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Note on Periodic Instantons

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It is shown that under suitable assumptions the actions of periodic instantons may be identified with gauge anomalies, and the actions with even topological charges are given by weights and magnetic charges.

§1. Introduction

For periodic instantons, interesting results have been obtained by Garland and Murray [3], [4]. On the other hand gauge anomalies have been investigated by Alvarez-Gaumé and Ginsparg [1] and others. By their work we know that gauge anomalies are deeply concerned with homotopy theory.

In this article we discuss periodic instantons which satisfy suitable conditions ((3), (4) in Section 2). In Section 2, using a conformal compactification [7] the action is identified with the gauge anomaly in [1] up to the constant $4\pi^2$. In Section 3 we consider periodic instantons with even topological charges. There we refer to [6] for lifting group actions on bundles. For the $SU(2)$ instantons we can directly verify the existence of a lifting action, but for a use in future we discuss from a general view point.

Troughout the article S^1 denotes the circle with length 2π, and w.n. means a winding number.

§2. Periodic instantons

Let $\tilde{A}(\theta, x)$ be a connection on the space $S^1 \times R^3$. Then it can be described as $\tilde{A} = \Phi d\theta + \sum_{i=1}^{3} A_i dx^i$, where Φ, A_i: $R^3 \to \Omega su(2)$ the Lie algebra of all smooth maps from the circle group into the Lie algebra $su(2)$. Here we assume that the connection satisfies the following conditions.

(1) $\displaystyle\int_{S^1 \times R^3} |F_{\tilde{A}}|^2 < \infty$, $F_{\tilde{A}}$ is the curvature,

(2) $F_{\tilde{A}}$ is self-dual,

(3) $|\dot{A}| \longrightarrow 0$ as $|x| \longrightarrow \infty$, $\dot{A}_i = \dfrac{\partial A_i}{\partial \theta} d\theta \wedge dx^i$ for $\theta \in S^1$,

(4) $A \longrightarrow$ a diagonal form as $|x| \longrightarrow \infty$.

By (1), (2) the connection can be extended to a connection on a bundle over a conformal compactification $S^1 \times R^3 \cup S^2_\infty$ ([7]). By [7]

$$S^1 \times R^3 \cup S^2_\infty = S^1 \times D^3_N \cup S^2_\infty \times ([0, \infty) \times {}_f S^1) \cup S^2_\infty,$$

where D^3_N denotes a 3 disc with sufficiently large radius and $[0, \infty) \times {}_f S^1$ is a warped product space with function $f(\tau) = e^{-2\tau}$ for $\tau \in [0, \infty)$. Then the compactification is S^1-homeomorphic to the unreduced join $S^1 \times D^3 \cup D^2 \times S^2 = S^1 * S^2$.

REMARK. By a direct verification we can see that Chakrabarti's 1-pole solutions [2] and the solution in [5] fulfills the assumptions (1) $\sim$ (4). Let $\{E_a\}$ be the basis of the Lie algebra $su(2)$, $A_a = \Sigma A_{ai}\, dx^i$ and $A = \Sigma E_a \otimes A_a$. Then by (3) $\dot{A}_{ai} \to 0$ for each a, i. Further, by (1) and (2), $|\tilde{A}|$ is bounded ([7]).

By [3] (actually using Bianchi identity, integration by parts and Stokes theorem),

$$\int_{S^1 \times R^3} |F_{\tilde{A}}|^2 = 2 \int_{S^1 \times S^2_\infty} \mathrm{tr}\left(F_A \wedge \Phi d\theta + \frac{1}{2} \mu_0 A \wedge \dot{A} \right),$$

where μ_0 is a constant,

Let $s_0 : S^1 \times D^3 \to P$, $s_\infty : D^2 \times S^2 \to P$ be local sections of the principal $SU(2)$-bundle, and $\hat{A}$ be the Ehresmann connection. Then we have

$$s_0^*(\hat{A}) = \tilde{A},\ s_0^*(\hat{A}) = g^{-1}(s_\infty^*(\hat{A}))g + g^{-1}dg,$$

where $g : S^1 \times S^2 \to SU(2)$ is the corresponding gauge transformation. By [7],

$$\langle s_\infty^*(\hat{A}), \partial/\partial\theta \rangle \longrightarrow 0 \quad \text{at infinity } S^2_\infty. \quad \text{(See also [4]).}$$

Then $\Phi d\theta \to g^{-1} d_\theta g$ asymptotically. By the remark above

$$\int_{S^1 \times S^2} A \wedge \dot{A} \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty.$$

Now we can assume that near the limit set S^2_∞ $A = g^*(A_0)$ holds for the restriction A_0 on S^2_∞ of the extended connection, because near S^2_∞ the space is a product $D^2 \times S^2$. By the assumption (4) $A = g^*(A_0) \to$ a diagonal form. Then $F_A = F_{g^*(A_0)} \to d_x g^*(A_0)$, where d_x is the exterior derivative in R^3. Thus

$$\begin{aligned} \int_{S^1 \times R^3} |F_{\tilde{A}}|^2 &= 2 \int_{S^1 \times S^2} - \mathrm{tr}(g^{-1} d_\theta g \wedge g^* F_{A_0}) \\ &= -2 \int_{S^1 \times S^2} \mathrm{tr}(g^{-1} d_\theta g \wedge d_x g^*(A_0)). \end{aligned}$$

Here we refer to [1] for the gauge anomaly of the gauge transformation g: $S^1 \times S^2 \to SU(2)$. In our case the non zero modes condition is not necessary. But by a siutable gauge on the limit set S^2_∞ we can choose a connection such as

$$k\begin{pmatrix} i(-ydx+xdy) & 0 \\ 0 & -i(-ydx+xdy) \end{pmatrix} \quad \text{in a local form, } k \text{ is an integer,}$$

whose Dirac operator satisfies the non zero condition.

By [1] the gauge anomaly is given as follows.

$$\left(\frac{i}{2\pi}\right)^2 \int_{S^1\times S^2} \mathrm{tr}(g^{-1}d_\theta g \wedge d_x g^*(A_0))$$
$$= \text{w.n. } (g: S^1 \times S^2 \to SU(2)) \text{ (13.83}a\text{ and 13.103 in [8]).}$$

Hence we have

THEOREM. $$\int_{S^1\times R^3} |F_{\tilde{A}}|^2 = 8\pi^2 \times \text{w.n.} \quad (g: S^1 \times S^2 \to SU(2)).$$

§3. Instantons with even topological charges

For a compact Hausdorff space Y, let $X = S^1 * Y$ and $B = S^0 * X$ be unreduced joins. Let $\alpha: S^1 \to G$ be a homomorphism of the circle group to a compact connected Lie group. Then $\mathrm{Ad}(\alpha(\ ))$ gives an S^1-action on the group G. Denote by G_α the centralizer of the subgroup $\alpha(S^1)$ in G and G^α the quotient space G/G_α. The map

$$g^\alpha: G^\alpha \ni [h] \longrightarrow h\alpha(\ )h^{-1}\alpha(\ )^{-1} \in \Omega' G$$

is an injective map, where $\Omega' G$ is the based loop space. Consider the diagram of homotopy sets

$[S^1 * Y, G] \xrightarrow{\cong} [(e * Y, Y), (\Omega' G, e_0)] \xrightarrow{\cong} [S^0 * Y, \Omega' G] \xleftarrow{g^\alpha_*} [S^0 * Y, G^\alpha]$, where e is the unit element, e_0 the constant loop and $\cong$ means an isomorphism. Then we have

THEOREM (2 in [6]). If $[\chi'] \in [S^1 * Y, G]$ belongs to im g^α_*, then there is a bundle action on the bundle which corresponds to $[\chi']$.

Now consider the case $G = SU(2n-2)$, $B = S^0 * (S^1 * S^{2n-3})$, $n \geqq 2$, and $\alpha: S^1 \ni e^{i\theta} \to (e^{i\theta})^{n-1} \times (e^{-i\theta})^{n-1} \in G$.

PROPOSITION (4 in [6]). Im $g^\alpha_* \supset 2\pi_{2n-1}(SU(2n-2))$.

We have $G^\alpha = SU(2n-2)/S(U(n-1) \times U(n-1))$, and

$$[S^0 * S^{2n-3}, SU(2n-2)/S(U(n-1)\times U(n-1))] \xrightarrow[\cong]{\partial} \ker(i_0)_* \\ \cap \\ [S^{2n-3}, S(U(n-1)\times U(n-1))].$$

where $i_0\colon S(U(n-1)\times U(n-1)) \to SU(2n-2)$ is the inclusion map. Then the restriction on the fixed sphere S^{2n-2} of any C^{2n-2}-bundle of even topological charge splits as an S^1-vector bundle, i.e. $E_1 \oplus E_1^-$ with structure group $S(U(n-1)\times U(n-1))$, where the bundle $E_1 \oplus E_1^-$ is trivial as an $SU(2n-2)$-bundle.

Let $\gamma\colon S^2_\infty \to SU(2)/S(U(1)\times U(1))$ be a map. Instead of α above if we use a map $\alpha_p\colon S^1 \ni e^{i\theta} \to (e^{ip\theta})^{n-1} \times (e^{-ip\theta})^{n-1} \in SU(2)$, then

$$\Phi_\infty d\theta = g^{-1}d_\theta g = \mathrm{ip}\left\{\begin{pmatrix} e & 0 \\ 0 & \bar{e}\end{pmatrix}\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\gamma^{-1}\begin{pmatrix} \bar{e} & 0 \\ 0 & e\end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right\}d_\theta,$$

where $e = e^{ip\theta}$. Hence by the theorem in Section 2 we have

COROLLARY. $YM(\tilde{A}) = \dfrac{1}{2}\displaystyle\int_{S^1\times R^3} |F_{\tilde{A}}|^2 = 8\pi^2 p c_1(E_1)$,

where $c_1(E_1)$ is the first Chern class of E_1.

REMARK 1. Choose a representative $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix} \in SU(2)$ in the class γ, then

$$\gamma\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\gamma^{-1} = \begin{pmatrix} |\alpha|^2 - |\beta|^2 & -2\alpha\beta \\ -2\bar{\alpha}\bar{\beta} & |\beta|^2 - |\alpha|^2\end{pmatrix}.$$

Therefore γ is a stabilizer of $\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ if and only if $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix} \in S(U(1) \times U(1))$. Thus we can identify the Higgs field Φ with the map γ and its homotopy class with the magnetic charge $c_1(E_1)$.

REMARK 2. By [5] a periodic instanton is given by the periodic solution to the equation of motion, that is

$$\phi = 1 + \frac{\lambda^2}{2}\,\frac{\sinh r}{r}\,\frac{1}{\cosh r - \cos\theta},$$

where $r = |x|$. Let β be the period. In the integral $\displaystyle\int_0^\beta \frac{\nabla\phi}{\phi}d\theta$, the coefficient of x^i is given by

$$\left[\left(\frac{\lambda^2}{2}\frac{r\cosh r-\sinh r}{r^3}+\frac{\sinh r}{r}\right)\left(2\frac{1}{\sqrt{a^2-1}}\,\mathrm{Tan}^{-1}\sqrt{\frac{a+1}{a-1}}\tan\frac{\theta}{2}\right)\right.$$

$$\left.-\frac{2}{r}\,\mathrm{Tan}^{-1}\sqrt{\frac{a'+1}{a'-1}}\tan\frac{\theta}{2}\right]_0^{2\pi},$$

where $a=\cosh r+\dfrac{\lambda^2}{2r}\sinh r$, $a'=\cosh r$. Let τ be Pauli matrices and $\hat{x}=\dfrac{x}{r}$, and put

$$-\frac{\sqrt{-1}}{2}\int_0^\beta\frac{\nabla\phi}{\phi}d\theta=f(r)\sqrt{-1}\,\hat{x}\cdot\tau.$$

Then calculating the improper integral we obtain

$$\lim_{r\to 0}f(r)=\pi,\ \text{and}\ \lim_{r\to\infty}f(r)=0.$$

Thus we have a Skyrmion with barion number 1. Then it seems to be interesting to study carolons from a view point of Skyrmions [9].

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