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Extremal Problems with respect to Ideal Boundary Components of an Infinite Network II

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Several extremum problems will be studied with the constraint qualification related to ideal boundary components of an infinite network. We shall give a generalized inverse relation between the extremal length and the extremal width of the network relative to ideal boundary components.

§ 1. Introduction

In the previous paper [2], we introduced a notion of ideal boundary components of an infinite network $N = \{X, Y, K, r\}$. For a set Γ of paths in N, the extremal length $\lambda_p(\Gamma)$ of order p $(1 < p < \infty)$ is defined by

$$\lambda_p(\Gamma)^{-1} = \inf\{H_p(W);\ W \in E_p(\Gamma)\},$$

where $H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p$ and $E_p(\Gamma)$ is the set of all $W \in L^+(Y)$ such that $H_p(W) < \infty$ and

$$\textstyle\sum_P r(y) W(y) := \sum_{y \in C_Y(P)} r(y) W(y) \geq 1$$

for all $P \in \Gamma$. For a set Λ of cuts in N, the extremal width $\mu_q(\Lambda)$ of Λ of order q $(1 < q < \infty)$ is defined by

$$\mu_q(\Lambda)^{-1} = \inf\{H_q(W);\ W \in E_q^*(\Lambda)\},$$

where $E_q^*(\Lambda)$ is the set of all $W \in L^+(Y)$ such that $H_q(W) < \infty$ and

$$\textstyle\sum_Q W(y) := \sum_{y \in Q} W(y) \geq 1$$

for all $Q \in \Lambda$. In the preceding paper, we proved the following generalized inverse relation:

$$(*) \qquad [\lambda_p(\Gamma)]^{1/p} [\mu_q(\Lambda)]^{1/q} = 1 \qquad \text{with } 1/p + 1/q = 1\ (1 < p < \infty)$$

for $\Gamma = \mathbf{P}_{A,\alpha}$ (the set of paths from a finite subset A of X to an ideal boundary component α of N) and $\Lambda = \mathbf{Q}_{A,\alpha}$ (the set of cuts between A and α). In this paper, for two ideal boundary components α and β of N, we shall prove the relation $(*)$ in the case where Γ is the set $\mathbf{P}_{\alpha,\beta}$ of paths from α to β and Λ is the set $\mathbf{Q}_{\alpha,\beta}$ of cuts between α and β. The definitions of $\mathbf{P}_{\alpha,\beta}$ and $\mathbf{Q}_{\alpha,\beta}$ will be given in §2. We shall discuss the duality between the min-work problem with respect to $\mathbf{P}_{\alpha,\beta}$ and the related max-potential problem. Several convex programming problems will be studied with the constraints related to α and β.

For notation and terminology, we mainly follow [2].

§2. Preliminaries

Let p and q be positive numbers such that $1/p+1/q=1$ and $1<p<\infty$. Assume that $G=\{X, Y, K\}$ is an infinite graph which is connected, locally finite and has no self-loop with the countable set X of nodes, the countable set Y of arcs and the node-arc incidence function K. Let r be a strictly positive real valued function on Y. We call the pair $N=\{G, r\}$ an infinite network. For a subset A of X, denoted by $i(A)$ the set of interior nodes of A and by $b(A):=A-i(A)$ the set of boundary nodes of A. Recall that $a\in i(A)$ if and only if all neighboring nodes of a belong to A, i.e., $X(a)\subset A$.

Denote by $ibc(N)$ the set of all ideal bundary components of N as in [2]. A sequence $\{N_n^*\}(N_n^*=\langle X_n^*, Y_n^*\rangle)$ of infinite subnetworks of N is called a determining sequence of $\alpha\in ibc(N)$ if each N_n^* is an end (cf. [2]) of N and the following conditions hold:

(2.1) $\quad N_{n+1}^*$ is a subnetwork of N_n^* and $X_{n+1}^*\subset i(X_n^*)$;

(2.2) $\quad \bigcap_{n=1}^{\infty} X_n^*=\phi$.

It should be noted that each $b(X_n^*)$ is a finite set by definition.

Denote by $\mathbf{Z}$ the set of all integers, by $\mathbf{Z}^+$ the set of all non-negative integers and put $\mathbf{Z}^-=-\mathbf{Z}^+=\{-n; n\in\mathbf{Z}^+\}$. We regard them directed sets with respect to the natural order if we take them as index sets of paths.

To introduce a notion of paths from $\alpha\in ibc(N)$ to $\beta\in ibc(N)$, we begin with

DEFINITION 2.1. Let J be any one of directed sets $\mathbf{Z}$, $\mathbf{Z}^+$ and $\mathbf{Z}^-$. An infinite path P in N is a triple $\{\varphi, \psi, p\}$ of mappings φ and ψ from J into X and Y respectively and a function p on Y satisfying the conditions:

(P.1) $\quad \varphi^{-1}(x)$ is a finite set (possibly, empty set);
(P.2) $\quad \psi$ is one-to-one and $e(\psi(i))=\{\varphi(i), \varphi(i+1)\}$ for each i;
(P.3) $\quad p(\psi(i))=-K(\varphi(i), \psi(i))$ for each $i\in J$,
$\quad\quad p(y)=0$ for $y\in Y-\psi(J)$.

For simplicity, we set

$$\varphi(k)=x_k,\ \psi(k)=y_k,\ \varphi(J)=C_X(P) \text{ and } \psi(J)=C_Y(P)$$

and call the triple $\{C_X(P), C_Y(P), p\}$ a path as in [2]. In case $J = \mathbf{Z}^+$, P is called a path from $\varphi(0) = x_0$ (the initial node) to the point at infinity ∞. Denote by $\mathbf{P}_{x_0,\infty}$ the set of all paths from x_0 to ∞. In case $J = \mathbf{Z}^-$, P is called a path from ∞ to $\varphi(0) = x_0$ (the terminal node). Denote by $\mathbf{P}_{\infty,x_0}$ the set of all paths from ∞ to x_0. In case $J = \mathbf{Z}$, P is called a path from ∞ to ∞. Denote by $\mathbf{P}_{\infty,\infty}$ the set of all paths from ∞ to ∞.

For a path $P = \{\varphi, \psi, p\} \in \mathbf{P}_{x_0,\infty}$, we define the opposite path P^- of P by $P^- = \{\varphi', \psi', p'\}$ such that $\varphi'(-n) = \varphi(n)$ for $n \in \mathbf{Z}^+$, $\psi'(-n) = \psi(n)$ and $p'(\psi(-n)) = -p(\psi(n))$ for $n \in \mathbf{Z}^+$. Note that $P^- \in \mathbf{P}_{\infty,x_0}$ and $C_X(P^-)$ and $C_Y(P^-)$ are equal to $C_X(P)$ and $C_Y(P)$ respectively as sets ignoring the order. We define the opposite path P^- of $P \in \mathbf{P}_{\infty,x_0} \cup \mathbf{P}_{\infty,\infty}$ similarly.

For two paths P_1 and P_2, the sum $P_1 + P_2$ is well-defined in case the terminal node of P_1 coincides with the initial nodes of P_2 (cf. [2]). If $P_1 \in \mathbf{P}_{\infty,x_0}$ and $P_2 \in \mathbf{P}_{x_0,\infty}$, then $P_1 + P_2 \in \mathbf{P}_{\infty,\infty}$.

Hereafter, let $\alpha, \beta \in ibc(N)$, $\alpha \neq \beta$ and $\{N_n^*\}$ ($N_n^* = \langle X_n^*, Y_n^* \rangle$) and $\{\bar{N}_n^*\}$ ($\bar{N}_n^* = \langle \bar{X}_n^*, \bar{Y}_n^* \rangle$) be determining sequences of α and β respectively such that $X_1^* \cap \bar{X}_1^* = \phi$.

A path $P \in \mathbf{P}_{x,\infty}$ is called a path from x to α if $C_X(P) - X_n^*$ is a finite set (possibly, empty set) for each n. Denote by $\mathbf{P}_{x,\alpha}$ the set of all paths from x to α and put $\mathbf{P}_{A,\alpha} = \bigcup_{x \in A} \mathbf{P}_{x,\alpha}$ for a subset A of X. Let $\mathbf{P}_\alpha = \mathbf{P}_{X,\alpha}$.

DEFINITION 2.2. A path $P \in \mathbf{P}_{\infty,\infty}$ is called a path from α to β if there exist $x_0 \in X$ and paths P_1 and P_2 such that

$$P = P_1^- + P_2, \; P_1 \in \mathbf{P}_{x_0,\alpha} \text{ and } P_2 \in \mathbf{P}_{x_0,\beta}.$$

Denote by $\mathbf{P}_{\alpha,\beta}$ the set of all paths from α to β.

For a finite nonempty subset A of X such that $A \cap \bar{X}_1^* = \phi$, the set of cuts between A and β is defined by

$$\mathbf{Q}_{A,\beta} = \bigcup_{n=1}^{\infty} \mathbf{Q}_{A,\bar{X}_n^*},$$

where $\mathbf{Q}_{A,\bar{X}_n^*}$ is the set of all cuts between A and $\bar{X}_n^*$ (cf. [2]). Notice that $\{\mathbf{Q}_{X_m^*,\bar{X}_n^*}\}$ is increasing with respect to both m and n. So we set

$$\mathbf{Q}_{\alpha,\beta} = \bigcup_{m=1}^{\infty} (\bigcup_{n=1}^{\infty} \mathbf{Q}_{X_m^*,\bar{X}_n^*}) = \bigcup_{m=1}^{\infty} \mathbf{Q}_{X_m^*,\beta}.$$

and call its element a cut between α and β. Clearly,

$$\mathbf{Q}_{\alpha,\beta} = \bigcup_{n=1}^{\infty} \mathbf{Q}_{X_n^*,\bar{X}_n^*}.$$

Needless to say, these definitions do not depend on the choice of determining sequences of α and β.

§3. Max-potential and min-work problems

Let α and β be distinct ideal boundary components of N and let $c \in L^+(Y)$. We shall study the duality between the following min-work problem (MWP) and max-potential problem (MPP) related to α, β and c:

(MWP) Minimize $\sum_P c(y)$ subject to $P \in \mathbf{P}_{\alpha,\beta}$.

(MPP) Maximize $\delta_c(u; \alpha, \beta)$

$$:= \inf\{u(P); P \in \Gamma_c(\alpha)\} - \sup\{u(P); P \in \Gamma_c(\beta)\}$$

subject to $u \in S_c^*$

$$:= \{u \in L(X); |\sum_{x\in X} K(x, y)u(x)| \le c(y) \text{ on } Y\}.$$

Here $\Gamma_c(\alpha) = \{P \in \mathbf{P}_\alpha; \sum_P c(y) < \infty\}$ and $u(P)$ for $P \in \mathbf{P}_\alpha$ denotes the limit value of $u(x)$ as x tends to α along P if it exists. It is clear that $u(P)$ exists for any $u \in S_c^*$ and $P \in \Gamma_c(\alpha) \cup \Gamma_c(\beta)$. Note that $\delta_c(u; \alpha, \beta)$ is the potential drop of u between α and β relative to c. Denote by $N(\mathbf{P}_{\alpha,\beta}; c)$ and $N^*(\alpha, \beta; c)$ the values of (MWP) and (MPP) respectively.

For a subset A of X, β and c, let $N(\mathbf{P}_{A,\beta}; c)$ be the value of the min-work problem as in [2], i.e.,

$$N(\mathbf{P}_{A,\beta}; c) = \inf\{\textstyle\sum_P c(y); P \in \mathbf{P}_{A,\beta}\}.$$

By the same argument as in the proof of [2; Lemma 2.1], we obtain

LEMMA 3.1. *$\{N(\mathbf{P}_{b(X_n^*),\beta}; c)\}$ converges increasingly to $N(\mathbf{P}_{\alpha,\beta}; c)$ as $n \to \infty$.*

By the relation: $N(\mathbf{P}_{b(X_m^*),\beta}; c) = N(\mathbf{P}_{X_m^*,\beta}; c)$, we have

COROLLARY 3.2. *$N(\mathbf{P}_{X_m^*,\beta}; c)\} \to N(\mathbf{P}_{\alpha,\beta}; c)$ as $n \to \infty$.*

Now we show the following duality theorem for (MWP) and (MPP):

THEOREM 3.3. *If $\Gamma_c(\alpha) \neq \phi$ and $\Gamma_c(\beta) \neq \phi$, then $N(\mathbf{P}_{\alpha,\beta}; c) = N^*(\alpha, \beta; c)$ holds and* (MPP) *has an optimal solution.*

PROOF. Let $u \in S_c^*$ and $P \in \mathbf{P}_{\alpha,\beta}$ with $\sum_P c(y) < \infty$. Then there exist $x_0 \in X$, $P_1 \in \mathbf{P}_{x_0,\alpha}$ and $P_2 \in \mathbf{P}_{x_0,\beta}$ such that $P = P_1^- + P_2$. Let $C_X(P_1) = \{x_0, x_1, x_2, \ldots\}$, $C_X(P_2) = \{x_0, x_1', x_2', \ldots\}$, $C_Y(P_1) = \{y_0, y_1, y_2, \ldots\}$, $C_Y(P_2) = \{y_0', y_1', y_2', \ldots\}$, $e(y_i) = \{x_i, x_{i+1}\}$ and $e(y_i') = \{x_i', x_{i+1}'\}$ for each $i \in \mathbf{Z}^+$ with $x_0' = x_0$. Then

$$\begin{aligned}\textstyle\sum_P c(y) &= \textstyle\sum_{P_1} c(y) + \sum_{P_2} c(y)\\ &\ge \textstyle\sum_{i=1}^n \{|u(x_i) - u(x_{i-1})| + |u(x_i') - u(x_{i-1}')|\}\\ &\ge u(x_n) - u(x_n')\end{aligned}$$

for every n, so that

$$\sum_P c(y) \geq u(P_1) - u(P_2) \geq \delta_c(u; \alpha, \beta).$$

Hence $N(\mathbf{P}_{\alpha,\beta}; c) \geq N^*(\alpha, \beta; c)$.

To prove the converse inequality, define $\hat{u} \in L(X)$ by

$$\hat{u}(x) = \inf\{\sum_P c(y);\ P \in \mathbf{P}_{x,\beta}\} = N(\mathbf{P}_{x,\beta}; c)$$

for $x \in X$. Notice that $\hat{u}(x) < \infty$ by our assumption $\Gamma_c(\beta) \neq \phi$. By the same way as in the proof of [2; Theorem 2.1], we see that $\hat{u} \in S_c^*$, $\hat{u}(P) = 0$ for every $P \in \Gamma_c(\beta)$ and

$$\inf\{\hat{u}(x);\ x \in b(X_m^*)\} = N(\mathbf{P}_{b(X_m^*),\beta}; c)$$

for every m. We shall prove that $N(\mathbf{P}_{\alpha,\beta}; c) \leq \delta_c(\hat{u}; \alpha, \beta)$. Let $P \in \Gamma_c(\alpha)$ with $C_X(P) = \{x_0, x_1, x_2,\ldots\}$. Then $\hat{u}(P) = \lim_{n\to\infty} \hat{u}(x_n)$. For $t > \hat{u}(P)$, there exists n_0 such that $\hat{u}(x_n) < t$ for all $n \geq n_0$. For each m large enough, there exists $j_m (> n_0)$ such that $x_{j_m} \in b(X_m^*)$, since $P \in \mathbf{P}_\alpha$, so that

$$t > \hat{u}(x_{j_m}) \geq \inf\{\hat{u}(x);\ x \in b(X_m^*)\} = N(\mathbf{P}_{b(X_m^*),\beta}; c).$$

By Lemma 3.1, $t \geq N(\mathbf{P}_{\alpha,\beta}; c)$ and hence $\hat{u}(P) \geq N(\mathbf{P}_{\alpha,\beta}; c)$. Therefore,

$$N^*(\alpha, \beta; c) \geq \delta_c(\hat{u}; \alpha, \beta) = \inf_{P\in\Gamma_c(\alpha)} \hat{u}(P) \geq N(\mathbf{P}_{\alpha,\beta}; c).$$

It follows that $N(\mathbf{P}_{\alpha,\beta}; c) = N^*(\alpha, \beta; c)$ and that $\hat{u}$ is an optimal solution of (MPP).

§4. The extremal length $\lambda_p(\mathbf{P}_{\alpha,\beta})$

Related to the extremal length $\lambda_p(\mathbf{P}_{\alpha,\beta})$ of $\mathbf{P}_{\alpha,\beta}$ of order p we consider the following convex programming problem on $L(X)$:

(4.1) Minimize $D_p(u) := H_p(du)$

subject to $u \in L(X)$, $u(\alpha) = 1$ and $u(\beta) = 0$.

Here $du(y) = -r(y)^{-1}\sum_{x\in X} K(x, y)u(x)$ is a discrete derivative of u and $u(\alpha) = t$ implies that $u(P)$ exists and is equal to t for p-almost every $P \in \mathbf{P}_\alpha$, i.e., $\lambda_p(\mathbf{P}_\alpha - \Gamma) = \infty$ with $\Gamma = \{P \in \mathbf{P}_\alpha;\ u(P) \text{ exists and } u(P) = t\}$. Denote by $d_p(\alpha, \beta)$ the value of Problem (4.1). Notice that $d_p(\alpha, \beta) < \infty$.

We have

THEOREM 4.1. *If* $\lambda_p(\mathbf{P}_{\alpha,\beta}) < \infty$, *then* $d_p(\alpha, \beta) = \lambda_p(\mathbf{P}_{\alpha,\beta})^{-1}$.

PROOF. First we shall prove that $\lambda_p(\mathbf{P}_{\alpha,\beta})^{-1} \leq d_p(\alpha, \beta)$. Let $u \in L(X)$ such that $D_p(u) < \infty$, $u(\alpha) = 1$ and $u(\beta) = 0$. Put

$\Gamma(\alpha; u) = \{P \in \mathbf{P}_\alpha;\ u(P) = 1\}$,

$\Gamma(\beta; u) = \{P \in \mathbf{P}_\beta; u(P) = 0\}$,

$\Gamma(\alpha, \beta; u) = \{P \in \mathbf{P}_{\alpha,\beta}; P = P_1^- + P_2, P_1 \in \mathbf{P}_\alpha, P_2 \in \mathbf{P}_\beta, u(P_1) = 1, u(P_2) = 0\}$.

Then $\lambda_p(\mathbf{P}_\alpha - \Gamma(\alpha; u)) = \lambda_p(\mathbf{P}_\beta - \Gamma(\beta; u)) = \infty$ by our assumption, so that $\lambda_p(\mathbf{P}_{\alpha,\beta} - \Gamma(\alpha, \beta; u)) = \infty$ by [1; Lemma 2.3]. Let $W = |du|$. Then $H_p(W) < \infty$ and $\sum_P r(y)W(y) \geq 1$ for all $P \in \Gamma(\alpha, \beta; u)$ by the same reasoning as in the proof of Theorem 3.3. Namely, $W \in E_p(\Gamma(\alpha, \beta; u))$. Thus by [1; Lemma 2.2]

$$\lambda_p(\mathbf{P}_{\alpha,\beta})^{-1} = \lambda_p(\Gamma(\alpha, \beta; u))^{-1} \leq H_p(W) = D_p(u),$$

so that $\lambda_p(\mathbf{P}_{\alpha,\beta})^{-1} \leq d_p(\alpha, \beta)$.

Next we prove the converse inequality. Let $W \in E_p(\mathbf{P}_{\alpha,\beta})$. Then,

$$\sum_P r(y)W(y) < \infty \text{ for } p\text{-almost every } P \in \mathbf{P}_\alpha \cup \mathbf{P}_\beta$$

(cf. [2; Lemma 1.1]). Take $c = rW$. Then $\Gamma_c(\alpha) \neq \phi$ and $\Gamma_c(\beta) \neq \phi$ by our assumption $\lambda_p(\mathbf{P}_{\alpha,\beta}) < \infty$. We can find $u \in L(X)$ such that $u(\beta) = 0$, $u \in S_c^*$ and $\delta_c(u; \alpha, \beta) = N(\mathbf{P}_{\alpha,\beta}; c) \geq 1$ by Theorem 3.3. Define $v \in L(X)$ by $v(x) = \min(u(x), 1)$. Then $v(P) = 1$ for every $P \in \Gamma_c(\alpha)$, $v(\beta) = 0$ and $|dv(y)| \leq |du(y)| \leq W(y)$ on Y. Since $\lambda_p(\mathbf{P}_\alpha - \Gamma_c(\alpha)) = \infty$, we have $v(\alpha) = 1$ and

$$d_p(\alpha, \beta) \leq D_p(u) \leq H_p(W).$$

Therefore, $d_p(\alpha, \beta) \leq \lambda_p(\mathbf{P}_{\alpha,\beta})^{-1}$.

By the same reasoning as in the proof of [2; Theorem 2.4] with Lemma 3.1, we obtain the following property (stability) of extremal length:

THEOREM 4.2. *For every determining sequence* $\{N_n^*\}$ $(N_n^* = \langle X_n^*, Y_n^* \rangle)$ *of* α, $\lambda_p(\mathbf{P}_{b(X_m^*),\beta}) \to \lambda_p(\mathbf{P}_{\alpha,\beta})$ *as* $n \to \infty$.

§5. Extremal width $\mu_q(\mathbf{Q}_{\alpha,\beta})$

We prepare

LEMMA 5.1. *Let A and B be mutually disjoint nonempty subsets of X and $\beta \in ibc(N)$ such that $A \cap \bar{X}_1^* = \phi$. Then $E_q^*(\mathbf{Q}_{A,B}) = E_q^*(\mathbf{Q}_{b(A),B})$ and $E_q^*(\mathbf{Q}_{A,B}) = E_q^*(\mathbf{Q}_{b(A),\beta})$.*

PROOF. By the obvious relations $\mathbf{Q}_{A,B} \subset \mathbf{Q}_{b(A),B}$ and $\mathbf{Q}_{A,\beta} \subset \mathbf{Q}_{b(A),\beta}$, we have $E_q^*(\mathbf{Q}_{A,B}) \supset E_q^*(\mathbf{Q}_{b(A),B})$ and $E_q^*(\mathbf{Q}_{A,\beta}) \supset E_q^*(\mathbf{Q}_{b(A),\beta})$. For the proof of the converse relation, it suffices to note that every $Q \in \mathbf{Q}_{b(A),B}$ (resp. $\mathbf{Q}_{b(A),\beta}$) contains $Q' \in \mathbf{Q}_{A,B}$ (resp. $\mathbf{Q}_{A,\beta}$). For $Q \in \mathbf{Q}_{b(A),B}$ with $Q = Q(b(A)) \ominus Q(B)$, let $Q'(A) = Q(b(A)) \cup A$ and $Q'(B) = Q(B) - A$. Then $Q' = Q'(A) \ominus Q'(B) \in \mathbf{Q}_{A,B}$ and $Q' \subset Q$. For $Q \in \mathbf{Q}_{b(A),\beta}$, there exists n such that $Q \in \mathbf{Q}_{b(A),\bar{X}_n^*}$. By the above observation, we can find $Q'' \in \mathbf{Q}_{A,\bar{X}_n^*} (\subset \mathbf{Q}_{A,\beta})$ such that $Q'' \subset Q$.

COROLLARY 5.2. *The following equalities hold*:

(1) $\mu_q(\mathrm{Q}_{A,B}) = \mu_q(\mathrm{Q}_{b(A),B}) = \mu_q(\mathrm{Q}_{b(A),b(B)})$;

(2) $\mu_q(\mathrm{Q}_{A,\beta}) = \mu_q(\mathrm{Q}_{b(A),\beta})$.

In order to study some properties of $\mu_q(\mathrm{Q}_{\alpha,\beta})$, we need the notion of flows. For $w \in L(Y)$ and a subset A of X, let

$$I(w; x) = \sum_{y \in Y} K(x, y) w(y),$$
$$I(w; A) = \sum_{x \in A} I(w; x)$$

provided that $\sum_{x \in A} |I(w; x)| < \infty$.

For mutually disjoint nonempty subsets A and B of X, the set $F(A, B)$ of flows from A to B is the set of $w \in L(Y)$ such that

$$I(w; x) = 0 \text{ for all } x \in X - A - B \text{ and } I(w; A) + I(w; B) = 0.$$

Denote by $L_0(Y)$ the set of $w \in L(Y)$ with finite support and by $F_q(A, B)$ the closure of $F_0(A, B) := F(A, B) \cap L_0(Y)$ in the Banach space $L_q(Y; r) := \{w \in L(Y); H_q(w) < \infty\}$ with the norm $[H_q(\cdot)]^{1/q}$.

In case there exists n_0 such that $A \cap \bar{X}^*_{n_0} = \phi$, we have

$$F_q(A, \bar{X}^*_n) \supset F_q(A, \bar{X}^*_{n+1}),$$

so we put $F_q(A, \beta) = \bigcap_{n=n_0}^{\infty} F_q(A, \bar{X}^*_n)$ and call its element a flow from A to β. This set does not depend on the choice of the determining sequence of β.

Let $\mathscr{F}$ be any one of $F_0(A, B)$, $F_q(A, B)$ and $F_q(A, \beta)$ and consider the following extremum problem:

$$\text{Find } d_q^*(\mathscr{F}) := \inf\{H_q(w);\ w \in \mathscr{F} \text{ and } I(w; A) = -1\}.$$

LEMMA 5.3. *Let* $N^* = \langle X^*, Y^* \rangle$ *and* $\bar{N}^* = \langle \bar{X}^*, \bar{Y}^* \rangle$ *be ends of* N *such that* $X^* \cap \bar{X}^* = \phi$. *Then* $d_q^*(F_0(X^*, \bar{X}^*)) = d_q^*(F_0(b(X^*), \bar{X}^*))$.

PROOF. Since $F_0(X^*, \bar{X}^*) \supset F_0(b(X^*), \bar{X}^*)$,

$$d_q^*(F_0(X^*, \bar{X}^*)) \leq d_q^*(F_0(b(X^*), \bar{X}^*)).$$

On the other hand, let $w \in F_0(X^*, \bar{X}^*)$ with $I(w; X^*) = -1$. Define $w' \in L(Y)$ by

$$w'(y) = 0 \text{ on } i(Y^*) := \bigcup_{x \in i(X^*)} Y(x);$$
$$w'(y) = w(y) \text{ on } Y - i(Y^*).$$

Then, $w' \in F_0(b(X^*), \bar{X}^*)$ and $I(w'; b(X^*)) = -1$. In fact, clearly

$$I(w'; x) = 0 \qquad \text{for } x \in i(X^*).$$

For $x \in X - (X^* \cup \bar{X}^*)$, $Y(x) \cap i(Y^*) = \phi$ and $I(w'; x) = I(w; x) = 0$. By the relation

$$\sum_{x \in X^*} \sum_{y \in i(Y^*)} K(x, y) w(y) = \sum_{y \in i(Y^*)} w(y) \sum_{x \in X^*} K(x, y) = 0,$$

we have

$$I(w'; b(X^*)) = \sum_{x \in X^*} \sum_{y \in Y - i(Y^*)} K(x, y) w(y) = I(w; X^*) = -1.$$

Therefore w' is a feasible solution for $d_q^*(F_0(b(X^*), \bar{X}^*))$, and hence

$$d_q^*(F_0(b(X^*), \bar{X}^*)) \le H_q(w') \le H_q(w).$$

Thus $d_q^*(F_0(b(X^*), \bar{X}^*)) \le d_q^*(F_0(X^*, \bar{X}^*))$.

It is easily seen that

$$d_q^*(F_0(A, B)) = d_q^*(F_q(A, B)).$$

Therefore we obtain

COROLLARY 5.4. $d_q^*(F_0(X^*, \bar{X}^*)) = d_q^*(F_q(b(X^*), b(\bar{X}^*)))$.

Now we prove a stability of extremal width:

THEOREM 5.5. *$\mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*}) \to \mu_q(\mathbf{Q}_{\alpha, \beta})$ as $n \to \infty$.*

PROOF. Noting that $\mathbf{Q}_{X_n^*, \bar{X}_n^*} \subset \mathbf{Q}_{X_{n+1}^*, \bar{X}_{n+1}^*} \subset \mathbf{Q}_{\alpha, \beta}$, we have

$$\mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*}) \ge \mu_q(\mathbf{Q}_{X_{n+1}^*, \bar{X}_{n+1}^*}) \ge \mu_q(\mathbf{Q}_{\alpha, \beta}),$$

so that $\lim_{n \to \infty} \mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*}) \ge \mu_q(\mathbf{Q}_{\alpha, \beta})$. To show the converse inequality, we may assume that $\lim_{n \to \infty} \mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*}) > 0$ and $\mu_q(\mathbf{Q}_{\alpha, \beta}) < \infty$. By [3; Theorem 4.1 and Proposition 4.2] and Corollary 5.2, there exists $w_n \in F_q(b(X_n^*), b(\bar{X}_n^*))$ such that $I(w_n; b(\bar{X}_n^*)) = -1$ and

$$H_q(w_n) = d_q^*(F_q(b(X_n^*), b(\bar{X}_n^*))) = \mu_q(\mathbf{Q}_{b(X_n^*), b(\bar{X}_n^*)})^{-1} = \mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*})^{-1},$$

since $b(X_n^*)$ and $b(\bar{X}_n^*)$ are finite sets. Notice that $\{H_q(w_n)\}$ is bounded by our assumption. For each w_n, there exists $w'_n \in F_0(b(X_n^*), b(\bar{X}_n^*))$ such that $I(w'_n; b(X_n^*)) = -1$ and $H_q(w_n - w'_n) < 1/n$. Then $w'_n \in F_0(X_n^*, \bar{X}_n^*)$, $I(w'_n; X_n^*) = -1$ and

$$1 = |I(w'_n; X_n^*)| \le \sum_Q |w'_n(y)|$$

for all $Q \in \mathbf{Q}_{X_n^*, \bar{X}_n^*}$. Nemely, $|w'_n| \in E_q^*(Q_{X_n^*, \bar{X}_n^*})$, and hence $\mu_q(Q_{X_n^*, \bar{X}_n^*})^{-1} \le H_q(w'_n)$. Therefore

$$\lim_{n \to \infty} H_q(w'_n) = \lim_{n \to \infty} H_q(w_n) = \lim_{n \to \infty} \mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*})^{-1}.$$

If $m > n$, then $(w'_n + w'_m)/2$ is a feasible solution of $d_q^*(F_0(X_n^*, \bar{X}_n^*))$. By Clarkson's inequality (cf. [2]) and Corollary 5.4, we see that $\{w'_n\}$ is a Cauchy sequence in $L_q(Y; r)$. Thus we can find $w' \in L_q(Y; r)$ such that $H_q(w'_n - w') \to 0$ as $n \to \infty$. On the other hand, it follows from [2; Lemma 4.3] that there exist $\Lambda \subset \mathbf{Q}_{\alpha,\beta}$ and a subsequence $\{w'_{n_k}\}$ of $\{w'_n\}$ such that $\mu_q(\mathbf{Q}_{\alpha,\beta} - \Lambda) = \infty$ and

$$\sum_Q |w'_{n_k}(y) - w'(y)| \longrightarrow 0 \text{ as } k \longrightarrow \infty \qquad \text{for all } Q \in \Lambda.$$

By [2; Lemma 4.2], $\mu_q(\Lambda) = \mu_q(\mathbf{Q}_{\alpha,\beta})$. Let $Q \in \Lambda$. Then there exists n_0 such that $Q \in \mathbf{Q}_{X_n^*, \bar{X}_n^*}$ for all $n \geq n_0$. By the above observation $\sum_Q |w'_n(y)| \geq 1$. Thus,

$$1 - \sum_Q |w'(y)| \leq \sum_Q |w'_{n_k}(y)| - \sum_Q |w'(y)|$$
$$\leq \sum_Q |w'_{n_k}(y) - w'(y)| \longrightarrow 0$$

as $k \to \infty$, so that $1 \leq \sum_Q |w'(y)|$, i.e., $|w'| \in E_q^*(\Lambda)$. Consequently,

$$\mu_q(\mathbf{Q}_{\alpha,\beta})^{-1} = \mu_q(\Lambda)^{-1} \leq H_q(w') = \lim_{n\to\infty} H_q(w'_n) = \lim_{n\to\infty} \mu_q(\mathbf{Q}_{X_n^*, \bar{X}_n^*})^{-1}.$$

This completes the proof.

COROLLARY 5.6. *$\mu_q(\mathbf{Q}_{X_m^*, \bar{X}_n^*}) \to \mu_q(\mathbf{Q}_{\alpha,\beta})$ as $m \to \infty$ and $n \to \infty$.*

By [2; Theorem 4.1 and Corollary 4.1], we have

(5.1) $\quad \mu_q(\mathbf{Q}_{b(X_m^*), \bar{X}_n^*}) \longrightarrow \mu_q(\mathbf{Q}_{b(X_m^*),\beta})$ as $n \longrightarrow \infty$;

(5.2) $\quad [\lambda_p(\mathbf{P}_{b(X_m^*),\beta})]^{1/p} [\mu_q(\mathbf{Q}_{b(X_m^*),\beta})]^{1/q} = 1.$

Combining Theorems 4.2 and 5.5 with (5.1) and (5.2), we obtain

THEOREM 5.7. $\quad [\lambda_p(\mathbf{P}_{\alpha,\beta})]^{1/p} [\mu_q(\mathbf{Q}_{\alpha,\beta})]^{1/q} = 1.$

References

[1] T. Kayano and M. Yamasaki, Boundary limit of discrete Dirichlet potentials, Hiroshima Math. J. **14** (1984), 401–406.

[2] A. Murakami and M. Yamasaki, Extremal problems with respect to ideal boundary components of an infinite network, ibid. **19** (1989), 77–87.

[3] T. Nakamura and M. Yamasaki, Generalized extremal length of an infinite network, ibid. **6** (1976), 95–111.