(A)$ and $u_Q(x) = 0$ on $Q(B)$. For a cut $Q \in Q_{A,B}$ and $w \in L(Y)$, let us define a cut-value $J(w; Q)$ of w on Q by

$$J(w; Q) = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) u_Q(x)$$

if the sum is well-defined.

We prove the following key lemma.

LEMMA 2.3. *Let $w \in F(A, B)$ and $Q \in Q_{A,B}$. Then the equality $I(w) = -J(w; Q)$, i.e.,*

$$\sum_{x \in X} u_Q(x) \sum_{y \in Y} K(x, y) w(y) = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) u_Q(x)$$

holds if any one of the following conditions is fulfilled:

(1) $w \in F_0(A, B)$ *and* $Q \in Q_{A,B}$.

(2) $w \in F_p(A, B)$ *and* $Q \in Q_{A,B}^{(f)}$.

PROOF. If Condition (1) is fulfilled, then our assertion is merely a change of order of summation. Assume Condition (2). There exists a sequence $\{w_n\}$ in $F_0(A, B)$ such that $H_p(w - w_n) \to 0$ as $n \to \infty$. Since $w_n \in L_0(Y)$, we have $I(w_n) = -$

$J(w_n; Q)$ by (1). Noting that $w_n(y) \to w(y)$ as $n \to \infty$ for each $y \in Y$ and that $Q \in Q^{(f)}_{A,B}$ and

$$|\sum_{x \in X} K(x, y) u_Q(x)| = 0 \quad \text{for} \quad y \notin Q,$$

we see that $J(w_n; Q) \to J(w; Q)$ as $n \to \infty$. Since $I(w_n) \to I(w)$ as $n \to \infty$ by the above observation, we conclude that $I(w) = -J(w; Q)$.

We proved the following max-flow min-cut theorem in [5]:

THEOREM 2.4. $M(W; F_0(A, B)) = M^*(W; Q_{A,B})$.

§3. Extremal width $\mu_p(\Lambda)$ of a set Λ of cuts

We recall the extremal width of a set of cuts which was defined in [3].

For a set Λ of cuts, we define the extremal width $\mu_p(\Lambda)$ of Λ (of order p) as the inverse of the value of the following convex programming problem:

(CP) Minimize $H_p(W)$

subject to $W \in L^+(Y)$ and $\sum_{y \in Q} W(y) \geq 1$ for all $Q \in \Lambda$.

Denote by $E(\Lambda)$ the set of all feasible solutions of (CP). Then

$$\mu_p(\Lambda)^{-1} = \inf\{H_p(W); W \in E(\Lambda)\}.$$

The following properties of $\mu_p(\Lambda)$ were proved in [3]:

LEMMA 3.1. *If* $\Lambda_1 \subset \Lambda_2$, *then* $\mu_p(\Lambda_1) \geq \mu_p(\Lambda_2)$.

LEMMA 3.2. $\sum_{n=1}^{\infty} \mu_p(\Lambda_n)^{-1} \geq \mu_p(\bigcup_{n=1}^{\infty} \Lambda_n)^{-1}$.

We say that a set Λ of cuts is p-exceptional if $\mu_p(\Lambda) = \infty$. By the above lemmas, we see that any subset of a p-exceptional set is p-exceptional and that the countable union of p-exceptional sets is also p-exceptional.

LEMMA 3.3. *A set* Λ *of cuts is p-exceptional if and only if there exists* $W \in L^+(Y)$ *such that* $H_p(W) < \infty$ *and* $\sum_{y \in Q} W(y) = \infty$ *for all* $Q \in \Lambda$. *We call this* W *a penalty function for* Λ.

COROLLARY 3.4. *A p-exceptional set of cuts does not contain a finite cut.*

LEMMA 3.5. *Let* Λ *be a set of* $Q_{A,B} - Q^{(f)}_{A,B}$. *Then* Λ *is p-exceptional if any one of the following conditions is fulfilled*:

(1) $\sum_{y \in Y} r(y) < \infty$;

(2) $r(y)$ *is bounded and* Λ *contains at most countable cuts.*

PROOF. Assume Condition (1) and let $W=1$ on Y. Then $\sum_{y\in Q} W(y)=\infty$ for all $Q\in\Lambda$ and $H_p(W)=\sum_{y\in Y} r(y)<\infty$, so that $\mu_p(\Lambda)=\infty$. Next assume Condition (2). On account of Lemma 3.2, it suffices to show that $\{Q\}$ is p-exceptional for every $Q\in\Lambda$. Let $Q=\{y_k;\ k=1,2,\cdots\}\in\Lambda$ and define $W\in L(Y)$ by

$$W(y_y)=[kr(y_k)^{1/p}]^{-1} \text{ for each } k \text{ and } W(y)=0 \text{ on } Y-Q.$$

It is clear that $H_p(W)<\infty$. By our assumption, there exists $t>0$ such that $r(y_k)\le t$ for all k, so that

$$\textstyle\sum_{y\in Q} W(y)\ge t^{-1/p}\sum_{k=1}^{\infty} k^{-1}=\infty.$$

Thus $\{Q\}$ is p-exceptional.

The following lemma which was proved in [3] will play an important role in our study.

LEMMA 3.6. *Let Λ be a set of cuts and assume that a sequence $\{W_n\}$ of nonnegative functions converges to 0 in $L_p(Y; r)$, i.e., $H_p(W_n)\to 0$ as $n\to\infty$. Then there exist a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ and an p-exceptional subset Λ' of Λ such that* $\lim_{k\to\infty}\sum_{y\in Q} W_{n_k}(y)=0$ *for every* $Q\in\Lambda-\Lambda'$.

§4. Main results

Denote by $Q_{A,B}^{(\infty)}$ the totality of p-exceptional subsets of $Q_{A,B}$. Then every $\Lambda\in Q_{A,B}^{(\infty)}$ is a subset of $Q_{A,B}-Q_{A,B}^{(f)}$ by Corollary 3.4.

Given $W\in L^+(Y)$ and mutually disjoint nonempty finite subsets A and B of X, consider the following maximin cut problem:

(MM) Find $M^{\#}(W; Q_{A,B})=\sup\{M^*(W; Q_{A,B}-\Lambda);\ \Lambda\in Q_{A,B}^{(\infty)}\}$.

Here $M^*(W; Q_{A,B}-\Lambda)$ is a min-cut problem defined in §2.

We shall prove the following duality theorem:

THEOREM 4.1. *Let $W\in L^+(Y)$ and $H_p(W)<\infty$. Then the following equality holds:*

$$M(W; F_p(A, B))=M^{\#}(W; Q_{A,B}).$$

PROOF. Let w be a feasible solution of our max-flow problem, i.e., $w\in F_p(A, B)$ and $|w(y)|\le W(y)$ on Y. Then there exists a sequence $\{w_n\}$ in $F_0(A, B)$ such that $H_p(w-w_n)\to 0$ as $n\to\infty$. For any $Q\in Q_{A,B}$, we have $I(w_n)=-J(w_n; Q)$ by Lemma 2.3, so that

$$|I(w_n)|\le \textstyle\sum_{y\in Y}|w_n(y)|\,|\sum_{x\in X} K(x, y)u_Q(x)|=\sum_{y\in Q}|w_n(y)|.$$

Put $W_n(y)=|w_n(y)-w(y)|$. Since $H_p(W_n)\to 0$ as $n\to\infty$, we can find by Lemma 3.6

a subsequence $\{W_{n_k}\}$ of $\{W_n\}$ and a p-exceptional subset Λ_0 of $Q_{A,B}$ such that $\sum_{y\in Q} W_{n_k}(y) \to 0$ as $k\to\infty$ for every $Q\in Q_{A,B}-\Lambda_0$. Note that $I(w_{n_k})\to I(w)$ as $k\to\infty$. From the relation

$$\sum_{y\in Q}|w_{n_k}(y)| - \sum_{y\in Q}|w(y)| \le \sum_{y\in Q} W_{n_k}(y),$$

it follows that

$$|I(w)| \le \limsup_{k\to\infty} \sum_{y\in Q}|w_{n_k}(y)| \le \sum_{y\in Q}|w(y)| \le \sum_{y\in Q} W(y)$$

for every $Q\in Q_{A,B}-\Lambda_0$, and hence

$$I(w) \le M^*(W; Q_{A,B}-\Lambda_0) \le M^\#(W; Q_{A,B}).$$

Therefore $M(W; F_p(A, B)) \le M^\#(W; Q_{A,B})$. To prove the converse inequality, let t be any number such that $t < M^\#(W; Q_{A,B})$. There is a p-exceptional subset Λ_1 of $Q_{A,B}$ such that $M^*(W; Q_{A,B}-\Lambda_1) > t$, By Lemma 3.3, we can find a penalty function W' for Λ_1, i.e., $W'\in L^+(Y)$ such that $H_p(W')<\infty$ and $\sum_{y\in Q} W'(y)=\infty$ for all $Q\in\Lambda_1$. For any $\varepsilon>0$, we see easily that

$$M^*(W+\varepsilon W'; Q_{A,B}) \ge M^*(W; Q_{A,B}-\Lambda_1) > t,$$

which is the so-caled penalty method. By Theorem 2.4,

$$M(W+\varepsilon W'; F_0(A, B)) = M^*(W+\varepsilon W'; Q_{A,B}).$$

Since $M(W+\varepsilon W'; F_0(A, B)) > t$, there exists $w_\varepsilon\in F_0(A, B)$ such that $I(w_\varepsilon)>t$ and $|w_\varepsilon(y)| \le W(y)+\varepsilon W'(y)$ on Y. Noting that $H_p(w_\varepsilon) \le 2^p[H_p(W)+\varepsilon^p H_p(W')]$ and taking $\varepsilon=1/n$ for $n=1, 2,\dots$, we can find a weakly convergent subsequence of $\{w_\varepsilon\}$. Denote it by $\{w_n\}$ and let w be the limit. Since $F_p(A, B)$ is convex and strongly closed, it is weakly closed (cf. [1]). Therefore $w\in F_p(A, B)$. Since $w_n(y)\to w(y)$ as $n\to\infty$ for every $y\in Y$, $|w(y)|\le W(y)$ on Y. By Remark 2.2, we see that $I(w_n)\to I(w)$ as $n\to\infty$. Thus $I(w)\ge t$, and $M(W; F_p(A, B))\ge t$. Therefore we have

$$M^\#(W; Q_{A,B}) \le M(W; F_p(A, B)).$$

THEOREM 4.2. *Let* $W\in L^+(Y)$ *and* $H_p(W)<\infty$. *Then there exists an optimal solution* w *of* (M$F_p(A, B)$), *i.e.,* $w\in F_p(A, B)$ *such that* $|w(y)|\le W(y)$ *on* Y *and* $I(w) = M(W; F_p(A, B))$.

PROOF. There exists a sequence $\{w_n\}$ in $F_p(A, B)$ such that $|w_n(y)|\le W(y)$ on Y and $I(w_n)\to M(W; F_p(A, B))$ as $n\to\infty$. Since $\{H_p(w_n)\}$ is bounded, we can find a weakly convergent subsequence of $\{w_n\}$. Denote it again by $\{w_n\}$ and let w be the limit. Since $F_p(A, B)$ is weakly closed, we see that $w\in F_p(A, B)$. It follows that $|w(y)|\le W(y)$ on Y. By Remark 2.2, $I(w)=M(W; F_p(A, B))$.

THEOREM 4.3. *There exists an optimal solution* Λ^* *of* (MM), *i.e.,* $\Lambda^*\in Q_{A,B}^{(\infty)}$ *such* *that* $M^{\#}(W; Q_{A,B}) = M^*(W; Q_{A,B} - \Lambda^*)$.

PROOF. There is a sequence $\{\Lambda_n\}$ of p-exceptional subsets of $Q_{A,B}$ such that $M^{\#}(W; Q_{A,B}) - 1/n < M^*(W; Q_{A,B} - \Lambda_n)$ for every n. Let Λ^* be the union of $\{\Lambda_n\}$. Then Λ^* is also a p-exceptional subset of $Q_{A,B}$ by Lemma 3.2 and contains Λ_n, so that

$$M^*(W; Q_{A,B} - \Lambda_n) \leq M^*(W; Q_{A,B} - \Lambda^*) \leq M^{\#}(W; Q_{A,B}).$$

Therefore $M(W; Q_{A,B} - \Lambda^*) = M^{\#}(W; Q_{A,B})$.

Now we obtain max-flow min-cut theorems:

THEOREM 4.4. *Let* $W \in L^+(Y)$ *and* $H_p(W) < \infty$. *If* $\sum_{y \in Y} r(y) < \infty$, *then* $M(W; F_p(A, B)) = M^*(W; Q^{(f)}_{A,B})$.

PROOF. By Lemma 3.5 (1), we see that $\Lambda_0 = Q_{A,B} - Q^{(f)}_{A,B}$ is a p-exceptional set. By Corollary 3.4, $\Lambda \subset \Lambda_0$ for every $\Lambda \in Q^{(\infty)}_{A,B}$. Thus we have

$$M^*(W; Q^{(f)}_{A,B}) = M^*(W; Q_{A,B} - \Lambda_0) \geq M^*(W; Q_{A,B} - \Lambda),$$

and hence $M^*(W; Q^{(f)}_{A,B}) = M^{\#}(W; Q_{A,B})$. Our assertion follows from Theorem 4.1.

By Lemma 3.5(2), we obtain:

Theorem 4.5. *Let* $W \in L^+(Y)$ *and* $H_p(W) < \infty$ *and assume that* $r(y)$ *is bounded. If* $Q_{A,B} - Q^{(f)}_{A,B}$ *is at most countable, then*

$$M(W; F_p(A, B)) = M^*(W; Q^{(f)}_{A,B}).$$

If we omit the condition that both A and B are finite sets, then Theorems 4.1 and 4.4 do not hold in general. This is shown by

EXAMPLE 4.6. Let $X = \{x_n, x'_n; n \in Z^+\}$ and $Y = \{y_{n+1}, y'_{n+1}, y''_n; n \in Z^+\}$ and define K by

$$K(x_n, y_{n+1}) = K(x'_n, y'_{n+1}) = K(x_n, y''_n) = -1,$$

$$K(x_{n+1}, y_{n+1}) = K(x'_{n+1}, y'_{n+1}) = K(x'_n, y''_n) = 1$$

for all $n \in Z^+$ and

$$K(x, y) = 0 \quad \text{for any other pair,}$$

where Z^+ is the set of all nonnegative integers. Let us take $A = \{x_n; n \in Z^+\}$ and $B = \{x'_n; n \in Z^+\}$ and assume that $\sum_{y \in Y} r(y) < \infty$. Then $F(A, B) = F_p(A, B)$ for any $p > 1$. Now we define $W \in L^+(Y)$ by

$$W(y_{n+1}) = W(y'_{n+1}) = 0 \quad \text{and} \quad W(y''_n) = 2^{-n}$$

for all $n \in Z^{+}$. Then $M(W; F_p(A, B)) = M^*(W; Q_{A,B}) = 2$. Since $Q_{A,B}^{(f)}$ is empty, we have $M^{\#}(W; Q_{A,B}) = M^*(W; Q_{A,B}^{(f)}) = \infty$ by Theorem 4.4.

References

[1] V. Barbu and Th. Precupanu, Convexity and optimization in Banach spaces, Sijthoff & Noordhoff International Publishers, 1978.

[2] L. R. Ford and D.R. Fulkerson, Flows in networks, Princeton Univ. Press, Princeton, N. J., 1962.

[3] A. Murakami and M. Yamasaki, Extremum problems with respect to ideal boundary components of an infinite network, Hiroshima Math. J. **19** (1989), 77–87.

[4] T. Nakamura and M. Yamasaki, Minimal cut problems on an infinite network, ibid. **7** (1977), 597–603.

[5] T. Nakamura and M. Yamasaki, Extremal length of an infinite network which is not necessarily locally finite, ibid. **7** (1977), 813–826.

[6] G. Strang, Maximum flow through a domain, Math. Programming **26** (1983), 123–143.

[7] M. Yamasaki, Extremum problems on an infinite network, Hiroshima Math. J. **5** (1975), 223–250.