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A Class of Double Lie Algebras on Simple Lie Algebras and Projectivity of Simple Lie Groups

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The concept of projective double Lie algebras on a Lie algebra is introduced, and they are investigated on real simple Lie algebras of odd-dimension. The results are applied to classifying all geodesic homogeneous local Lie loops in projective relation with any odd-dimensional real simple Lie group.

Introduction

In the previous paper [4], we have investigated how to determine geodesic homogeneous local Lie loops in projective relation with a given Lie group (G, μ), and shown that any one of them is brought from a double Lie algebra (a Lie algebra with the same underlying vector space) $\mathfrak{h}$ on the Lie algebra $\mathfrak{g}$ of (G, μ) satisfying the relation;

$$\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} \subset Der\,\mathfrak{g},$$

where $\mathrm{ad}_{\mathfrak{h}}$ denotes the adjoint representation of the Lie algebra $\mathfrak{h}$.

In this paper, all such double Lie algebras on any odd-dimensional (real) simple Lie algebra are determined in the main theorem (Theorem 1). Applying this to the theorems in [4], we classify all geodesic homogeneous local Lie loops which are in projective relation with an arbitrarily given real simple Lie group (G, μ) of odd-dimension, and show that they are Akivis local loops (Theorem 3).

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§1. Main theorem

Let V be a finite-dimensional vector space over a field Φ, $\mathfrak{g} = (V, [\ ,\ ]_{\mathfrak{g}})$ a Lie algebra with the underlying vector space V.

DEFINITION. A Lie algebra $\mathfrak{h} = (V, [\ ,\ ]_{\mathfrak{h}})$ on V will be called a *projective double Lie algebra* on $\mathfrak{g}$ if the relation

$$\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} \subset Der\,\mathfrak{g} \tag{1.1}$$

holds, where $\mathrm{ad}_{\mathfrak{h}}$ denotes the adjoint representation of the Lie algebra $\mathfrak{h}$ and $Der\,\mathfrak{g}$ the Lie algebra of derivations of $\mathfrak{g}$.

EXAMPLE 1. For any fixed element p of Φ, let $\mathfrak{g}_p=(\mathrm{V}, [\,,\,]_p)$ be a double Lie algebra on $\mathfrak{g}$ with the bracket operation given by

$$[X, Y]_p := p[X, Y]_{\mathfrak{g}} \tag{1.2}$$

for $X, Y \in \mathrm{V}$. Then, $\mathfrak{g}_p$ gives a projective double Lie algebra on $\mathfrak{g}$.

In this section, the projective double Lie algebras on a real simple Lie algebra will be investigated. It should be noted that all discussions in §§1–2 are valid for complex Lie algebras.

The main theorem is as follows;

THEOREM 1. *Let* $\mathfrak{g}=(\mathrm{V}, [\,,\,]_{\mathfrak{g}})$ *be an odd-dimensional real simple Lie algebra. Then, any projective double Lie algebra on* $\mathfrak{g}$ *must be the Lie algebra* $\mathfrak{g}_p$ *obtained from* $\mathfrak{g}$ *by* (1.2), *for some real number* p.

To prove this, we first show the following;

PROPOSITION. *Every projective double Lie algebra* $\mathfrak{h}$ *on a real simple Lie algebra* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}$, *unless it is an abelian Lie algebra.*

PROOF. Let $\mathfrak{h}=(\mathrm{V}, [\,,\,]_{\mathfrak{h}})$ be a projective double Lie algebra on a simple Lie algebra $\mathfrak{g}=(\mathrm{V}, [\,,\,]_{\mathfrak{g}})$, that is, $\mathfrak{h}$ satisfies the relation (1.1). Since $\mathfrak{g}$ is simple, it is isomorphic to $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}=Der\,\mathfrak{g}$ under the adjoint representation of $\mathfrak{g}$. Then, by (1.1), there exists a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that

$$\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} = \mathrm{ad}_{\mathfrak{g}}\mathfrak{a} \cong \mathfrak{a} \tag{1.3}$$

and so that

$$\mathfrak{a} \cong \mathfrak{h}/\mathfrak{z}(\mathfrak{h}), \tag{1.4}$$

where $\mathfrak{z}(\mathfrak{h})$ denotes the center of $\mathfrak{h}$.

Let $\beta\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ be a nondegenerate bilinear form on $\mathfrak{g}$ which is $\mathfrak{g}$-invariant, i.e.,

$$\beta([X, Y]_{\mathfrak{g}}, Z) + \beta(Y, [X, Z]_{\mathfrak{g}}) = 0 \tag{1.5}$$

for X, Y, Z in $\mathfrak{g}$. Since $\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} \subset \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}$ holds, β is $\mathfrak{h}$-invariant and $\mathfrak{h}$ can be decomposed into a direct sum of minimal ideals mutually orthogonal with respect to β. In particular, we have a direct-sum decomposition of $\mathfrak{h}$;

$$\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{a}_1, \tag{1.6}$$

where $\mathfrak{a}_1 = (\mathfrak{z}(\mathfrak{h}))^{\perp}$ is a semi-simple ideal of $\mathfrak{h}$.

By (1.4) we get

$$\mathfrak{a}_1 \cong \mathfrak{a}. \tag{1.7}$$

The property (1.5) and the first equality of (1.3) imply

$$\begin{aligned}\beta([\mathfrak{g}, \mathfrak{a}]_{\mathfrak{g}}, \mathfrak{z}(\mathfrak{h})) &= \beta(\mathfrak{g}, [\mathfrak{a}, \mathfrak{z}(\mathfrak{h})]_{\mathfrak{g}}) \\ &= \beta(\mathfrak{g}, [\mathfrak{h}, \mathfrak{z}(\mathfrak{h})]_{\mathfrak{h}}) \\ &= \{0\}.\end{aligned}$$

Hence we see that $[\mathfrak{g}, \mathfrak{a}]_{\mathfrak{g}}$ is a subspace of $(\mathfrak{z}(\mathfrak{h}))^{\perp} = \mathfrak{a}_1$. Thus, we obtain from (1.7) the following;

$$\dim [\mathfrak{g}, \mathfrak{a}]_{\mathfrak{g}} \leq \dim \mathfrak{a}. \tag{1.8}$$

On the other hand, since the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is semi-simple by (1.7), the relations

$$\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}]_{\mathfrak{g}} \subset [\mathfrak{g}, \mathfrak{a}]_{\mathfrak{g}} \tag{1.9}$$

must be satisfied, which show

$$\dim \mathfrak{a} \leq \dim [\mathfrak{g}, \mathfrak{a}]_{\mathfrak{g}}. \tag{1.10}$$

From (1.8), (1.9) and (1.10) it follows that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, and that $\mathfrak{a} = \{0\}$ or $\mathfrak{a} = \mathfrak{g}$ because $\mathfrak{g}$ is simple. If $\mathfrak{a} = \{0\}$, then $\mathfrak{h} = \mathfrak{z}(\mathfrak{h})$, that is, the double Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$ is an abelian Lie algebra. If $\mathfrak{a} = \mathfrak{g}$, then $\mathfrak{z}(\mathfrak{h}) = \{0\}$ and $\mathfrak{h} = \mathfrak{a}_1 \cong \mathfrak{a} = \mathfrak{g}$, which completes the proof. q.e.d.

§2. The proof of the main theorem

We prove Theorem 1. Assume that $\mathfrak{g} = (V, [\,,\,]_{\mathfrak{g}})$ is a real simple Lie algebra of odd-dimension and $\mathfrak{h}$ a projective double Lie algebra on $\mathfrak{g}$. If $\mathfrak{h}$ is an abelian Lie algebra, it can be considered to be the Lie algebra $\mathfrak{g}_0$ given in Example 1 for $p = 0$. If $\mathfrak{h}$ is not abelian, then, by Proposition in §1, there exists an isomorphism $\phi: \mathfrak{h} \to \mathfrak{g}$ of Lie algebras. Since $\mathfrak{h}$ is assumed to satisfy the relation $\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} \subset \mathrm{ad}_{\mathfrak{g}}\mathfrak{g} = Der\,\mathfrak{g}$, we get $\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} = \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}$ and

$$\phi^{-1} \cdot \mathrm{ad}_{\mathfrak{g}}\mathfrak{g} \cdot \phi = \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}. \tag{2.1}$$

Conversely, if an invertible endomorphism $\phi \in GL(V)$ satifies (2.1), then the double Lie algebra $\mathfrak{h} = (V, [\,,\,]_{\mathfrak{h}})$, given by

$$[X, Y]_{\mathfrak{h}} := \phi^{-1}[\phi X, \phi Y]_{\mathfrak{g}} \tag{2.2}$$

satisfies $\mathrm{ad}_{\mathfrak{h}}\mathfrak{h} = \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}$, that is, $\mathfrak{h}$ is a projective double Lie algebra on $\mathfrak{g}$, and $\mathfrak{h}$ is isomorphic to $\mathfrak{g}$ under ϕ. Thus we can describe the set $\mathscr{P}(\mathfrak{g})$ of all non-abelian projective double Lie algebras on $\mathfrak{g}$ as follows:

$$\mathscr{P}(\mathfrak{g}) = \{\phi\mathfrak{g} \mid \phi^{-1}\cdot \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}\cdot\phi = \mathrm{ad}_{\mathfrak{g}}\mathfrak{g},\ \phi\in GL(V)\}, \tag{2.3}$$

where $\phi\mathfrak{g}$ denotes the double Lie algebra on $\mathfrak{g}$ with the bracket operation given by (2.2). It is evident that $\phi\mathfrak{g} = \mathfrak{g}$ if and only if ϕ is an automorphism of $\mathfrak{g}$,

Set

$$\mathscr{G}(\mathfrak{g}) := \{\phi\in GL(V) \mid \phi^{-1}\cdot \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}\cdot\phi = \mathrm{ad}_{\mathfrak{g}}\mathfrak{g}\}.$$

Then we can show that it is a subgroup of $GL(V)$ containing the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. For any $\phi\in\mathscr{G}(\mathfrak{g})$, we can define $\pi\phi\in GL(V)$ by the following equation;

$$[(\pi\phi)X, Y]_{\mathfrak{g}} = \phi[X, \phi^{-1}Y]_{\mathfrak{g}} \tag{2.4}$$

for $X, Y\in V$. Then, we get a group homomorphism $\pi\colon \mathscr{G}(\mathfrak{g})\to \mathrm{Aut}(\mathfrak{g})$ such that $\pi\phi = \phi$ if and only if $\phi\in \mathrm{Aut}(\mathfrak{g})$. Hence, it follows that each projective double Lie algebra $\mathfrak{h}\in\mathscr{P}(\mathfrak{g})$ is obtained from a unique element $\phi\in \mathrm{Ker}\,\pi$ by $\mathfrak{h} = \phi\mathfrak{g}$. If $\phi = p\cdot \mathrm{id}_V$ for some non-zero real number p, then it is evident that $\phi\in\mathrm{Ker}\,\pi$ and the corresponding projective double Lie algebra $\mathfrak{h} = \phi\mathfrak{g}$ on $\mathfrak{g}$ is given by

$$[X, Y]_{\mathfrak{h}} = p[X, Y]_{\mathfrak{g}}, \tag{2.5}$$

that is $\mathfrak{h} = \mathfrak{g}_p$ in Example 1. Conversely, suppose that ϕ is an element of $\mathrm{Ker}\,\pi$. Since V is assumed to be of odd-dimension, ϕ has a non-zero real eigenvalue, say p. Let W^p be the eigenspace of this eigenvalue. Then W^p is an ideal of $\mathfrak{g}$. In fact, for any $X\in\mathfrak{g}$ and $Y\in W^p$, the endomorphism $\phi\in\mathrm{Ker}\,\pi$ satisfies the relation

$$\phi[X, Y]_{\mathfrak{g}} = [X, \phi Y]_{\mathfrak{g}}, \tag{2.6}$$

which shows that $[X, Y]_{\mathfrak{g}}\in W^p$. Since $\mathfrak{g}$ is simple and W^p is non-empty, we get $W^p = \mathfrak{g}$, i.e., $\phi = p\cdot \mathrm{id}_V$.

Thus, Theorem 1 is proved. q.e.d.

REMARK. From (2.6) we see that the main theorem is valid for any central simple Lie algebra of arbitrary dimension (cf. [2] Ch. X).

§3. Projectivity of simple Lie groups

In this section, we give a result concerning the geodesic homogeneous local Lie loops in projective relation with a given simple Lie group of odd-dimension. This is an immediate consequence of the main theorem.

Let (G, μ) be a Lie group with the Lie algebra $\mathfrak{g}$ considered to be defined on the tangent space $V = T_e(G)$ of G at the identity element e. For any projective double

Lie algebra $\mathfrak{h} = (V, [\ ,\ ]_{\mathfrak{h}})$ on $\mathfrak{g}$, denote by $A(X)$, $X \in \mathfrak{h}$, the exponential of the endomorphism $\mathrm{ad}_{\mathfrak{h}} X$, i.e.,

$$A(X) = e^{\mathrm{ad}_{\mathfrak{h}} X}. \tag{3.1}$$

For any normal neighborhood of e, we can define an analytic local mutiplication $\tilde{\mu}$ by

$$\tilde{\mu}(\exp X, \exp Y) = \mu(\exp X, \exp A(X)Y) \tag{3.2}$$

as far as both of the left- and right-hand sides are defined, where $\exp : \mathfrak{g} \to G$ denotes the exponential map of the Lie group (G, μ). We can summarize Theorems 2.1, 2.16, 3.2 and 3.3 in [4] as follows:

THEOREM 2 ([4]). *Let (G, μ) be a Lie group with the Lie algebra $\mathfrak{g} = (V, [\ ,\ ]_{\mathfrak{g}})$ on the tangent space $V = T_e(G)$ at the identity e. A geodesic homogeneous local left Lie loop $\tilde{\mu}$ around e is in projective relation with (G, μ) if and only if it can be given by* (3.2) *with respect to a projective double Lie algebra $\mathfrak{h} = (V, [\ ,\ ]_{\mathfrak{h}})$ on $\mathfrak{g}$.*

EXAMPLE 2. For some real number p, let $\mathfrak{g}_p = (V, [\ ,\ ]_p)$ be the projective double Lie algebra on $\mathfrak{g}$ given by (1.2) in Example 1. Then, by Theorem 2 above, we get a geodesic homogeneous local Lie loop μ_p in projective relation with the Lie group (G, μ). The local multiplication μ_p defined by (3.2) has the following expression;

$$\mu_p(x, y) = x^{p+1} y x^{-p}, \tag{3.3}$$

where $x^r = \exp rX$ for any real number r if it is well-defined for $x = \exp X$ (cf. [5]). This example has been found by M. Akivis [1], which will be called *Akivis local loop.*

Now, we can apply Theorem 1 to the Lie algebra of any simple Lie group of odd-dimension, and we get by Theorem 2 the following;

THEOREM 3. *Let (G, μ) be a (real) simple Lie group of odd-dimension. Then, a geodesic homogeneous local (left) Lie loop $\tilde{\mu}$ at the identity e which is given by* (3.2) *is in projective relation with (G, μ) if and only if $\tilde{\mu}$ is an Akivis local loop, that is, it can be expressed by μ_p in Example 2 above for some real number p.*

PROOF. Since the Lie algebra $\mathfrak{g} = (V, [\ ,\ ]_{\mathfrak{g}})$ of (G, μ) given on the tangent space $V = T_e(G)$ at e is a real simple Lie algebra, Theorem 1 assures that $\mathfrak{h}$ is a projective double Lie algebra on $\mathfrak{g}$ if and only if $\mathfrak{h} = \mathfrak{g}_p$ in Example 1, for some real number p. Then, any geodesic homogeneous local Lie loop $\tilde{\mu}$ at e must be expressed by μ_p in Example 2. Conversely, for any real number p, the local multiplication μ_p in Example 2 is a geodesic homogeneous local Lie loop which is in projective relation with the given Lie group (G, μ)(cf. [5]). q.e.d.

References

[1] M. Akivis, Geodesic loops and local triple systems in an affinely connected space (Russian), Sibir. Mat. Z. **19** (1978), 243–253.
[2] N. Jacobson, Lie Algebras, Interscience Pub., 1962.
[3] M. Kikkawa, Affine homogeneous structures on analytic loops, Mem. Fac. Sci., Shimane Univ. **21** (1987), 1–15.
[4] ———, Projectivity of homogeneous left loops on Lie groups II, Mem. Fac. Sci., Shimane Univ. **24** (1990), 1–16
[5] ———, Projectivity of homogeneous left loops, Proc. International Symposium on Non-associative Algebras and Related Topics Hiroshima, 1990, World Scientific (to appear).