Mem. Fac. Sci. Shimane Univ.,
25, pp. 45–51 Dec. 25, 1991

A Structure Theory of Freudenthal–Kantor Triple Systems IV

Noriaki KAMIYA

Department of Mathematics, Shimane University, Matsue 690 JAPAN
(Received September 4, 1991)

In this paper, we define the Jacobson radical and the Frattini subtriple system for Freudenthal–Kantor triple systems and investigate it. For the case of nilpotent triple system, it is shown that the Frattini subtriple system contains the Jacobson radical.

Introduction

The notion of radical plays an important role in the theory of algebras and triple systems. It seems to be interesting for us to know how the radical behaves in Freudenthal–Kantor triple systems. Thus in particular, we investigate the Jacobson radical and the Frattini subtriple system of the triple system in this article.

Throughout this paper, we shall be concerned with algebras and triple systems which are finite dimensional over a field Φ of characteristic different from 2 or 3, unless otherwise specified. We shall mainly employ the notation and terminology in [7, 8].

§1. Preliminaries

For $\varepsilon = \pm 1$, a triple system $U(\varepsilon)$ with triple product $\langle -, -, - \rangle$ is called a Freudenthal–Kantor triple system if

$$[L(a, b), L(c, d)] = L(\langle abc \rangle, d) + \varepsilon L(c, \langle bad \rangle) \qquad (U-1)$$

$$K(\langle abc \rangle, d) + K(c, \langle abd \rangle) + K(a, K(c, d)b) = 0 \qquad (K-1)$$

where $L(a, b)c = \langle abc \rangle$ and $K(a, b)c = \langle acb \rangle - \langle bca \rangle$.

PROPOSITION 1.1 ([8]). *Let $U(\varepsilon)$ be a Freudenthal–Kantor triple system. If P is a linear transformation of $U(\varepsilon)$ such that $P\langle xyz \rangle = \langle Px, Py, Pz \rangle$ and $P^2 = -\varepsilon Id$, then $(U(\varepsilon), [-, -, -])$ is a Lie triple system with respect to the triple product*

$$[xyz] := \langle xPyz \rangle - \langle yPxz \rangle + \langle xPzy \rangle - \langle yPzx \rangle. \qquad (1-1)$$

In particular, if $\varepsilon = -1$, $K(x, y) = 0$ and $P = Id$ (that is, $U(\varepsilon)$ is a Jordan triple system), then the triple product becomes $[xyz] = \langle xyz \rangle - \langle yxz \rangle$.

COROLLARY. ([7, 17]) *Let $U(\varepsilon)$ be a Freudenthal–Kantor triple system. Then* *the vector space* $T(\varepsilon) = U(\varepsilon) \oplus U(\varepsilon)$ *becomes a Lie triple system with respect to the triple product defined by*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} L(a, d) - L(c, b) & K(a, c) \\ -\varepsilon K(b, d) & \varepsilon(L(d, a) - L(b, c)) \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix}. \qquad (1-2)$$

The Lie triple system $T(\varepsilon)$ defined above is called the Lie triple system associated with a Freudenthal-Kantor triple system $U(\varepsilon)$. Then the standard imbedding Lie algebra $L(\varepsilon) = L(T(\varepsilon), T(\varepsilon)) \oplus T(\varepsilon)$ associated with $T(\varepsilon)$ is defined as follows;

$$[X_1, X_2] := ([H_1, H_2] + L(x_1, x_2)) \oplus H_1 x_2 - H_2 x_1 \qquad (1-3)$$

where $X_i = H_i + x_i$, $H_i \in L(T(\varepsilon), T(\varepsilon))$, $x_i \in T(\varepsilon)(i = 1, 2)$.

An element of the Lie algebra $L(\varepsilon)$ is expressed as a linear combination of

$$\begin{pmatrix} L(a, b) & K(c, d) \\ -\varepsilon L(e, f) & \varepsilon L(b, a) \end{pmatrix} \oplus \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (1-4)$$

PROPOSITION 1.2 ([7]). *For a Freudenthal-Kantor triple system* $U(\varepsilon)$, *the Lie triple system* $T(\varepsilon)$ *associated with* $U(\varepsilon)$ *and the standard imbedding Lie algebra* $L(\varepsilon)$, *we have*

(a) $R(T(\varepsilon)) = T(R(U(\varepsilon)))$

(b) $R(T(\varepsilon)) = R(L(\varepsilon)) \cap T(\varepsilon)$

(c) $R(L(\varepsilon)) = L(T(\varepsilon), R(T(\varepsilon))) \oplus R(T(\varepsilon))$

where $R(U(\varepsilon))$ *(resp.* $R(T(\varepsilon))$, $R(L(\varepsilon))$) *is the solvable radical of* $U(\varepsilon)$. *(resp.* $T(\varepsilon)$, $L(\varepsilon)$)

PROPOSITION 1.3 ([11]). *Let* $U(\varepsilon)$ *be a Freudenthal-Kantor triple system with a left neutral pair* (u, v). *(i.e.* $L(u, v) = Id$). *Then there exists a subtriple system* S *of* $U(\varepsilon)$ *such that* $U(\varepsilon) = S \oplus R(U(\varepsilon))$ *and* $S \cong U(\varepsilon)/R(U(\varepsilon))$.

We remark that the notion of Freudenthal-Kantor triple systems contains that of Jordan triple systems [13], generalized Jordan triple systems of second order [12], structurable algebras [2] and J-ternary algebras [1, 4].

§2. Jacobson radical

In this section, we shall define the Jacobson radical for a Freudenthal-Kantor triple system and investigate it.

DEFINITION. The *Jacobson radical* of Freudenthal-Kantor triple system $U(\varepsilon)$ is defined by the intersection of all maximal ideals of $U(\varepsilon)$, with the convention that this intersection is $U(\varepsilon)$ if there are no maximal ideals. We denote it by $J_R(U(\varepsilon))$.

Similarly, in the case of a Lie triple system T, we denote it by $J_R(T)$. (cf. [11])

This notion of Jacobson radical is different from that of [5] due to Prof. L. Hogben and K. McCrimmon for Jordan algebras.

LEMMA 2.1 *The Jacobson radical* $J_R(U(\varepsilon))$ *of* $U(\varepsilon)$ *is contained in the derived subtriple system* $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$.

PROOF. In the case $U(\varepsilon)=\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$, this is trivial. So we assume that $U(\varepsilon)\neq\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$. If $x\notin\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$, then there is a subspace M of $U(\varepsilon)$ which is complementary to the subspace Φx spanned by x and so M contains $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$. Then M is a maximal ideal of $U(\varepsilon)$. Since $J_R(U(\varepsilon))\subseteq M$, $x\notin J_R(U(\varepsilon))$. Therefore we have $J_R(U(\varepsilon))\subseteq\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$.

THEOREM 2.2 *Let* $U(\varepsilon)$ *be a Freudenthal-Kantor triple system with a left neutral pair* (u, v). *Then we have*

$$J_R(U(\varepsilon))=\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle\cap R(U(\varepsilon)),$$

where $R(U(\varepsilon))$ *is the solvable radical of* $U(\varepsilon)$.

PROOF. If I is a maximal ideal of $U(\varepsilon)$, then the factor triple system $U(\varepsilon)/I$ is simple or $\langle U(\varepsilon)/I,\ U(\varepsilon)/I,\ U(\varepsilon)/I\rangle=0$. In the former case, since $U(\varepsilon)/I$ is simple, I contains $R(U(\varepsilon))$. From Proposition 1.3, $U(\varepsilon)$ is decomposed to

$$U(\varepsilon)=B_0\oplus R(U(\varepsilon)) \qquad (2-1)$$

(B_0 is a semisimple subtriple system of $U(\varepsilon)$). Hence I is of the form $M+R(U(\varepsilon))$, where M is a maximal ideal of B_0. Since the semisimple subtriple system B_0 can be expressed as the direct sum of simple ideals (cf. Theorem 2.5 in [7]), the Jacobson radical of B_0 is 0. Hence the intersection of all such maximal ideals of $U(\varepsilon)$ equals to $R(U(\varepsilon))$. In the latter case, since $\langle U(\varepsilon)/I,\ U(\varepsilon)/I,\ U(\varepsilon)/I\rangle=0$, I contains $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$. Hence the intersection of all such maximal ideals of $U(\varepsilon)$ contains $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$. Considering two cases, we have

$$\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle\cap R(U(\varepsilon))\subseteq J_R(U(\varepsilon))\subseteq R(U(\varepsilon)). \qquad (2-2)$$

By Lemma 2.1 we have

$$J_R(U(\varepsilon))\subseteq\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle. \qquad (2-3)$$

Combining $(2-2)$ with $(2-3)$, we obtain

$$J_R(U(\varepsilon))=\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle\cap R(U(\varepsilon)).$$

This completes the proof.

COROLLARY. *If* $U(\varepsilon)$ *is a perfect (that is* $U(\varepsilon)=\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$*) Freudenthal-Kantor triple system with a left neutral pair, then we have*

$$J_R(U(\varepsilon)) = R(U(\varepsilon)).$$

In particular, if $U(\varepsilon)$ *is semisimple, then* $J_R(U(\varepsilon)) = 0$.

Corollary. *If* $U(\varepsilon)$ *is a L-solvable Freudenthal-Kantor triple system with a left neutral pair, then*

$$J_R(U(\varepsilon)) = \langle U(\varepsilon)\, U(\varepsilon)\, U(\varepsilon) \rangle.$$

THEOREM 2.3. *Let* $U(\varepsilon)$ *be a Freudenthal-Kantor trile system with a left neutral pair and* $T(\varepsilon)$ *be the Lie triple system associated with* $U(\varepsilon)$. *Then,*

$$J_R(T(\varepsilon)) = T(J_R(U(\varepsilon))).$$

PROOF. From the definition of Lie triple system $T(\varepsilon)$ associated with a Freudenthal-Kantor triple system $U(\varepsilon)$, we have

$$[T(\varepsilon)\, T(\varepsilon)\, T(\varepsilon)] = \langle U(\varepsilon)\, U(\varepsilon)\, U(\varepsilon) \rangle \oplus \langle U(\varepsilon)\, U(\varepsilon)\, U(\varepsilon) \rangle. \tag{2-4}$$

Hence we have the following

$$J_R(T(\varepsilon)) = [T(\varepsilon)\, T(\varepsilon)\, T(\varepsilon)] \cap R(T(\varepsilon))$$

(by Theorem A in [12])

$$= (\langle U(\varepsilon)\, U(\varepsilon)\, U(\varepsilon) \rangle \oplus \langle U(\varepsilon)\, U(\varepsilon)\, U(\varepsilon) \rangle) \cap (R(U(\varepsilon)) \oplus R(U(\varepsilon)))$$

(by (2 − 4) and Prop. 1.2)

$$= J_R(U(\varepsilon)) \oplus J_R(U(\varepsilon))$$

(by Theorem 2.2)

$$= T(J_R(U(\varepsilon))).$$

This completes the proof.

THEOREM 2.4. *Let* $U(\varepsilon)$ *be a Freudenthal-Kantor triple system with a left neutral pair and* $L(\varepsilon)$ *be the standard imbedding Lie algebra. Then,*

$$J_R(L(\varepsilon)) \cap U(\varepsilon) = J_R(U(\varepsilon))$$

where $J_R(L(\varepsilon))$ *is the Jacobson radical of* $L(\varepsilon)$.

PROOF. From Theorem B in [12], it follows that

$$J_R(L(\varepsilon)) = L(T(\varepsilon), R(T(\varepsilon))) \oplus J_R(T(\varepsilon)). \tag{2-5}$$

By Theorem 2.3, we have

$$J_R(T(\varepsilon)) = T(J_R(U(\varepsilon))). \tag{2-6}$$

Combining $(2-5)$ with $(2-6)$, we obtain

$$J_R(L(\varepsilon)) = L(T(\varepsilon), R(T(\varepsilon))) \oplus (J_R(U(\varepsilon)) \oplus J_R(U(\varepsilon))).$$

Hence we get

$$J_R(L(\varepsilon)) \cap U(\varepsilon) = J_R(U(\varepsilon)).$$

This completes the proof.

§3. Frattini subtriple system

In this final section, we shall define the Frattini subtriple system for Freudenthal-Kantor triple systems and consider it.

DEFINITION. A proper subtriple system M of a Freudenthal-Kantor triple system $U(\varepsilon)$ is called the *maximal subtriple system* $U(\varepsilon)$ if the only subtriple system properly containing M is $U(\varepsilon)$ itself.

DEFINITION. The *Frattini subtriple system* $F(U(\varepsilon))$ of $U(\varepsilon)$ is the intersection of all maximal subtriple system of $U(\varepsilon)$. It means that $F(U(\varepsilon)) = U(\varepsilon)$ if $U(\varepsilon)$ has no maximal subtriple system.

PROPOSITION 3.1. *The Frattini subtriple system $F(U(\varepsilon))$ of $U(\varepsilon)$ is contained in the derived subtriple system $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$.*

PROOF. If $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle = U(\varepsilon)$, then it is trival. Let $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle \neq U(\varepsilon)$. Then we assume that $x \in F(U(\varepsilon))$, $x \notin \langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$. Furthermore, assume that $U(\varepsilon) = B + \langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle + \Phi x$, where B is a complement of $\Phi x + \langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$ in $U(\varepsilon)$ as the vector space over a field Φ. Let $M = \langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle + B$. Then M is a maximal subtriple system of $U(\varepsilon)$. From the assumption that x is an element of $F(U(\varepsilon))$, it must belong to such subtriple system M. This is a contradiction. Hence, $F(U(\varepsilon))$ is contained in $\langle U(\varepsilon)U(\varepsilon)U(\varepsilon)\rangle$. This completes the proof.

For every ideal A of $U(\varepsilon)$, we define A^n by

$$A^n := A^{n-1} * U(\varepsilon), \ A^0 = A,$$

where $A * U(\varepsilon) = \langle AU(\varepsilon)U(\varepsilon)\rangle + \langle U(\varepsilon)AU(\varepsilon)\rangle + \langle U(\varepsilon)U(\varepsilon)A\rangle$.

An ideal A of a Freudenthal-Kantor triple system $U(\varepsilon)$ is called *nilpotent* if there exists a positive integer n such that $A^n = (0)$. The least such n is said to be the *class of nilpotency* of A.

DEFINITION. The *upper central series* of a nilpotent Freudenthal-Kantor triple system $U(\varepsilon)$ is a sequence $\{Z_i\}$ of ideals Z_i of $U(\varepsilon)$ defined inductively as follows, $Z_0 = \{0\}$ and, for all $i \geq 0$, Z_{i+1}/Z_i is the center of $U(\varepsilon)/Z_i$. i.e.,

$$Z_1(U(\varepsilon)) := \{x \in U(\varepsilon) | x * U(\varepsilon) = 0\}$$

$$Z_i(U(\varepsilon))/Z_{i-1}(U(\varepsilon)) := Z_1(U(\varepsilon)/Z_{i-1}(U(\varepsilon))) \ (i \geq 1).$$

If n is the class of nilpotency for $U(\varepsilon)$, then Z_n coincides with $U(\varepsilon)$.

DEFINITION. If H is any subtriple system of a Freudenthal–Kantor triple system $U(\varepsilon)$, its *idealizer* $I(H)$ is the set

$$H \cup \{x \in U(\varepsilon) | x * H \subseteq H\}.$$

We note the following:

$$\{x \in U(\varepsilon) | x * H \subseteq H\} \supseteq \{x \in U(\varepsilon) | x * U(\varepsilon) \subseteq H\},$$

$$\langle xI(H)I(H) \rangle + \langle I(H)xI(H) \rangle + \langle I(H)I(H)x \rangle \subseteq x * I(H) \subseteq H, \qquad \textit{for all } x \in H.$$

PROPOSITION 3.2. *Let $U(\varepsilon)$ be a nilpotent Freudenthal–Kantor triple system. Then every proper subtriple system H* of *$U(\varepsilon)$ is a proper subtriple system of its idealizer $I(H)$.*

PROOF. We define a sequence $\{H_i\}$ of subtriple system of $U(\varepsilon)$ such that $H_0 = H$ and H_{i+1} is the idealizer in $U(\varepsilon)$ of H_i for all $i \geq 0$. By the induction, we can easily show that $H_i \supseteq Z_i$ for all $i \geq 0$. Since $Z_n = U(\varepsilon)$ if n is the class of nilpotency, we get $H_n = U(\varepsilon)$. This implies that $H_0 \subsetneqq H_1$.

PROPOSITION 3.3. *Let $U(\varepsilon)$ be a Freudenthal–Kantor triple system and M be a maximal subtriple system of $U(\varepsilon)$. If M contains $U(\varepsilon)^n$, for some integer n, then M is an ideal of $U(\varepsilon)$.*

PROOF. Let M be a maximal subtriple system such that $M \supseteq U(\varepsilon)^n$. Since $U(\varepsilon)^n$ is an ideal, we can define the quotient triple systems $\overline{U(\varepsilon)}$ and $\overline{M}$ by $\overline{U(\varepsilon)} = U(\varepsilon)/U(\varepsilon)^n$, $\overline{M} = M/U(\varepsilon)^n$. Then $\overline{U(\varepsilon)} \supseteq \overline{M}$ and $(\overline{U(\varepsilon)})^n = 0$. Hence $\overline{U(\varepsilon)}$ is nilpotent. By Proposition 3.2, $I(\overline{M}) \supsetneqq \overline{M}$. Since $M/U(\varepsilon)^n$ is a maximal subtriple system of $U(\varepsilon)/U(\varepsilon)^n$, it must be that $I(M/U(\varepsilon)^n) = U(\varepsilon)/U(\varepsilon)^n = \overline{U(\varepsilon)}$. Hence $\overline{M}$ is an ideal of $\overline{U(\varepsilon)}$. Therefore this completes the proof.

Form Proposition 3.3, we have the following theorems.

THEOREM 3.4. *Let $U(\varepsilon)$ be a nilpotent Freudenthal–Kantor triple system and M be a maximal subtriple system of $U(\varepsilon)$. Then M is an ideal of $U(\varepsilon)$.*

THEOREM 3.5. *Let $U(\varepsilon)$ be a nilpotent Freudenthal–Kantor triple system. Then the Frattini subtriple system $F(U(\varepsilon))$ of $U(\varepsilon)$ is an ideal of $U(\varepsilon)$. In particular, $F(U(\varepsilon))$ contains the Jacobson radical $J_R(U(\varepsilon))$.*

References

[1] B. N. Allison,; *A construction of Lie algebras from J-ternary algebras,* Amer. J. Math. **98** (1976), 285–294.

[2] B. N. Allison,; *A class of nonassociative algebras with involution containing the class of Jordan algebras,* Math. Ann. **237** (1978), 133–156.

[3] J. R. Faulkner and J. C. Ferrar,; *On the structure of sympletic ternary algebras,* Nederl. Akd. Wetensch. Proc. Ser. **A75** (= Indag. Math. 34.), (1972), 247–256

[4] W. Hein,; *A construction of Lie algebras by triple systems,* Trans. Amer. Math. Soc. **205** (1975), 79–95.

[5] L. Hogben and K. McCrimmon,; *Maximal modular inner ideals and the Jacobson radical of a Jordan algebra.* J. Alg., **68** (1981), 155–169.

[6] N. Jacobson,; *Lie algebras,* Interscience, New York, 1962,

[7] N. Kamiya,; *A structure theory of Freudenthal-Kantor triple systems,* J. Alg., **110** (1987), 108–123.

[8] N. Kamiya,; *A structure theory of Freudenthal-Kantor triple systems II,* Comm. Math. Univ. Sancti Pauli., **38** (1989), 41–60.

[9] N. Kamiya,; *A structure theory of Freudenthal-Kantor triple systems III,* Mem. Fac. Sci., Shimane Univ., **23** (1989), 33–51.

[10] N. Kamiya,; *The construction of all simple Lie Algebra over C from balanced Freudenthal-Kantor triple systems,* Contributions to general algebra **7** (1991), 205–213.

[11] N. Kamiya,; *The Wedderburn principal theorem for Freudenthal-Kantor triple systems,* preprint.

[12] N. Kamiya,; *On the Jacobson radicals of Lie triple systems,* Yokohama Math. J., **28** (1980), 107–109.

[13] N. Kamiya,; *The nil radical for Freudenthal-Kantor triple systems,* Nova Journal of Algebra and Geometry, to appear.

[14] I. L. Kantor,; *Models of exceptional Lie algebras.* Soviet Math. Dokl. **14** (1973), 254–258.

[15] O. Loos,; *Lectures on Jordan triples,* The Univ. of British Columbia, Vancouver 1971.

[16] K. Yamaguti,; *Remarks on characterization of the points and symplecta in the metasymplectic geometry,* Mem. Fac. Ed. Kumamoto Univ. Natur. Sci., **11** (1976), 1–8.

[17] K. Yamaguti,; *On the metasymplectic geometry and triple systems,* Sûrikaisekikenkyûsho Kôkyûroku **306** (1977), 55–92 [In Japanese].