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Representations of $\mathscr{P}$-Congruences on $\mathscr{P}$-Regular Semigroups

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One of the purposes of this paper is to characterize congruences on $\mathscr{P}$-regular semigroups by using *$\mathscr{P}$-congruence pairs*, which is the $\mathscr{P}$-regular version of congruence pairs for regular semigroups. The other is to study congruences on a $\mathscr{P}$-regular semigroup S with the same trace, the restriction of a congruence to the set of idempotents of S. Also the special types of congruences on $\mathscr{P}$-regular semigroups are observed.

§ 1. Introduction

Let S be a regular semigroup and E the set of idempotents of S. Let $P \subseteq E$. If S satisfies the following, it is called a *$\mathscr{P}$-regular semigroup* and P is called a *C-set* in S:

(1) $P^2 \subseteq E$,
(2) $qPq \subseteq P$ for any $q \in P$,
(3) for any $a \in S$, there exists $a^+ \in V(a)$ (the set of all inverses of a) such that $a^+P^1a \subseteq P$ and $aP^1a^+ \subseteq P$.

In such a case, S is denoted by $S(P)$. Let $a \in S(P)$ and $a^+ \in V(a)$. If a^+ satisfies $a^+P^1a \subseteq P$ and $aP^1a^+ \subseteq P$, then a^+ is called a *$\mathscr{P}$-inverse* of a, and the set of all $\mathscr{P}$-inverses of a is denoted by $V_P(a)$. An element of a C-set P in S is called a *projection.* The class of $\mathscr{P}$-regular semigroups contains both the classes of orthodox semigroups and regular $*$-semigroups. A good account of the concept of $\mathscr{P}$-regularity can be seen in [7] and [8].

Hereafter $S(P)$ means a $\mathscr{P}$-regular semigroup S with a C-set P in S. A congruence on S is sometimes called a *$\mathscr{P}$-congruence* on $S(P)$. Let ρ be a $\mathscr{P}$-congruence on $S(P)$, and put $\bar{x} = x\rho$ for any $x \in S$, $\bar{S} = \{\bar{x}: x \in S\}$ and $\bar{P} = \{\bar{q}: q \in P\}$. Then $\bar{S}(\bar{P})$ is also a $\mathscr{P}$-regular semigroup with a C-set $\bar{P}$. So $\bar{S}(\bar{P})$ is called the *factor $\mathscr{P}$-regular semigroup* of $S(P)$ mod. ρ, and it is denoted by $S(P)/(\rho)_{\mathscr{P}}$.

Let ρ be a $\mathscr{P}$-congruence on $S(P)$. Then it is called an *orthodox $\mathscr{P}$-congruence*

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on $S(P)$ if $S(P)/(\rho)_{\mathscr{P}}$ is an orthodox semigroup, and it is called a *strong $\mathscr{P}$-congruence* on $S(P)$ if it saitsfies the condition:

For any $a \in S(P)$ and $e \in P$, $a\rho e$ implies $a^+\rho e$ for all $a^+ \in V_P(a)$.

Also it is called a *strong orthodox $\mathscr{P}$-congruence* on $S(P)$ if it is both an orthodox $\mathscr{P}$-congruence and a strong $\mathscr{P}$-congruence on $S(P)$. As was seen in [8], if ρ is a strong $\mathscr{P}$-congruence on $S(P)$, then $S(P)/(\rho)_{\mathscr{P}}$ is a regular $*$-semigroup with the set $\{e\rho : e \in P\}$ of projections; in fact, the $*$-operation $\bigstar$ on $S(P)/(\rho)_{\mathscr{P}}$ is given by $(a\rho)^{\bigstar} = a^+\rho(a \in S(P),\ a^+ \in V_P(a))$.

The set $\{a \in S(P) : a\rho e$ for some $e \in E\}$ is called the *kernel* of ρ, and it is denoted by $\ker \rho$. Also the set $\{a \in S(P) : a\rho e$ for some $e \in P\}$ is called the *$\mathscr{P}$-kernel* of ρ, and it is denoted by $\mathscr{P}$-$\ker \rho$. It is well-known that $\ker \rho$ is the union of idempotent ρ-classes (for example, see [3]). The restriction $\rho \cap (E \times E)$ $[\rho \cap (P \times P)]$ of ρ is called the $[\mathscr{P}$-$]$ *trace* of ρ, and it is denoted by $[\mathscr{P}$-$]$ $\mathrm{tr}\,\rho$.

For any subset A *of* $S(P)$, define the terminology as follows:

A is $[\mathscr{P}$-$]$ *full* if $E \subseteq A\,[P \subseteq A]$,
A is a *$\mathscr{P}$-subset* if $V_P(a) \subseteq A$ for any $a \in A$,
A is a *$\mathscr{P}$-self-conjugate* if $x^+Ax \subseteq A$ for any $x \in S(P)$ and $x^+ \in V_P(x)$,
A is *weakly closed* if $a^2 \in A$ for any $a \in A$.

The following is due to [7] and [8].

RESULT 1.1. *Let $a, b \in S(P)$, $e \in E$ and $p \in P$. Then*
(i) $V_P(b)V_P(a) \subseteq V_P(ab)$, (ii) $a^+ \in V_P(a)$ *implies* $a \in V_P(a^+)$,
(iii) $V_P(e) \subseteq E$, (iv) $p \in V_P(p)$.

The next statement can be found in [2].

RESULT 1.2. *Let ρ be a $\mathscr{P}$-congruence on $S(P)$ and $a, b \in S(P)$. Then $a\rho b$ if and only if*

$$ba' \in \ker \rho,\quad aa'\rho bb'aa',\quad b'b\rho b'ba'a$$

for some $a' \in V(a)$ and $b' \in V(b)$.

Strong $\mathscr{P}$-congruences on $\mathscr{P}$-regular semigroups were studied in [4]. In this paper, a generalization of [4] is presented.

In § 2, for a given $\mathscr{P}$-congruence ρ on a $\mathscr{P}$-regular semigroup $S(P)$, the maximum and the minimum $\mathscr{P}$-congruences on $S(P)$ whose traces coincide with $\mathrm{tr}\,\rho$ are determined, and the properties for these $\mathscr{P}$-congruences are given.

The concept introduced in § 3 is "$\mathscr{P}$-congruence pairs". This concept is a characterization of the pair $(\mathrm{tr}\,\rho, \ker \rho)$ associated with a given $\mathscr{P}$-congruence ρ on $S(P)$, and the pair uniquely determines the $\mathscr{P}$-congruence κ such that $\mathrm{tr}\,\kappa = \mathrm{tr}\,\rho$

and $\ker\kappa = \ker\rho$.

In the last two sections, orthodox $\mathscr{P}$-congruences and strong orthodox $\mathscr{P}$-congruences on $\mathscr{P}$-regular semigroups are investigated by the similar argument in §2 and §3.

We use the notation and terminology of [3] and [8] unless otherwise stated.

§2. $\mathscr{P}$-congruences with the same trace

Let $S(P)$ be a $\mathscr{P}$-regular semigroup. For any $\mathscr{P}$-congruence ρ on $S(P)$, define a relation $\rho_{\max}$ on $S(P)$ as follows:

$$\begin{aligned}\rho_{\max} = \{(a, b)\colon & \text{ there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \\ & \text{such that } aea^+ \rho beb^+ aea^+,\ beb^+ \rho aea^+ beb^+, \\ & a^+ ea\rho a^+ eab^+ eb \text{ and } b^+ eb\rho b^+ eba^+ ea \text{ for all } e \in P\}.\end{aligned}$$

REMARK. Suppose that $a\rho_{\max}b$. Then there exist $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ such that $aea^+ \rho beb^+ aea^+$, $beb^+ \rho aea^+ beb^+$, $a^+ ea\rho a^+ eab^+ eb$ and $b^+ eb\rho b^+ eba^+ ea$ for every $e \in P$. Let $a^* \in V_P(a)$ and $b^* \in V_P(b)$. Note that $ae = aea^+ ae$ and $be = beb^* be$. Then

$$\begin{aligned}aea^* &= (aea^+) aea^* \rho beb^+ aea^+ aea^* \\ &= beb^* (beb^+ aea^+) aea^* \rho beb^* aea^+ aea^* = beb^* aea^*.\end{aligned}$$

Similarly we have

$$beb^* \rho aea^* beb^*,\ a^* ea\rho a^* eab^* eb,\ b^* eb\rho b^* eba^* ea.$$

Thus

$$\begin{aligned}\rho_{\max} = \{(a, b)\colon & aea^+ \rho beb^+ aea^+,\ beb^+ \rho aea^+ beb^+, \\ & a^+ ea\rho a^+ eab^+ eb \text{ and } b^+ eb\rho b^+ eba^+ ea \\ & \text{for any } a^+ \in V_P(a),\ b^+ \in V_P(b) \text{ and } e \in P\}.\end{aligned}$$

LEMMA 2.1. *Let ρ be a $\mathscr{P}$-congruence on $S(P)$ and $a, b \in S(P)$. If $a\rho_{\max}b$, then*

$$aa^+ \rho bb^+ aa^+,\ bb^+ \rho aa^+ bb^+,\ a^+ a\rho a^+ ab^+ b,\ b^+ b\rho b^+ ba^+ a$$

for any $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$.

PROOF. Suppose that $a\rho_{\max}b$. Let $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$. Since $aea^+ \rho beb^+ aea^+$ for any $e \in P$ by Remark above, we have

$$aa^+ = a(a^+ a)a^+ \rho b(a^+ a)b^+ a(a^+ a)a^+ = ba^+ ab^+ aa^+,$$

so that

$$bb^+aa^+\rho(bb^+)(ba^+ab^+aa^+) = ba^+ab^+aa^+\rho aa^+.$$

By the similar argument, we have the remainder.

THEOREM 2.2. *For any $\mathscr{P}$-congruence ρ on a $\mathscr{P}$-regular semigroup $S(P)$, $\rho_{\max}$ is the greatest $\mathscr{P}$-congruence on $S(P)$ whose trace coincides with* $\operatorname{tr}\rho$.

PROOF. Obviously $\rho_{\max}$ is an equivalence. Suppose that $a\rho_{\max}b$ and $c \in S(P)$. Let $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, $c^+ \in V_P(c)$ and $e \in P$. Then, by Remark above,

$$acec^+a^+ = a(cec^+)a^+\rho b(cec^+)b^+a(cec^+)a^+ = bcec^+b^+acec^+a^+$$

and

$$\begin{aligned} c^+a^+eacc^+b^+ebc &= c^+(a^+ea)cc^+b^+ebcc^+c \\ &\rho\ c^+a^+ea(b^+ebcc^+)(b^+ebcc^+)c \\ &= c^+(a^+eab^+eb)c \\ &\rho\ c^+a^+eac. \end{aligned}$$

Likewise, $bcec^+b^+\rho acec^+a^+bcec^+b^+$ and $c^+b^+ebc\rho c^+b^+ebcc^+a^+eac$. Thus $\rho_{\max}$ is right compatible. Similarly, $\rho_{\max}$ is left compatible, so that $\rho_{\max}$ is a $\mathscr{P}$-congruence on $S(P)$.

Next we shall show that $\operatorname{tr}\rho = \operatorname{tr}\rho_{\max}$. Let $e, f \in E$. Suppose that $e\rho f$. Then $ep\ \rho\ fp$ for every $p \in P$, which implies

$$((epe^+)\rho,\ (fpf^+)\rho) \in \mathscr{R} \text{ in } S(P)/(\rho)_{\mathscr{P}},$$

where $e^+ \in V_P(e)$ and $f^+ \in V_P(f)$. So we have

$$epe^+\rho fpf^+epe^+,\ fpf^+\rho epe^+fpf^+.$$

Dually, $e^+pe\rho e^+pef^+ef$ and $f^+pf\rho f^+pfe^+pe$. Thus $e\rho_{\max}f$. Conversely, assume that $e\rho_{\max}f$. It follows from Lemma 2.1 that, in $S(P)/(\rho)_{\mathscr{P}}$,

$$(e\rho)\mathscr{R}((ee^+)\rho)\mathscr{R}((ff^+)\rho)\mathscr{R}(f\rho),\ (e\rho)\mathscr{L}((e^+e)\rho)\mathscr{L}((f^+f)\rho)\mathscr{L}(f\rho)$$

for any $e^+ \in V_P(e)$ and $f^+ \in V_P(f)$. Thus two idempotents $e\rho$ and $f\rho$ of $S(P)/(\rho)_{\mathscr{P}}$ are $\mathscr{H}$-equivalent, so that $e\rho f$. Therefore $\operatorname{tr}\rho = \operatorname{tr}\rho_{\max}$.

Finally, let σ be any $\mathscr{P}$-congruence on $S(P)$ such that $\operatorname{tr}\sigma = \operatorname{tr}\rho$. Suppose that $a, b \in S(P)$ and $a\sigma b$. Let $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ and $e \in P$. Since $ae\sigma be$, we have $((aea^+)\sigma,\ (beb^+)\sigma) \in \mathscr{R}$ in $S(P)/(\sigma)_{\mathscr{P}}$, so that

$$aea^+\sigma beb^+aea^+,\ beb^+\sigma aea^+beb^+.$$

Therefore we have

$$aea^+ \rho beb^+ aea^+, \ beb^+ \rho aea^+ beb^+$$

since $\operatorname{tr}\sigma = \operatorname{tr}\rho$. Dually,

$$a^+ ea \rho a^+ eab^+ eb, \ b^+ eb \rho b^+ eba^+ ea.$$

Thus $a\rho_{\max} b$, and hence $\rho_{\max}$ is the greatest $\mathscr{P}$-congruence on $S(P)$ whose trace is $\operatorname{tr}\rho$. So we have the theorem.

COROLLARY 2.3. *For any orthodox $\mathscr{P}$-congruence ρ on a $\mathscr{P}$-regular semigroup $S(P)$, $\rho_{\max}$ is the greatest orthodox $\mathscr{P}$-congruence on $S(P)$ whose trace coincides with $\operatorname{tr}\rho$.*

PROOF. Let ρ be an orthodox $\mathscr{P}$-congruence on $S(P)$. It is sufficient to show that $\rho_{\max}$ is an orthodox $\mathscr{P}$-congruence on $S(P)$. Let $e, f \in E$, $e^+ \in V_P(e)$, $f^+ \in V_P(f)$ and $p \in P$. Since ρ is an orthodox $\mathscr{P}$-congruence on $S(P)$, we have

$$(ef)p(f^+e^+)\rho(ef)^2p(f^+e^+)^2, \ (f^+e^+)p(ef)\rho(f^+e^+)^2p(ef)^2.$$

So $(ef, (ef)^2) \in \rho_{\max}$, and hence $\rho_{\max}$ is an orthodox $\mathscr{P}$-congruence on $S(P)$.

From now on, denote the maximum idempotent separating congruence on a semigroup T by μ_T.

COROLLARY 2.4 (compare with [7, Proposition 4.1]). *The maximum idempotent separating $\mathscr{P}$-congruence $\mu_{S(P)}$ on a $\mathscr{P}$-regular semigroup $S(P)$ is given as follows:*

$$\begin{aligned} \mu_{S(P)} = \{(a, b)\colon & \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \\ & \text{such that } aea^+ = beb^+ aea^+, \ beb^+ = aea^+ beb^+, \\ & a^+ea = a^+eab^+eb, \ b^+eb = b^+eba^+ea \text{ for all } e \in P\} \\ = \{(a, b)\colon & aea^+ = beb^+ aea^+, \ beb^+ = aea^+ beb^+, \\ & a^+ea = a^+eab^+eb, \ b^+eb = b^+eba^+ea \\ & \text{for any } a^+ \in V_P(a), \ b^+ \in V_P(b) \text{ and } e \in P\}. \end{aligned}$$

Let S be an orthodox semigroup and E the band of idempotents of S. Then it is easy to check that $S(E)$ is a $\mathscr{P}$-regular semigroup with a C-set E in S. So we have immediately

COROLLARY 2.5 ([1, Theorem 4.2]). *Let ρ be a congruence on an orthodox semigroup S with the band E of idempotents of S. Then*

$$\begin{aligned} \rho_{\max} = \{(a, b)\colon & \text{there exist } a' \in V(a) \text{ and } b' \in V(b) \\ & \text{such that } aea' \rho beb' aea', \ beb' \rho aea' beb', \end{aligned}$$

$$a'ea\rho a'eab'eb,\ b'eb\rho b'eba'ea \ \textit{for any}\ e\in E\}$$

$$= \{(a, b)\colon aea'\rho beb'aea',\ beb'\rho aea'beb',$$

$$a'ea\rho a'eab'eb,\ b'eb\rho b'eba'ea$$

$$\textit{for any}\ a'\in V(a),\ b'\in(b)\ \textit{and}\ e\in E\}$$

is the greatest congruence on S whose trace coincides with tr ρ.

On the other hand, the minimum $\mathscr{P}$-congruence on $S(P)$ with the same trace is given as follws:

THEOREM 2.6. *For any $\mathscr{P}$-congruence ρ on a $\mathscr{P}$-regular semigroup $S(P)$, define a relation ρ_0 on $S(P)$ by*

$$\rho_0 = \{(a, b)\colon \textit{there exist}\ x, y\in S(P)^1\ \textit{and}\ e, f\in E$$

$$\textit{such that}\ a = xey,\ b = xfy\ \textit{and}\ e\rho f\}.$$

Then $\rho_{\min} = \rho_0^t$, the transitive closure of ρ_0, is the least $\mathscr{P}$-congruence on $S(P)$ whose trace coincides with tr ρ. *In other words, the least $\mathscr{P}$-congruence on $S(P)$ with* tr ρ *as its trace is the $\mathscr{P}$-congruence on $S(P)$ generated by* tr ρ.

PROOF. It is obvious that $\rho_{\min}$ is a $\mathscr{P}$-congruence on $S(P)$ whose trace is tr ρ. Let σ be any $\mathscr{P}$-congruence on $S(P)$ such that tr σ = tr ρ. Suppose that $a, b\in S(P)$ and $a\rho_0 b$. Then there exist $x, y\in S(P)^1$ and $e, f\in E$ such that $a = xey$, $b = xfy$ and $e\rho f$. Since tr σ = tr ρ, we have $e\sigma f$, so that $a\sigma b$. Hence $\rho_0 \subseteq \sigma$, and so $\rho_{\min} \subseteq \sigma$. Therefore $\rho_{\min}$ is the least $\mathscr{P}$-congruence on $S(P)$ whose trace is tr ρ. Thus we have the theorem.

The following corollary gives us the characterization, which is different from both [1, Theorem 4.1] and [6, Theorem 3.3], of the least congruence on an orthodox semigroup with the same trace.

COROLLARY 2.7. *For any congruence ρ on an orthodox semigroup S, the congruence generated by* tr ρ *is the least congruence on S whose trace coincides with* tr ρ.

Several properties of $\rho_{\max}$ and $\rho_{\min}$ are presented.

PROPOSITION 2.8. *Let ρ be a $\mathscr{P}$-congruence on $S(P)$ and $e\in E$. Then we have*

$$e\rho = e\rho_{\max} \cap \ker \rho.$$

PROOF. Let ρ be a $\mathscr{P}$-congruence on $S(P)$ and $e\in E$. Suppose that $a\in e\rho_{\max}\cap \ker\rho$. Then $a\rho_{\max} e$ and $a\rho f$ for some $f\in E$. Since $a\rho_{\max} e$, by using Lemma 2.1, in $S(P)/(\rho)_{\mathscr{P}}$,

$$(a\rho)\mathscr{R}((aa^+)\rho)\mathscr{R}((ee^+)\rho)\mathscr{R}(e\rho),\ (a\rho)\mathscr{L}((a^+a)\rho)\mathscr{L}((e^+e)\rho)\mathscr{L}(e\rho)$$

for any $a^+\in V_P(a)$ and $e^+\in V_P(e)$. So two idempotents $a\rho=f\rho$ and $e\rho$ of $S(P)/(\rho)_{\mathscr{P}}$ are $\mathscr{H}$-equivalent, and hence $a\rho e$. Thus $e\rho_{\max}\cap\ker\rho\subseteq e\rho$. The reverse inclusion obviously holds.

Proposition 2.9. *For any $\mathscr{P}$-congruence ρ on $S(P)$, $\rho=\rho_{\max}$ if and only if $S(P)/(\rho)_{\mathscr{P}}$ is a fundamental $\mathscr{P}$-regular semigroup.*

Proof. As was seen in Corollary 2.4, the maximum idempotent separating $\mathscr{P}$-congruence $\mu_{T(Q)}$ on a $\mathscr{P}$-regular semigroup $T(Q)$ is given by

$$\mu_{T(Q)}=\{(a,b)\colon aea^+=beb^+aea^+,\ beb^+=aea^+beb^+,$$
$$a^+ea=a^+eab^+eb,\ b^+eb=b^+eba^+ea$$
$$\text{for any } a^+\in V_Q(a),\ b^+\in V_Q(b) \text{ and } e\in Q\}.$$

Suppose that $\rho=\rho_{\max}$. Then $(a\rho, b\rho)\in\mu_{S(P)/(\rho)_{\mathscr{P}}}$ implies, for any $e\in P$,

$$(a\rho)(e\rho)(a^+\rho)=(b\rho)(e\rho)(b^+\rho)(a\rho)(e\rho)(a^+\rho),$$
$$(b\rho)(e\rho)(b^+\rho)=(a\rho)(e\rho)(a^+\rho)(b\rho)(e\rho)(b^+\rho),$$
$$(a^+\rho)(e\rho)(a\rho)=(a^+\rho)(e\rho)(a\rho)(b^+\rho)(e\rho)(b\rho),$$
$$(b^+\rho)(e\rho)(b\rho)=(b^+\rho)(e\rho)(b\rho)(a^+\rho)(e\rho)(a\rho),$$

where $a^+\in V_P(a)$ and $b^+\in V_P(b)$. So, for any $e\in P$

$$aea^+\rho beb^+aea^+,\quad beb^+\rho aea^+beb^+,$$
$$a^+ea\rho a^+eab^+eb,\quad b^+eb\rho b^+eba^+ea,$$

and hence $(a,b)\in\rho_{\max}=\rho$, that is, $a\rho=b\rho$. Thus $\mu_{S(P)/(\rho)_{\mathscr{P}}}$ is the identity relation on $S(P)/(\rho)_{\mathscr{P}}$. Therefore $S(P)/(\rho)_{\mathscr{P}}$ is fundamental. Conversely, suppose that $S(P)/(\rho)_{\mathscr{P}}$ is fundamental. Let $a\rho_{\max}b$. Then $(a\rho, b\rho)\in\mu_{S(P)/(\rho)_{\mathscr{P}}}$ from the definition of $\rho_{\max}$. Since $\mu_{S(P)/(\rho)_{\mathscr{P}}}$ is the equality relation on $S(P)/(\rho)_{\mathscr{P}}$, we have $a\rho=b\rho$, so that $\rho_{\max}\subseteq\rho$. Therefore $\rho=\rho_{\max}$.

For any $\mathscr{P}$-congruences ρ and σ on $S(P)$ such that $\rho\subseteq\sigma$, define a relation σ/ρ on $S(P)/(\rho)_{\mathscr{P}}$ by

$$\sigma/\rho=\{(a\rho, b\rho)\colon (a,b)\in\sigma\}.$$

Proposition 2.10. *For any $\mathscr{P}$-congruence ρ on $S(P)$, $\rho_{\max}/\rho$ is the maximum idempotent separating $\mathscr{P}$-congruence on $S(P)/(\rho)_{\mathscr{P}}$.*

Proof. Clear.

Let Λ be the lattice of all $\mathscr{P}$-congruences on $S(P)$. Define a relation Θ on

Λ as follows: for any $\rho, \sigma \in \Lambda$,

$$\rho\Theta\sigma \text{ if and only if } \operatorname{tr}\rho = \operatorname{tr}\sigma.$$

It immediately follows from Theorems 2.2 and 2.6 that $\rho\Theta$, the Θ-class containing $\rho \in \Lambda$, is the interval $[\rho_{\min}, \rho_{\max}]$ of Λ.

PROPOSITION 2.11 (5, Theorem 5.1]). *If $\mathscr{P}$-congruences ρ and σ on $S(P)$ are Θ-equivalent, then $\rho\sigma = \sigma\rho$. Therefore, for any $\rho \in \Lambda$, $\rho\Theta$ is a complete modular sublattice of Λ.*

PROOF. Suppose that $\operatorname{tr}\rho = \operatorname{tr}\sigma$. Let $(a, b) \in \rho\sigma$. Then there exists $c \in S(P)$ such that $a\rho c$ and $c\sigma b$. Choose $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ and $c^+ \in V_P(c)$. Then $aa^+\rho cc^+aa^+$ and $b^+b\sigma b^+bc^+c$, which imply that $aa^+\sigma cc^+aa^+$ and $b^+b\rho b^+bc^+c$ since $\operatorname{te}\rho = \operatorname{tr}\sigma$. Also $c^+a\rho c^+c$ and $cc^+\sigma bc^+$. Then

$$a = (aa^+)a\sigma cc^+aa^+a = (cc^+)a\sigma bc^+a,$$

$$b = b(b^+b)\rho bb^+bc^+c = b(c^+c)\rho bc^+a,$$

and so $(a, b) \in \sigma\rho$. Hence $\rho\sigma \subseteq \sigma\rho$. Likewise $\sigma\rho \subseteq \rho\sigma$.

PROPOSITION 2.12. *Let $\xi \in \Lambda$, and let Γ be the lattice of all idempotent separating $\mathscr{P}$-congruences on $S(P)/(\xi_{\min})_{\mathscr{P}}$. Then the mapping $\rho \mapsto \rho/\xi_{\min}$ is a complete isomorphism of $\xi\Theta$ onto Γ.*

PROOF. Clear.

§3. $\mathscr{P}$-congruence pairs

LEMMA 3.1. *Let ρ be a $\mathscr{P}$-congruence on $S(P)$ and $a, b \in S(P)$. Then $a\rho b$ if and only if*

$$ab^+ \in \ker\rho,\ aa^+\rho bb^+aa^+,\ b^+b\rho b^+ba^+a$$

for some $a^+ \in V(a)$ and $b^+ \in V(b)$,

PROOF. Noting that $\ker\rho$ is a $\mathscr{P}$-subset of $S(P)$ by Result 1.1 (iii), this is trivial from Result 1.2.

Let ξ be an equivalence on E. Then ξ is called a *normal equivalence* on E if it satisfies the following: for any $a \in S(P)$ and $e, f, g, h, i, j, k \in E$,

(a) if $e\xi f$ and aea^+, $afa^+ \in E$ for some $a^+ \in V_P(a)$, then $aea^+\xi afa^+$,
(b) if $e\xi f$, $g\xi h$ and $eg, fh \in E$, then $eg\xi fh$,
(c) if $\square \neq (e\xi)(f\xi) \cap E \subseteq h\xi$, $\square \neq (f\xi)(g\xi) \cap E \subseteq i\xi$ and $\square \neq (e\xi)(i\xi) \cap E \subseteq j\xi$ $[\square \neq (h\xi)(g\xi) \cap E \subseteq k\xi]$, then $\square \neq (h\xi)(g\xi) \cap E$ $[\square \neq (e\xi)(i\xi) \cap E]$ and $j\xi k$.

Let ξ be a normal equivalence on E. Define a partial binary operation $\circ$ on E/ξ as follows: for any $e, f, g \in E$,

$$e\xi \circ f\xi = g\xi, \quad \text{where} \quad \square \neq (e\xi)(f\xi) \cap E \subseteq g\xi.$$

It is easy to verify that the partial binary operation $\circ$ is well-defined. The partial groupoid E/ξ satisfies the following:

(w) if $e\xi \circ f\xi, f\xi \circ g\xi$ and $e\xi \circ (f\xi \circ g\xi)[(e\xi \circ f\xi) \circ g\xi]$ are defined in E/ξ, then $(e\xi \circ f\xi) \circ g\xi [e\xi \circ (f\xi \circ g\xi)]$ is defined in E/ξ and $(e\xi \circ f\xi) \circ g\xi = e\xi \circ (f\xi \circ g\xi)$. In this case, the element $e\xi \circ (f\xi \circ g\xi)(= (e\xi \circ f\xi) \circ g\xi)$ is simply denoted by $e\xi \circ f\xi \circ g\xi$.

Let K be a weakly closed full $\mathscr{P}$-subset of $S(P)$ and ξ a normal equivalence on E. Then the pair (ξ, K) is called a *$\mathscr{P}$-congruence pair* for $S(P)$ if it satisfies the following: for any $a, b, c \in S(P)$, $c^+ \in V_P(c)$, $e, f, g \in E$ and $p \in P$,

(C1) $a \in K$ implies $a^+ a \xi a^+ a^+ aa$ for any $a^+ \in V_P(a)$,

(C2) $aefb \in K$ and $e\xi \circ f\xi = (a^+ a)\xi$ for some $a^+ \in V_P(a)$ imply $ab \in K$,

(C3) $ab^+ \in K$ and $aa^+ \xi bb^+ aa^+$, $b^+ b \xi b^+ ba^+ a$ for some $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ imply $apb^+ \in K$ and $apa^+ \xi bpb^+ apa^+$, $b^+ pb \xi b^+ pba^+ pa$,

(C4) $a, b \in K$, $aa^+ \xi ee^+ aa^+$, $ee^+ \xi aa^+ ee^+$, $a^+ a \xi a^+ ae^+ e$, $e^+ e \xi e^+ ea^+ a$, $bb^+ \xi ff^+ bb^+$, $ff^+ \xi bb^+ ff^+$, $b^+ b \xi b^+ bf^+ f$, $f^+ f \xi f^+ fb^+ b$ and $e\xi \circ f\xi = g\xi$ for some $a^+ \in V_P(a)$ $b^+ \in V_P(b)$, $e^+ \in V_P(e)$ and $f^+ \in V_P(f)$ imply $ab \in K$,

(C5) $ap \in K$ and $aa^+ \xi paa^+$, $p \xi pa^+ a$ for some $a^+ \in V_P(a)$ imply $cac^+ \in K$.

For any $\mathscr{P}$-congruence pair (ξ, K) for $S(P)$, define a relation $\kappa_{(\xi,K)}$ on $S(P)$ as follows:

(4)
$$\kappa_{(\xi,K)} = \{(a, b)\colon ab^+ \in K \text{ and } aa^+ \xi bb^+ aa^+, \; bb^+ \xi aa^+ bb^+, \; a^+ a \xi a^+ ab^+ b, \; b^+ b \xi b^+ ba^+ a \text{ for some } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b)\}.$$

The following lemma enables us to substitute "some" in the definition above by "any".

LEMMA 3.2. *Let (ξ, K) be a $\mathscr{P}$-congruence pair for $S(P)$ and $a, b \in S(P)$. Suppose that $(a, b) \in \kappa_{(\xi,K)}$. Then $ab^* \in K$ and*

$$aa^* \xi bb^* aa^*, \quad bb^* \xi aa^* bb^*, \quad a^* a \xi a^* ab^* b, \quad b^* b \xi b^* ba^* a$$

for any $a^* \in V_P(a)$ *and* $b^* \in V_P(b)$. *Further*

$$aa^* \xi ab^* ba^*, \ bb^* \xi ba^* ab^*, \ a^* a \xi a^* bb^* a, \ b^* b \xi b^* aa^* b$$

for any $a^* \in V_P(a)$ *and* $b^* \in V_P(b)$.

PROOF. Suppose that $(a, b) \in \kappa_{(\xi, K)}$. Then $aa^+ \xi bb^+ aa^+$ and $ab^+ \in K$ for some $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$. Choose $a^* \in V_P(a)$ and $b^* \in V_P(b)$. Since K is a $\mathscr{P}$-subset of $S(P)$, $ab^+ \in K$ implies $ab^* \in K$. Also

$$\begin{aligned} aa^* &= (aa^+) aa^* \xi bb^+ aa^+ aa^* \\ &= bb^* (bb^+ aa^+) aa^* \xi bb^* aa^+ aa^* = bb^* aa^* \end{aligned}$$

since ξ is a normal equivalence on E. Similarly, $bb^* \xi aa^* bb^*$, $a^* a \xi a^* ab^* b$ and $b^* b \xi b^* ba^* a$. The second statement is easily verified.

Now we can determine $\mathscr{P}$-congruences on $S(P)$ by $\mathscr{P}$-congruence pairs.

THEOREM 3.3. *Let* $S(P)$ *be a* $\mathscr{P}$*-regular semigroup. For any* $\mathscr{P}$*-congruence pair* (ξ, K) *for* $S(P)$, $\kappa_{(\xi, K)}$ *is a* $\mathscr{P}$*-congruence on* $S(P)$ *such that* $\operatorname{tr} \kappa_{(\xi, K)} = \xi$ *and* $\ker \kappa_{(\xi, K)} = K$. *Conversely, for any* $\mathscr{P}$*-congruence* ρ *on* $S(P)$, $(\operatorname{tr} \rho, \ker \rho)$ *is a* $\mathscr{P}$*-congruence pair for* $S(P)$ *and* $\rho = \kappa_{(\operatorname{tr}_\rho, \ker_\rho)}$.

PROOF. Let (ξ, K) be a $\mathscr{P}$-congruence pair for $S(P)$ and $\kappa_{(\xi, K)} = \kappa$. Obviously, κ is reflexive and symmetric. Suppose that $a \kappa b$ and $b \kappa c$. Let $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ and $c^+ \in V_P(c)$. Set

$$\begin{aligned} &x = ab^+ \in K, \ x^+ = ba^+ \in V_P(x), \ y = bc^+ \in K, \\ &y^+ = cb^+ \in V_P(y), \ e = bb^+, \ f = cc^+. \end{aligned}$$

Note that $e \in V_P(e)$ and $f \in V_P(f)$. Then, by Lemma 3.2,

$$\begin{aligned} xx^+ e &= ab^+ ba^+ bb^+ \xi aa^+ bb^+ \xi bb^+ = e, \\ exx^+ &= bb^+ ab^+ ba^+ \xi bb^+ aa^+ \xi aa^+ \xi xx^+, \\ x^+ xe &= ba^+ ab^+ bb^+ = x^+ x, \\ ex^+ x &= bb^+ ba^+ ab^+ = x^+ x \xi bb^+ = e \end{aligned}$$

and

$$\begin{aligned} yy^+ f &= bc^+ cb^+ cc^+ \xi bb^+ cc^+ \xi cc^+ = f, \\ fyy^+ &= cc^+ bc^+ cb^+ \xi cc^+ bb^+ \xi bb^+ \xi yy^+, \\ y^+ yf &= cb^+ bc^+ cc^+ = y^+ y, \\ fy^+ y &= cc^+ cb^+ bc^+ = y^+ y \xi cc^+ = f. \end{aligned}$$

Also $e\xi \circ f\xi = (bb^+)\xi \circ (cc^+)\xi = (cc^+)\xi = f\xi$. So, by (C4), we have $xy = ab^+bc^+ \in K$. Since $ab^+bc^+ = a(a^+ab^+b)c^+ \in K$ and $a^+a\xi a^+ab^+b$, we have $ac^+ \in K$ by using (C2). Further, since ξ is a normal equivalence on E,

$$aa^+cc^+\xi bb^+(aa^+bb^+)cc^+\xi bb^+bb^+cc^+ = bb^+cc^+\xi cc^+.$$

Similarly, $aa^+\xi cc^+aa^+$, $a^+a\xi a^+ac^+c$ and $c^+c\xi c^+ca^+a$. Thus $a\kappa c$, so that κ is transitive.

To prove that κ is left compatible, suppose that $a\kappa b$ and $c \in S(P)$. Choose $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ and $c^+ \in V_P(c)$. Set

$$x = ab^+ \in K,\ x^+ = ba^+ \in V_P(x).$$

Then

$$x = xbb^+ \in K,$$

$$bb^+xx^+ = bb^+ab^+ba^+\xi bb^+aa^+\xi aa^+\xi xx^+ \qquad \text{(by Lemma 3.2)},$$

$$bb^+x^+x = bb^+ba^+ab^+ = ba^+ab^+\xi bb^+ \qquad \text{(by Lemma 3.2)},$$

so that $cxc^+ = cab^+c^+ \in K$ by (C5). It immediately follows from (C3) that $b^+c^+cb\xi b^+c^+cba^+c^+ca$. Since $ba^+ \in K$ and $a^+a\xi a^+ab^+b$, $bb^+\xi aa^+bb^+$, we have $a^+c^+ca\xi a^+c^+cab^+c^+cb$, again by (C3). Moreover, in the partial groupoid E/ξ satisfying (w),

$$\begin{aligned}(c^+c)\xi &\circ (bb^+)\xi \circ (c^+c)\xi \circ (aa^+)\xi \\ &= [(c^+c)\xi \circ (bb^+)\xi] \circ [(c^+c)\xi \circ (bb^+)\xi] \circ (aa^+)\xi \\ &= (c^+c)\xi \circ [(bb^+)\xi \circ (aa^+)\xi] \\ &= (c^+c)\xi \circ (aa^+)\xi,\end{aligned}$$

so that $c^+caa^+\xi c^+cbb^+c^+caa^+$, which implies $caa^+c^+\xi cbb^+c^+caa^+c^+$ since ξ is a normal equivalence on E. Likewise $cbb^+c^+\xi caa^+c^+cbb^+c^+$. Hence we have $ca\kappa cb$, and thus κ is left compatible. Also it is easy to prove that κ is right compatible by using (C3). Therefore κ is a $\mathscr{P}$-congruence on $S(P)$.

Next we show that $\xi = \operatorname{tr}\kappa$. Let $e, f \in E$ and $e^+ \in V_P(e)$, $f^+ \in V_P(f)$. Suppose that $e\xi f$. Then

$$\begin{aligned}(ff^+ee^+)\xi &= (ff^+)\xi \circ e\xi \circ e^+\xi \\ &= (ff^+)\xi \circ f\xi \circ e^+\xi = f\xi \circ e^+\xi = (ee^+)\xi.\end{aligned}$$

Similarly, $ff^+\xi ee^+ff^+$, $e^+e\xi e^+ef^+f$, $f^+f\xi f^+fe^+e$. Since K is full, $ee^+ff^+ \in K$. Also $e^+\xi \circ f\xi = e^+\xi \circ e\xi = (e^+e)\xi$. Therefore, by (C2), $ef^+ \in K$, so that $e\kappa f$. Conversely, suppose that $e\kappa f$. Then, since

$$(ee^+)\xi = (ff^+)\xi \circ (ee^+)\xi,$$

we have

$$e\xi = (ff^+)\xi \circ e\xi$$

by multiplying $e\xi$ from the right. So

$$f\xi \circ e\xi = [f\xi \circ (ff^+)\xi] \circ e\xi = (ff^+)\xi \circ e\xi = e\xi.$$

By the similar argument, $f\xi \circ e\xi = f\xi$. Therefore $e\xi = f\xi$, that is, $e\xi f$. Hence $\xi = \operatorname{tr}\kappa$.

Finally, we proceed to prove that $K = \ker\kappa$. Suppose that $a \in \ker\kappa$. Then $a\kappa e$ for some $e \in E$. Let $a^+ \in V_P(a)$ and $e^+ \in V_P(e)$. Set

$$x = ae^+ \in K,\ x^+ = ea^+ \in V_P(x),\ f = ee^+.$$

By using Lemma 3.2,

$$xx^+f = ae^+ea^+ee^+\xi aa^+ee^+\xi ee^+ = f,$$

$$fxx^+ = ee^+ae^+ea^+\xi ee^+aa^+\xi aa^+\xi xx^+,$$

$$x^+xf = ea^+ae^+ee^+ = x^+x,$$

$$fx^+x = ee^+ea^+ae^+ = x^+x\xi ee^+ = f,$$

$$f\xi \circ e\xi = (ee^+)\xi \circ e\xi = e\xi,$$

so that $xe = ae^+e \in K$ by (C4). Also set

$$y = ae^+e \in K,\ y^+ = e^+ea^+ \in V_P(y),\ z = a^+a = z^+,$$

$$g = a^+ae^+e \in E,\ g^+ = e^+ea^+a \in V_P(g).$$

From Result 1.1 (iv), $z^+ \in V_P(z)$. Then, by simple calculations,

$$yy^+\xi ee^+yy^+,\ ee^+\xi yy^+ee^+,\ y^+y\xi y^+ye^+e,\ e^+e\xi e^+ey^+y,$$

$$zz^+\xi gg^+zz^+,\ gg^+\xi zz^+gg^+,\ z^+z\xi z^+zg^+g,\ g^+g\xi g^+gz^+z.$$

Further, $(ee^+)(e^+ea^+ae^+e) \in P^2 \subseteq E$ and

$$\begin{aligned}[(ee^+)(e^+ea^+ae^+e)]\xi &= (ee^+)\xi \circ (e^+ea^+ae^+e)\xi \\ &= [(ee^+)\xi \circ (e^+e)\xi] \circ (a^+ae^+e)\xi \\ &= e\xi \circ g\xi.\end{aligned}$$

So we have $yz = ae^+ea^+a \in K$ by (C4). Since

$$ae^+ea^+a = a(a^+ae^+e)a^+a \in K,\ a^+a\xi a^+ae^+e,$$

it follows from (C2) that $a = a(a^+a) \in K$. Thus $\ker\kappa \subseteq K$. Conversely, suppose that $a \in K$. Let $a^+ \in V_P(a)$. Then $a^+ \in K$ since K is a $\mathscr{P}$-subset of $S(P)$. By (C1), we have

$$aa^+\xi aaa^+a^+, \quad a^+a\xi a^+a^+aa.$$

Set

$$e = aa^+a^+a \in E, \; e^+ = a^+aaa^+ \in V_P(e),$$

$$x = a^+a^+ \in K, \; x^+ = aa \in V_P(a^+a^+).$$

Then it is easily verified that

$$aa^+\xi ee^+aa^+, \; ee^+\xi aa^+ee^+, \; a^+a\xi a^+ae^+e, \; e^+e\xi e^+ea^+a,$$

$$xx^+\xi e^+exx^+, \; e^+e\xi xx^+e^+e, \; x^+x\xi x^+xee^+, \; ee^+\xi ee^+x^+x,$$

and $e\xi \circ e^+\xi = (ee^+)\xi$, so that $a(a^+a^+) \in K$ by (C4). Thus $a\kappa a^2$, and hence $K \subseteq \ker\kappa$. Therefore $K = \ker\kappa$.

Conversely, let ρ be a $\mathscr{P}$-congruence on $S(P)$. Then $(\operatorname{tr}\rho, \ker\rho)$ is a $\mathscr{P}$-congruence pair for $S(P)$, and it follows from Lemma 3.1 that $\rho = \kappa_{(\operatorname{tr}_\rho, \ker_\rho)}$. Thus we have the theorem.

Let $\bar{C}$ be the set of $\mathscr{P}$-congruence pairs for $S(P)$. Define an order $\leq$ on $\bar{C}$ by

$$(\xi_1, K_1) \leq (\xi_2, K_2) \text{ if and only if } \xi_1 \subseteq \xi_2, \; K_1 \subseteq K_2.$$

Corollary 3.4. *The mappings*

$$(\xi, K) \longmapsto \kappa_{(\xi,K)}, \quad \rho \longmapsto (\operatorname{tr}\rho, \ker\rho)$$

are mutrually inverse order-preserving mappings of $\bar{C}$ onto Λ and of Λ onto $\bar{C}$, respectively. Therefore $\bar{C}$ forms a complete lattice.

§4. Orthodox $\mathscr{P}$-congruences

For a given orthodox $\mathscr{P}$-congruence ρ on $S(P)$, the maximum orthodox $\mathscr{P}$-congruence $\rho_{\max}$ on $S(P)$ whose trace is $\operatorname{tr}\rho$ was presented in Corollary 2.3. On the other hand, we have

Theorem 4.1. *For any othodox $\mathscr{P}$-congruence ρ on a $\mathscr{P}$-regular semigroup $S(P)$, define a relation ρ_0 on $S(P)$ by*

$$\rho_0 = \{(a, b)\colon \textit{there exist } x, y, u, v \in S(P)^1 \textit{ and } e, f \in E$$
$$\textit{such that } a = xey, \; b = ufv, \; x\eta u, \; y\eta v \textit{ and } e\rho f\},$$

where η is the least orthodox $\mathscr{P}$-congruence on $S(P)$. Then $\rho_{\min} = \rho_0^t$, the transitive closure of ρ_0, is the least orthodox $\mathscr{P}$-congruence on $S(P)$ whose trace coincides with $\operatorname{tr} \rho$.

PROOF. Obviously, ρ_0 is reflexive, symmetric and compatible, so that $\rho_{\min} = \rho_0^t$ is a $\mathscr{P}$-congruence on $S(P)$. Let $e, f \in E$ and $e^+ \in V_P(e)$, $f^+ \in V_P(f)$. Then we have

$$(ef)^2 = e(eff^+e^+)(ef)^2, \; ef = e(eff^+e^+)(ef), \; (ef, (ef)^2) \in \eta.$$

Thus $((ef)^2, ef) \in \rho_0$, and $((ef)^2, ef) \in \rho_{\min}$. Hence $\rho_{\min}$ is an orthodox $\mathscr{P}$-congruence on $S(P)$. Obviously, $\rho_{\min}$ is the least orthodox $\mathscr{P}$-congruence on $S(P)$ whose trace is $\operatorname{tr} \rho$.

Let Λ_1 be the lattice of all orthodox $\mathscr{P}$-congruences on $S(P)$, and define a relation Θ_1 on Λ_1 as follows: for any $\rho, \sigma \in \Lambda_1$,

$$\rho \Theta_1 \sigma \text{ if and only if } \operatorname{tr} \rho = \operatorname{tr} \sigma.$$

Of course, the results corresponding to Propositions 2.8–2.12 hold.

We now proceed to the next stage.

Let ξ be a normal equivalence on E. Then ξ is called an *orthodox normal equivalence* on E if it satisfies

(d) $$(e\xi)(f\xi) \cap E \neq \square \qquad \text{for any } e, f \in E.$$

In this case, E/ξ is a band under

$$e\xi \circ f\xi = g\xi, \quad \text{where} \quad \square \neq (e\xi)(f\xi) \cap E \subseteq g\xi.$$

Let K be both a $\mathscr{P}$-subset and a $\mathscr{P}$-full, $\mathscr{P}$-self-conjugate subsemigroup of $S(P)$ (therefore K is full). Also let ξ be an orthodox normal equivalence on E. Then the pair (ξ, K) is called an *orthodox $\mathscr{P}$-congruence pair* for $S(P)$ if it satisfies the conditions (C1), (C3) in §3 and

(C2)′ $aeb \in K$ and $e\xi a^+a$ for some $a^+ \in V_P(a)$ imply $ab \in K$,

for any $a, b \in S(P)$ and $e \in E$.

For any orthodox $\mathscr{P}$-congruence pair (ξ, K) for $S(P)$, define a relation $\kappa_{(\xi,K)}$ on $S(P)$ by (4) in §3. Of course, we can substitute "some" in (4) by "any".

Now we have

THEOREM 4.2. *Let $S(P)$ be a $\mathscr{P}$-regular semigroup. For any orthodox $\mathscr{P}$-congruence pair (ξ, K) for $S(P)$, $\kappa_{(\xi,K)}$ is an orthodox $\mathscr{P}$-congruence on $S(P)$ such that $\operatorname{tr} \kappa_{(\xi,K)} = \xi$ and $\ker \kappa_{(\xi,K)} = K$. Conversely, for any orthodox $\mathscr{P}$-congruence ρ on $S(P)$, $(\operatorname{tr} \rho, \ker \rho)$ is an orthodox $\mathscr{P}$-congruence pair for $S(P)$ and $\rho = \kappa_{(\operatorname{tr}_\rho, \ker_\rho)}$.*

Let $\bar{C}_1$ be the set of orthodox $\mathscr{P}$-congruence pairs for $S(P)$, and Λ_1 the lattice of all orthodox $\mathscr{P}$-congruences on $S(P)$. Define an order $\leq$ on $\bar{C}_1$ by

$$(\xi_1, K_1) \leq (\xi_2, K_2) \quad \text{if and only if} \quad \xi_1 \subseteq \xi_2,\ K_1 \subseteq K_2.$$

Then we have

COROLLARY 4.3. *The mappings*

$$(\xi, K) \longmapsto \kappa_{(\xi,K)}, \qquad \rho \longmapsto (\mathrm{tr}\,\rho, \ker \rho)$$

are mutually inverse order-preserving mappings of $\bar{C}_1$ *onto* Λ_1 *and of* Λ_1 *onto* $\bar{C}_1$, *respectively. Therefore* $\bar{C}_1$ *forms a complete lattice.*

§5. Strong orthodox $\mathscr{P}$-congruences

Firstly, strong orthodox $\mathscr{P}$-congruences on a $\mathscr{P}$-regular semigroup with the same $\mathscr{P}$-trace are discussed.

THEOREM 5.1. *Let* ρ *be a strong orthodox* $\mathscr{P}$*-congruence on a* $\mathscr{P}$*-regular semigroup* $S(P)$. *Then we have the following*:

(i) *The greatest strong orthodox* $\mathscr{P}$*-congruence* $\rho_{\max}$ *on* $S(P)$ *whose* $\mathscr{P}$*-trace coincides with* $\mathscr{P}$-tr ρ *is given by*

$$\begin{aligned}\rho_{\max} &= \{(a, b)\colon \text{there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \\ &\qquad \text{such that } aea^+ \rho beb^+,\ a^+ea\rho b^+eb \text{ for all } e \in P\} \\ &= \{(a, b)\colon aea^+ \rho beb^+,\ a^+ea\rho b^+eb \\ &\qquad \text{for any } a^+ \in V_P(a),\ b^+ \in V_P(b) \text{ and } e \in P\}.\end{aligned}$$

(ii) *Define a relation* ρ_0 *on* $S(P)$ *by*

$$\begin{aligned}\rho_0 = \{(a, b)\colon &\text{there exist } x, y, u, v \in S(P)^1 \text{ and } e, f \in P \\ &\text{such that } a = xey,\ b = ufv,\ x\tau u,\ y\tau v \text{ and } e\rho f\}\end{aligned}$$

where τ *is the least strong orthodox* $\mathscr{P}$*-congruence on* $S(P)$. *Then* $\rho_{\min} = \rho_0^t$, *the transitive closure of* ρ_0, *is the least strong orthodox* $\mathscr{P}$*-congruence on* $S(P)$ *whose* $\mathscr{P}$*-trace coincides with* $\mathscr{P}$-tr ρ.

PROOF. Obvious.

Let Λ_2 be the lattice of all strong orthodox $\mathscr{P}$-congruences on $S(P)$. Define a relation Θ_2 on Λ_2 as follows: for any $\rho, \sigma \in \Lambda_3$,

$$\rho\Theta_2\sigma \quad \text{if and only if} \quad \mathscr{P}\text{-tr}\,\rho = \mathscr{P}\text{-tr}\,\sigma.$$

Several properties of $\rho_{\max}$ and $\rho_{\min}$ are introduced without proof.

PROPOSITION 5.2. *Let ρ be a strong orthodox $\mathscr{P}$-congruence on $S(P)$ and $e \in P$. Then we have*

$$e\rho = e\rho_{\max} \cap \mathscr{P}\text{-}\ker \rho.$$

PROPOSITION 5.3. *For any strong orthodox $\mathscr{P}$-congruence ρ on $S(P)$, $\rho = \rho_{\max}$ if and only if $S(P)/(\rho)_{\mathscr{P}}$ is a fundamental orthodox $*$-semigroup.*

PROPOSITION 5.4. *Let ρ be a strong orthodox $\mathscr{P}$-congruence on $S(P)$. Then $\rho_{\max}/\rho$ is the maximum idempotent separating $\mathscr{P}$-congruence on $S(P)/(\rho)_{\mathscr{P}}$.*

PROPOSITION 5.5. *Let $\xi \in \Lambda_2$, and let Γ_2 be the lattice of all idempotent separating $\mathscr{P}$-congruences on $S(P)/(\xi_{\min})_{\mathscr{P}}$. Then the mapping $\rho \mapsto \rho/\xi_{\min}$ is a complete isomorphism of $\xi\Theta_2$ onto Γ_2.*

These propositions hold for strong $\mathscr{P}$-congruences on $S(P)$.

Next, the concept of strong orthodox $\mathscr{P}$-congruence pairs is introduced.

An orthodox normal equivalence ξ on E is called a *strong orthodox normal equivalence* on E if it satisfies

(e) $\quad e\xi p$ for any $e \in E$ and $p \in P$ implies $e^+ \xi p$ for any $e^+ \in V_P(e)$.

LEMMA 5.6. *Let ξ be a strong orthodox normal equivalence on E. Then $aa^+ \xi aa^*$ and $a^+a\xi a^*a$ for any $a \in S(P)$ and $a^+, a^* \in V_P(a)$.*

PROOF. Let $a \in S(P)$ and $a^+, a^* \in V_P(a)$. Then $aa^+ \xi aa^+$, $aa^* \in V_P(aa^+)$ and $aa^+ \in P$. So $aa^+ \xi aa^*$ by using (e). Likewise, $a^+a\xi a^*a$.

Let K be both a $\mathscr{P}$-subset and a $\mathscr{P}$-full, $\mathscr{P}$-self-conjugate subsemigroup of $S(P)$. Also let ξ be a strong orthodox normal equivalence on E. Then the pair (ξ, K) is called a *strong orthodox $\mathscr{P}$-congruence pair* for $S(P)$ if it satisfies the conditions (C1) in §3 and

(C2)″ $aeb \in K$ and $e\xi a^+a$ for some $a^+ \in V_P(a)$ imply $ab \in K$,

(C3)″ $ab^+ \in K$ and $aa^+ \xi bb^+$, $a^+a\xi b^+b$ for some $a^+ \in V_P(a)$ and $b^+ \in V_P(b)$ imply $aeb^+ \in K$ and $aea^+ \xi beb^+$, $a^+ea\xi b^+eb$,

for any $a, b \in S(P)$ and $e \in P$.

For any strong orthodox $\mathscr{P}$-congruence pair (ξ, K) for $S(P)$, define a relation $\kappa_{(\xi,K)}$ on $S(P)$ as follows:

$$\kappa_{(\xi,K)} = \{(a, b)\colon aa^+ \xi bb^+,\ a^+a\xi b^+b \text{ and } a^+b,\ ab^+ \in K \text{ for some } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b)\}.$$

REMARK. We can substitute "some" by "any".

Now we have the following theorem.

THEOREM 5.7. *Let* $S(P)$ *be a* $\mathscr{P}$*-regular semigroup. For any strong orthodox* $\mathscr{P}$*-congruence pair* (ξ, K) *for* $S(P)$, $\kappa_{(\xi,K)}$ *is a strong orthodox* $\mathscr{P}$*-congruence on* $S(P)$ *such that* $\operatorname{tr}\kappa_{(\xi,K)} = \xi$ *and* $\ker\kappa_{(\xi,K)} = K$. *Conversely, for any strong orthodox* $\mathscr{P}$*-ongruence* ρ *on a* $\mathscr{P}$*-regular semigroup* $S(P)$, $(\operatorname{tr}\rho, \ker\rho)$ *is a strong orthodox* $\mathscr{P}$*-congruence pair for* $S(P)$ *and* $\rho = \kappa_{(\operatorname{tr}_\rho, \ker_\rho)}$.

Let $\bar{C}_2$ be the set of strong orthodox $\mathscr{P}$-congruence pairs for $S(P)$. Define an order $\leq$ on $\bar{C}_2$ by

$$(\xi_1, K_1) \leq (\xi_2, K_2) \quad \text{if and only if} \quad \xi_1 \subseteq \xi_2,\ K_1 \subseteq K_2.$$

COROLLARY 5.8. *The mappings*

$$(\xi, K) \longmapsto \kappa_{(\xi,K)}, \qquad \rho \longmapsto (\operatorname{tr}\rho, \ker\rho)$$

are mutually inverse order-preserving mappings of $\bar{C}_3$ *onto* Λ_3 *and of* Λ_3 *onto* $\bar{C}_3$, *respectively. Therefore* $\bar{C}_3$ *forms a complete lattice.*

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