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Projectivity of Homogeneous Left Loops on Lie Groups II

(Local Theory)

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Let G be a Lie group. An analytic local multiplication μ at the identity element 1 of G is associated with a Lie algebra on the tangent space of G at 1. It is shown that μ is a geodesic homogeneous local left loop which is in projective relation with the group multiplication μ^0 (Theorems 2.1 and 2.16), and that any geodesic homogeneous local left loop in projective relation with μ^0 is given by such a μ (Theorem 3.3).

Introduction

The theory of analytic local loops was originated by A. I. Mal'cev [15] in 1955. He proved that any analytic Moufang loop is characterized by its tangent algebra which is called Mal'cev algebra. In 1964 the author introduced ([5]) local loops on any linearly connected manifolds, which are called now geodesic local loops, by means of parallel displacement of geodesic curves along geodesic curves. This concept was introduced, independently, by L. V. Sabinin [17] and developed by him since 1972 (cf. [17], [18], [19]). In [2], M. A. Akivis introduced the tangent algebra of any analytic local loop as an algebraic system with a bilinear product and a trilinear product satisfying some algebraic relations (cf. [1], [2], [3], [18]), which has been named Akivis algebra ([4]). However, the tangent Akivis algebra does not characterize the local loop in general.

On the other hand, generalizing the concept of Lie triple system of E. Cartan which is an algebraic system characterizing Riemannian symmetric space in local, K. Yamaguti [21] introduced in 1958 an algebraic system called general Lie triple system. It is also defined by a bilinear product and a trilinear product satisfying those relations which are presented by K. Nomizu [16] as relations of torsion tensor and curvature tensor of the canonical connection on reductive homogeneous spaces. In 1975, the concept of homogeneous Lie loops was introduced by the author [6] (cf. L. Sabinin [18]). It is a class of loops on differentiable manifolds characterized by their tangent algebras, called the tangent Lie triple algebras. The latter is a general Lie triple system on the tangent space at the identity which is given by the value of the torsion tensor and the curvature tensor of the canonical connection evaluated at the identity (cf. Definition 1.3). In 1986, K. H. Hofmann and K. Strambach [4] clarified the interrelation between the tangent Akivis algebra and the tangent Lie triple algebra of any homogeneous Lie loop. The theory of homogeneous Lie loops is a complete generalization of the theory of (real) Lie groups and Lie algebras to loop theory (cf. [6], [7]). Although this theory treats of global loop multiplications on manifolds, it was motivated by the local theory of geodesic loops in linearly connected manifolds. In this paper, we shall turn back to the theory of local loops and consider projective changes of group multiplication on any Lie group into homogeneous local loops, in a neighborhood of the identity element. Here, the word "projective" means that the system of geodesic curves (regarded as straight lines) is preserved. The changes are also restricted to preserve "homogeneity" of multiplications (cf. Definition 1.5).

After preparing some terminology and results for local left loops in §1, we introduce in §2 an analytic local left loop μ on a Lie group G. Each local multiplication μ is associated with some Lie algebra $\mathfrak{L}$ on the tangent space $T_0(G)$ at the identity 1 (cf. (2.2)). It is shown that the local multiplication μ is a geodesic homogeneous local left loop (Theorem 2.1), and that it is in projective relation with the group multiplication μ^0 of G (Theorem 2.16). In §3, it is proved that any geodesic homogeneous local left loop on a Lie group which is in projective relation with the group multiplication is reduced to the multiplication μ given above (Theorem 3.3). Finally, in §4, we shall remark that the construction of μ has an algebraic interpretation, by comparing the results developed in Part I ([14]).

§1. Homogeneous local left loops

In this section, we look at the theory of analytic homogeneous left loops developed in [6], [10], [11], [12] and [13] again from local point of view.

Let G be an analytic manifold of dimension n. For some fixed element 1 in G, we consider a local multiplication around 1.

DEFINITION 1.1. Let W be an open neighborhood of (1, 1) in $G \times G$. An analytic mapping

$$\mu: W \longrightarrow G$$

will be called a *homogeneous local left loop* at 1 if it satisfies the followings:

(i) $\mu(1, x) = x$, $\mu(x, 1) = x$ whenever $(1, x)$(resp. $(x, 1)$) belongs to W.

(ii) If $(x, 1)$ belongs to W, then there exists an open subset V of G containing 1, such that the left translation $L_x: V \rightarrow G$; $L_x y = \mu(x, y)$ is a diffeomorphism onto an open subset of G containing 1.

(iii) The multiplication μ has the *left inverse property*, that is, $L_x^{-1} = L_{x^{-1}}$ holds for $x^{-1} = L_x^{-1} 1$, and the map

$$L^{-1}: (x, y) \longmapsto L_x^{-1} y$$

is analytic in its domain.

(iv) Any left inner mapping $L_{x,y}$ satisfies

$$\mu(L_{x,y}z,\ L_{x,y}w) = L_{x,y}\mu(z,\ w)$$

whenever the left and the right sides of the equality are defined. Here, we call the map $L_{x,y}\colon z \mapsto L^{-1}_{\mu(x,y)}L_xL_y$ a *left inner mapping* if it is well-defined.

In the following, we denote by $T_0(G)$ the tangent space of G at the identity element 1. Since any left inner mapping $L_{x,y}$ is a local diffeomorphism leaving the identity 1 fixed, its differential $dL_{x,y}$ is an invertible linear endomorphism of $T_0(G)$.

Let $\mu\colon W \to G$ be a homogeneous local left loop in G at the identity 1, where W is an open submanifold of $G \times G$ containing $(1, 1)$. In any connected open submanifold U of G such that $1 \in U$ and $U \times U$ is contained in W, we can give a linear connection ∇_U associated with μ. In fact, for any vector fields X and Y on U, set

$$(1.1) \qquad (\nabla_X Y)_x = X_x Y - dL_x d\mu(dL_x^{-1} X_x,\ dL_x^{-1} Y_x)$$

at each point x in U, where $d\mu\colon T_0(G) \times T_0(G) \to T_0(G)$ is the bilinear map whose value $d\mu(X_0, Y_0)$ for X_0 and Y_0 in $T_0(G)$ is given by differentiating $\mu(u, v)$ in the direction (X_0, Y_0) at $(u, v) = (1, 1)$. Then, the vector field $\nabla_X Y$ on U satisfies the conditions for covariant differentiation, which defines a linear connection ∇_U on U. Moreover, if V is another open submanifold such that $(1, 1) \in V \times V$ in W, the associated linear connection ∇_V coincides with ∇_U on their common domain.

DEFINITION 1.2. The linear connection associated with μ which is given on any open submanifold U with $(1, 1) \in U \times U$ in W will be called the *canonical connection* of the homogeneous local left loop μ at 1.

REMARK. It is evident that if (G, μ) is a homogeneous Lie loop, then it is a homogeneous local left loop at its identity and the canonical connection defined above is reduced to the global one in the sense of [6] and [10].

We can see that the extensive theory of the canonical connection of homogeneous Lie loops and left loops developed in the articles [6]–[12] is still valid in analogous way for homogeneous *local* left loops, and that the theory of tangent Lie triple algebras of homogeneous left loops is available for local ones. For instance, if μ is a homogeneous local left loop at 1, the torsion tensor field S and the curvature tensor field R of the canonical connection ∇ are defined in some neighborhood of 1 and they satisfy $\nabla S = 0$ and $\nabla R = 0$, respectively, whose values S_0 and R_0 evaluated at 1 are given by

$$(1.2) \qquad S_0(X,\ Y) = d\mu(X,\ Y) - d\mu(Y,\ X),$$

$$(1.3) \qquad R_0(X,\ Y) = dL(X,\ Y) - dL(Y,\ X)$$

for any X and Y in $T_0(G)$ (cf. [11]). Here $dL(X, Y)$ denotes the endomorphism of $T_0(G)$ obtained by differentiating the linear endomorphism $dL_{u,v}$ of $T_0(G)$ in the directions (X, Y) at $(u, v) = (1, 1)$.

PROPOSITION 1.1. *The tangent space $T_0(G)$ at 1 forms a Lie triple algebra (general Lie triple system of* K. Yamaguti [21]) *with the operations;*

$$\langle X, Y\rangle = S_0(X, Y),$$
$$\langle X, Y, Z\rangle = R_0(X, Y)Z$$

for X, Y and Z in $T_0(G)$.

PROOF. It is obvious because $\nabla S=0$ and $\nabla R=0$ hold in a neighbourhood of 1. q.e.d.

DEFINITION 1.3. The Lie triple algebra obtained in Proposition 1.1 above will be called the *tangent Lie triple algebra* of the homogeneous local left loop μ at 1.

DEFINITION 1.4. (cf. [6]) A homogeneous local left loop μ at the identity 1 in G is said to be *geodesic* if, for each geodesic curve $x(t)$ of the canonical connection passing through $x(0) = 1$, the differential $dL_{x(t)}$: $T_0(G) \to T_{x(t)}(G)$ of the left translation $L_{x(t)}$ induces the parallel displacement along the geodesic curve.

It is easy to show;

PROPOSITION 1.2. *Every geodesic curve $x(t)$ through the identity $x(0) = 1$ in the geodesic homogeneous local left loop μ is a 1-parameter local subgroup* (i.e., *associative local subloop*) *of μ, that is,*

$$x(t+s) = \mu(x(t), x(s))$$

as far as both sides are well-defined.

In the same way as Theorem 7.8 in [6], we can show;

THEOREM 1.3. *Let μ (resp. $\tilde{\mu}$) be a geodesic homogeneous local left loop at 1 (resp. $\tilde{1}$) in an analytic manifold G (resp. $\tilde{G}$) and assume that* $\dim G = \dim \tilde{G}$. *Then, there exists a local isomorphism $\phi: U \to \tilde{U}$ of μ to $\tilde{\mu}$ if and only if there exists an automorphism Φ: $T_0(G) \to T_0(\tilde{G})$ of the tangent Lie triple algebras of μ and $\tilde{\mu}$ such that Φ is equal to the differential $d\phi$ of ϕ at the identity* 1.

In the theory of homogeneous (left) loops, it is very useful to associate the homogeneous system η with each homogeneous loop μ (cf. [8]) by

$$\eta(x, y, z) = L_x\mu(L_x^{-1}y, L_x^{-1}z). \tag{1.4}$$

For any homogeneous local left loop μ in this paper, we will consider the same operation η given by (1.4) as far as the right hand side is well-defined, and it will be called the *homogeneous system* associated with the *local* left loop μ. The fundamental equalities for homogeneous system hold also for this η in a neighborhood of 1 (cf. [9], [10]).

In [13] we have introduced the concept of projective relation in geodesic homogeneous left loops, and investigated how the relation holds on $\mathbf{R}^n$. Now, we consider two geodesic homogeneous local left loops μ and $\tilde{\mu}$ in the same analytic manifold G at the same identity 1. In the following, their canonical connections will be denoted by ∇ and $\tilde{\nabla}$, respectively.

DEFINITION 1.5. Two geodesic homogeneous left loops μ and $\tilde{\mu}$ at the same identity 1 are said to be *in projective relation* if they satisfy the following conditions (i) and (ii) in some neighborhood U of 1:

(i) Any geodesic curve of ∇ is a geodesic curve of $\tilde{\nabla}$, and vice versa.

(ii) The following mutual equalities are valid in U;

$$\tilde{\eta}(x, y, \eta(u, v, w)) = \eta(\tilde{\eta}(x, y, u), \tilde{\eta}(x, y, v), \tilde{\eta}(x, y, w)) \tag{1.5}$$

$$\eta(u, v, \tilde{\eta}(x, y, z)) = \tilde{\eta}(\eta(u, v, x), \eta(u, v, y), \eta(u, v, z)). \tag{1.6}$$

§ 2. Construction of homogeneous local left loops on Lie groups

Let G be a real Lie group with the group multiplication $xy = \mu^0(x, y)$. Then, μ^0 can be regarded as a homogeneous Lie loop. It is geodesic since the canonical connection is reduced to the $(-)$-connection of E. Cartan, whose curvature tensor R^0 vanishes identically while the tosion tensor S^0 gives the Lie bracket of the Lie algebra $\mathfrak{G}$ of G. In the following we denote the Lie bracket of $\mathfrak{G}$ by $[\,,]^0$. Then, we have

$$S^0(X, Y) = [X, Y]^0,$$
$$R^0(X, Y)Z = 0 \qquad \text{for any } X, Y, Z \text{ in } \mathfrak{G}.$$

Thus, the tangent Lie triple algebra of μ^0 is reduced to the Lie algebra $\mathfrak{G}$ of G.

In this section, we construct in G another homogeneous local left loop μ at the identity element 1 of G, and show that μ is in projective relation with μ^0. Here, we regard $\mathfrak{G}$ as a Lie algebra on the tangent space $T_0(G)$ at 1 with the Lie bracket $[\,,]^0$, that is, $\mathfrak{G} = (T_0(G), [\,,]^0)$. Following the notation in Varadarajan [20] (cf. Th. 2.15.4, p. 119) we denote the exponential map at 1 by $\exp\colon T_0(G) \to G$ and the product of $\exp X$ and $\exp Y$ in G by

$$\exp X \exp Y = \exp C(X : Y)$$

for any X and Y contained in some neighborhood $\mathfrak{A}$ of O in $T_0(G)$ so that $C\colon \mathfrak{A} \times \mathfrak{A} \to T_0(G)$ is an analytic map.

Now, assume that there is given another Lie algebra $\mathfrak{L}$ on $T_0(G)$ whose Lie bracket, denoted by the usual bracket $[\,,]$, satisfies a relation to $\mathfrak{G}$;

(2.1) $$[X, [Y, Z]^0] = [[X, Y], Z]^0 + [Y, [X, Z]]^0,$$

that is to say, every adjoint operation $\mathrm{ad}_{\mathfrak{L}}X$ of $\mathfrak{L}$ is a derivation of $\mathfrak{G}$. We denote the exponential of the endomorphism $\mathrm{ad}_{\mathfrak{L}}X$ of $T_0(G)$ by

$$A(X) = e^{\mathrm{ad}_{\mathfrak{L}}X}, \quad X \in T_0(G).$$

For the later use we show here some lemmas on $A(X)$ and $C(X: Y)$ above. In these lemmas all letters X, Y, Z denote arbitrary elements of $T_0(G)$.

LEMMA 1. *$A(X)$ is an automorphism of the Lie algebra $\mathfrak{G}$ and $A(X)^{-1} = A(-X)$.*

LEMMA 2. *$A(tX)sX = sX$ for $s, t \in \mathbf{R}$, $A(X)X = X$.*

LEMMA 3. *$A(A(X)Y) = A(X)A(Y)A(X)^{-1}$.*

LEMMA 4. $C(sX: tX) = (s+t)X$, $C(X: O) = C(O: X) = X$,

$$C(-X: -Y) = -C(Y: X),$$
$$C(X: C(Y: Z)) = C(C(X: Y): Z),$$
$$C(-Y: Z) = C(-C(X: Y): C(X: Z)).$$

LEMMA 5. $A(X)C(Y: Z) = C(A(X)Y: A(X)Z)$,

$$C(X: A(X)Y) = A(X)C(X: Y).$$

For the proof of Lemma 5 we use the following formula (Baker-Campbell-Hausdorff formula in Varadarajan [20] Lemma 2.15.3, p. 118):

$$C(X: Y) = \sum_{n=1}^{\infty} C_n(X: Y),$$

where

$$C_1(X: Y) = X + Y,$$
$$(n+1)C_{n+1}(X: Y) = \tfrac{1}{2}[X - Y, C_n(X: Y)]^0$$
$$+ \sum_{p \geqq 1, 2p \leqq n} K_{2p} \sum_{\substack{k_1, \dots, k_p > 0 \\ k_1 + \cdots + k_p = n}} [C_{k_1}(X: Y), [\cdots [C_{k_{2p}}(X: Y), X + Y]^0 \cdots]^0$$

and K_{2p}'s are rational numbers. The proofs of the other lemmas are omitted.

We associate with the Lie algebra $\mathfrak{L} = (T_0(G), [\,,])$ an analytic multiplication μ around the identity element 1 of the Lie group G as follows: For any normal neighborhood U of 1, set

(2.2) $$\mu(x, y) = \exp C(X: A(X)Y) \qquad \text{for } x = \exp X,\ y = \exp Y,$$

whenever x, y, $\exp A(X)Y$ and $\exp X \exp A(X)Y$ belong to U. Since the exponential map is an analytic diffeomorphism and since $A(X)Y$ and $C(X: Y)$ are analytic in X

and Y, we see that the multiplication μ is analytic in a neighborhood W of $(1, 1)$ in $G \times G$. It is clear that μ does not depend on the choice of the normal neighborhood U.

Hereafter, all equalities on μ should be understood to be assumed that both sides of the equalities are well-defined.

THEOREM 2.1. *Let G be a Lie group with the identity element* 1. *Assume that a Lie algebra $\mathfrak{L}$ on the tangent space $T_0(G)$ at* 1 *is given and satisfies the relation* (2.1). *Then, the analytic local multiplication μ around* 1, *given by* (2.2), *is a geodesic homogeneous local left loop at* 1.

To prove this theorem we show the following propositions:

PROPOSITION 2.2. *The identity* 1 *of G is a* (*two-sided*) *identity of the multiplication μ, that is,*

$$\mu(x, 1) = \mu(1, x) = x \qquad \text{for } x = \exp X.$$

PROOF. This is trivial by setting $X = O$ or $Y = O$ in (2.2). q.e.d.

PROPOSITION 2.3. *Every* 1-*parameter subgroup $x(t) = \exp tX (t \in \mathbf{R})$ of G is a* 1-*parameter subgroup of μ too, that is,*

$$\mu(x(t), x(s)) = x(t + s) \qquad \text{for } s, t \in \mathbf{R}.$$

PROOF. Lemma 2 shows this immediately. q.e.d.

COROLLARY 2.4. *The element $x^{-1} = \exp(-X)$ is the* (*two-sided*) *inverse of $x = \exp X$, with respect to μ.*

PROPOSITION 2.5. *The multiplication μ has the left inverse property, that is,*

$$\mu(x^{-1}, \mu(x, y)) = y \qquad \text{for } x = \exp X,\ y = \exp Y.$$

PROOF. By applying Lemmas 5, 2 and 4 to the left-hand side of the equation above, we have

$$\begin{aligned}
\mu(\exp(-X), \mu(\exp X, \exp Y)) &= \mu(\exp(-X), \exp C(X : A(X)Y)) \\
&= \exp C(-X : A(-X)C(X : A(X)Y)) \\
&= \exp C(-X : C(A(-X)X : A(-X)A(X)Y)) \\
&= \exp C(C(-X : X) : Y) \\
&= \exp Y.
\end{aligned}$$

q.e.d.

PROPOSITION 2.6. *The left inner mapping for $x = \exp X$, $y = \exp Y$ is given by*

$$L_{x,y}z = \exp A(X)A(-C(X : Y))A(Y)Z \qquad \text{for } z = \exp Z.$$

PROOF. From the definition of μ and Lemmas above we obtain;

$$\begin{aligned}\exp^{-1}(L^{-1}_{\mu(x,y)}L_xL_yz) &= C(-C(X: A(X)Y): A(-C(X: A(X)Y))C(X: A(X)\\ &\quad C(Y: A(Y)Z)))\\ &= A(X)C(-C(X: Y): A(-C(X: Y)C(X: C(Y: A(Y)Z)))\\ &= A(X)A(-C(X: Y))C(-C(X: Y): C(C(X: Y): A(Y)Z))\\ &= A(X)A(-C(X: Y))A(Y)Z.\end{aligned}$$

q.e.d.

COROLLARY 2.7. *$L_{x(t)}L_{x(s)} = L_{x(t+s)}$ for $x(t) = \exp tX$.*

PROOF. Proposition 2.6 for $x(t)$ and $x(s)$ shows $L_{x(t),x(s)} = \mathrm{id}$ since $A(sX)A(-C(sX: tX))A(tX) = \mathrm{I}$ (The identity map on $T_0(G)$). q.e.d.

PROPOSITION 2.8. *$L_{x,y}\mu(z, w) = \mu(L_{x,y}z, L_{x,y}w)$ for $x = \exp X$, $y = \exp Y$, $z = \exp Z$ and $w = \exp W$.*

PROOF. Set $L(X, Y) = A(X)A(-C(X: Y))A(Y)$. Then we get $\exp^{-1}L_{x,y}\mu(z, w) = L(X, Y)C(Z: A(Z)W) = C(L(X, Y)Z: L(X, Y)A(Z)W)$ by Proposition 2.6. On the other hand, by Lemma 3, we have

$$L(X, Y)A(Z) = A(L(X, Y)Z)L(X, Y),$$

which shows

$$\begin{aligned}L_{x,y}\mu(z, w) &= \exp C(L(X, Y)Z: A(L(X, Y)Z)L(X, Y)W)\\ &= \mu(\exp L(X, Y)Z, \exp L(X, Y)W)\\ &= \mu(L_{x,y}z, L_{x,y}w).\end{aligned}$$

q.e.d.

Summing up Propositions 2.2, 2.3, 2.5 and 2.8, we see that the local multiplication μ is a homogeneous local left loop at 1.

PROPOSITION 2.9. *The homogeneous local left loop μ is geodesic.*

PROOF. Choose a normal coordinate system of the Lie group G at 1, so that $x = \exp X$ has its i-th coordinate $x^i = X^i$ for any $X = X^j\partial_j^0$ in a neighborhood of O in $T_0(G)$, where $\{\partial_j^0\}$ is the natural basis of $T_0(G)$ with respect to this coordinate system. Then, by Proposition 2.2, the 1-parameter subgroup $x(t) = \exp tX$, $t \in \mathbf{R}$, satisfies $\dot{x}(t) = X$, $\ddot{X}(t) = O$ and

$$\begin{aligned}\eta(x(t), X_{x(t)}, X_{x(t)}) &= dL_{x(t)}d\mu(dL_{x(-t)}X_{x(t)}, dL_{x(-t)}X_{x(t)})\\ &= \frac{\partial^2}{\partial u\partial v}\Big|_{(0,0)}x(u+v-t)\\ &= O.\end{aligned}$$

Hence every 1-parameter subgroup $x(t) = \exp tX$, $t \in \mathbf{R}$, is a geodesic curve of the canonical connection ∇. Let $Y(t) = dL_{x(t)}Y$ be a vector field along the geodesic curve $x(t)$ given by left translations of any $Y = Y(0)$ in $T_0(G)$. Then we have

$$\begin{aligned}\eta(x(t), x(t+u), Y(t)) &= L_{x(t)}\mu(x(u), Y)\\ &= dL_{x(t)}dL_{x(u)}Y\\ &= Y(t+u)\end{aligned}$$

by Corollary 2.7. Hence we get

$$\nabla_{\dot{x}(t)}Y(t) = \frac{dY}{dt} - \eta(x(t), \dot{x}(t), Y(t)) = O$$

along the geodesic curve $x(t)$, that is, the left translation $L_{x(t)}$ induces the parallel displacement of Y along the geodesic curve $x(t)$ with respect to the canonical connection ∇ of μ. q.e.d.

Since μ is homogeneous, we can show

$$dL_p\frac{dY}{dt} = \eta(p, dL_p\dot{x}(t), dL_pY)$$

for any point p. This means

COROLLARY 2.10. *The parallel displacement of tangent vectors at* 1 *along the geodesic curve* $x(t) = \exp tX$ *is preserved by any left translation* L_p.

The proof of Theorem 2.1 is completed by Proposition 2.9 above.

REMARK. In [5] we have introduced a local multiplication in any differentiable manifold with a linear connection, which is called a *geodesic local loop*. Proposition 2.9 above shows that the homogeneous local left loop μ considered here is a geodesic local loop at 1 with respect to its canonical connection ∇. Moreover, Corollary 2.10 shows that the local multiplication μ_p at p given by

$$\mu_p(x, y) = \eta(p, x, y)$$

is a geodesic local loop at p with respect to ∇.

Now, we are at the stage of showing that the geodesic homogeneous local left loop μ at 1 is in projective relation with the group multiplication μ^0 of the Lie group G. We denote the group multiplication μ^0 by juxtaposition as usual, that is, $\mu^0(x, y) = xy$. The homogeneous system η^0 associated with μ^0 is given by

$$\eta^0(x, y, z) = yx^{-1}z.$$

On the other hand, the homogeneous system η associated with the homogeneous local left loop μ is given by (1.4) in a neighborhood of 1, that is,

$$\eta(x, y, z) = \mu(x, \mu(\mu(x^{-1}, y), \mu(x^{-1}, z))). \tag{2.3}$$

PROPOSITION 2.11. *The homogeneous system η is given by*

$$\eta(x, y, z) = x\mu(x^{-1}y, x^{-1}z).$$

PROOF. For $x = \exp X$, $y = \exp Y$ and $z = \exp Z$, (2.3) implies

$$\begin{aligned}\exp^{-1}\mu(x, y, z) &= C(X: A(X)A(A(-X)C(-X: Y))\\ &\qquad C(A(-X)C(-X: Y): A(-X)C(-X: Z)))\\ &= C(X: A(C(-X: Y))C(C(-X: Y)C(-X: Z)).\end{aligned}$$

Hence, we get

$$\begin{aligned}\eta(x, y, z) &= \exp X\, \mu(\exp(-X)\exp Y, \exp(-X)\exp Z)\\ &= x\mu(x^{-1}y, x^{-1}z).\end{aligned}$$

q.e.d.

PROPOSITION 2.12.

$$\eta^0(u, v, \eta(x, y, z)) = \eta(\eta^0(u, v, x), \eta^0(u, v, y), \eta^0(u, v, z))$$

PROOF. If we set $w = vu^{-1}$, then it is sufficient to show that the equation

$$w\eta(x, y, z) = \eta(wx, wy, wz) \tag{2.4}$$

holds in a neighborhood of 1. By Proposition 2.11 above, we have

$$\begin{aligned}\eta(wx, wy, wz) &= wx\mu((wx)^{-1}(wy), (wx)^{-1}(wz))\\ &= wx\mu(x^{-1}y, x^{-1}z)\\ &= w\mu(x, y, z).\end{aligned}$$

q.e.d.

PROPOSITION 2.13.

$$\mu(x, \eta^0(u, v, w)) = \eta^0(\mu(x, u), \mu(x, v), \mu(x, w))$$

PROOF. For $x = \exp X$, $u = \exp U$, $v = \exp V$ and $w = \exp W$, we have by using Lemma 4,

$$\begin{aligned}\exp^{-1}\mu(x, \eta^0(u, v, w)) &= C(X: A(X)C(V: C(-U: W)))\\ &= C(X: C(A(X)V: C(-A(X)U: A(X)W)))\\ &= C(C(X: A(X)V): C(-A(X)U: A(X)W))\\ &= C(C(X: A(X)V): C(-C(X: A(X)U): C(X: A(X)W)))\\ &= \exp^{-1}\mu(x, v)\mu(x, u)^{-1}\mu(x, w)\end{aligned}$$

$$= \exp^{-1}\eta^0(\mu(x, u), \mu(x, v), \mu(x, w)) \qquad \text{q.e.d.}$$

PROPOSITION 2.14.

$$\eta(x, y, \eta^0(u, v, w)) = \eta^0(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$$

PROOF. By Proposition 2.11 we have

$$\begin{aligned} \eta(x, y, \eta^0(u, v, w)) &= x\mu(x^{-1}y, x^{-1}\eta^0(u, v, w)) \\ &= x\mu(x^{-1}y, \eta^0(x^{-1}u, x^{-1}v, x^{-1}w)), \end{aligned}$$

and, by Proposition 2.13,

$$\begin{aligned} &\eta^0(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)) \\ &= \eta^0(x\mu(x^{-1}y, x^{-1}u), x\mu(x^{-1}y, x^{-1}v), x\mu(x^{-1}y, x^{-1}w)) \\ &= x\eta^0(\mu(x^{-1}y, x^{-1}u), \mu(x^{-1}y, x^{-1}v), \mu(x^{-1}y, x^{-1}w)) \\ &= x\mu(x^{-1}y, \eta^0(x^{-1}u, x^{-1}v, x^{-1}w)). \end{aligned} \qquad \text{q.e.d.}$$

PROPOSITION 2.15. *Any geodesic of μ through* 1 *is a geodesic of μ^0, and vice versa.*

PROOF. By Proposition 2.3 and the proof of Proposition 2.11, we see that all 1-parameter subgroups are the system of geodesic curves through the identity 1, with respect to both of the canonical connections of μ and μ^0, respectively. q.e.d.

From Propositions 2.12, 2.14 and 2.15 we obtain the following;

THEOREM 2.16. *Let G be a Lie group with the Lie algebra $\mathfrak{G}$. Assume that a Lie algebra $\mathfrak{L}$ is given on the tangent space $T_0(G)$ at the identity* 1, *such that the relation* (2.1) *is satisfied. Then, the geodesic homogeneous local left loop μ at* 1, *given by* (2.2), *is in projective relation with the group multiplication of the Lie group G.*

§3. Projectivity of Lie groups as homogeneous loops

Let G be a Lie group and $\mathfrak{G} = (T_0(G), [\,,]^0)$ its Lie algebra, where $T_0(G)$ denotes the tangent space at the identity 1. Assume that there exists a Lie algebra $\mathfrak{L} = (T_0(G), [\,,])$ on $T_0(G)$ satisfying (2.1), that is,

$$\mathrm{ad}_{\mathfrak{L}}X[Y, Z]^0 = [\mathrm{ad}_{\mathfrak{L}}X\,Y, Z]^0 + [Y, \mathrm{ad}_{\mathfrak{L}}X\,Z]^0$$

for X, Y, Z in $T_0(G)$. In §2, we have seen that the local multiplication μ given by (2.2) is a geodesic homogeneous local left loop at 1 (Theorem 2.1) and that μ is in projective relation with the group multiplication of G (Theorem 2.16). In this section, we will show that any geodesic homogeneous local left loop at 1 which is in projective relation with the group multiplication must be given by (2.2) associated with some Lie algebra $\mathfrak{L}$.

Choose a normal coordinate neighborhood at 1 in G, so that any element $x = \exp X$ in it has the coordinate $x^i = X^i$, with respect to the natural basis. By Baker-Campbell-Hausdorff formula in Lemma 5, we have

$$\mu^0(\exp X, \exp Y)^i = C^i(X: Y) = X^i + Y^i + (\tfrac{1}{2}[X, Y]^0)^i + \cdots.$$

Since the local multiplication μ is given by (2.2), we have

$$\begin{aligned}\mu^i(\exp sX, \exp tY) &= C^i(sX: A(sX)tY)\\ &= sX^i + t(A(sX)Y)^i + \tfrac{1}{2}st([X, A(sX)Y]^0)^i + \cdots.\\ &= sX^i + tY^i + st[X, Y]^i + \tfrac{1}{2}st([X, Y]^0)^i + O_3(s, t),\end{aligned}$$

from which the value of the bilinear map $d\mu: T_0(G) \times T_0(G) \to T_0(G)$ follows;

$$d\mu(X, Y) = \frac{\partial^2}{\partial s \partial t}|_{(0,0)}\mu(x(s), y(t)) = [X, Y] + \tfrac{1}{2}[X, Y]^0. \tag{3.1}$$

On the other hand, Proposition 2.6 implies, for $x(s) = \exp sX$ and $y(t) = \exp tY$,

$$\begin{aligned}dL_{x(s),y(t)} &= A(sX)A(-C(sX: tY))A(tY)\\ &= \{I + s\,\mathrm{ad}_{\mathfrak{L}}X + ..\}\left\{I - \mathrm{ad}_{\mathfrak{L}}\left(sX + tY + \frac{st}{2}[X, Y]^0\right) + ..\right\}\{I + t\,\mathrm{ad}_{\mathfrak{L}}Y + \cdots\}\\ &= I - st(\tfrac{1}{2}\mathrm{ad}_{\mathfrak{L}}[X, Y]^0 + \mathrm{ad}_{\mathfrak{L}}X \cdot \mathrm{ad}_{\mathfrak{L}}Y) - s^2\mathrm{ad}_{\mathfrak{L}}X \cdot \mathrm{ad}_{\mathfrak{L}}X\\ &\quad - t^2\mathrm{ad}_{\mathfrak{L}}Y \cdot \mathrm{ad}_{\mathfrak{L}}Y + O_3(s, t).\end{aligned}$$

By differentiating this once in s and t, respectively, and evaluating at (0, 0) we get

$$dL(X, Y) = -\tfrac{1}{2}\mathrm{ad}_{\mathfrak{L}}[X, Y]^0 - \mathrm{ad}_{\mathfrak{L}}X \cdot \mathrm{ad}_{\mathfrak{L}}Y. \tag{3.2}$$

Let S and R denote the torsion tensor and the curvature tensor, respectively, of the canonical connection ∇ of μ. Then, by (1.2) and (1.3), we have;

PROPOSITION 3.1. *The value S_0 and R_0 of the torsion and the curvature of ∇ at* 1 *are given, respectively, by*

$$S_0(X, Y) = [X, Y]^0 + 2[X, Y], \tag{3.3}$$

$$R_0(X, Y)Z = -[[X, Y]^0, Z] - [[X, Y], Z], \tag{3.4}$$

where $[\,,]^0$ *and* $[\,,]$ *denote the Lie bracket of* $\mathfrak{G}$ *and* $\mathfrak{L}$, *respectively.*

In [13], projectivity of geodesic homogeneous left loops is investigated and shown that if two geodesic homogeneous left loops μ and $\tilde{\mu}$ are in projective relation, then the (1, 2)-tensor fields $T = \nabla - \tilde{\nabla}$ and $-T = \tilde{\nabla} - \nabla$ are affine homogeneous structures of the canonical connections ∇ and $\tilde{\nabla}$, respectively (Proposition 1 in [13]). In this case, T satisfies $\nabla T = 0$, $\tilde{\nabla} T = 0$ and the following equations for the torsion S and the curvature R of ∇ (cf. Corollary to Proposition 1 in [13]);

(3.5) $T(X, X) = 0$

(3.6) $T(X, S(Y, Z)) = S(T(X, Y), Z) + S(Y, T(X, Z))$

(3.7) $T(X, R(Y, Z)W) = R(T(X, Y), Z)W + R(Y, T(X, Z))W + R(Y, Z)T(X, W)$

(3.8) $T(X, T(Y, Z)) = T(T(X, Y), Z) + T(Y, T(X, Z))$

(3.9) $R(X, Y)T(Z, W) = T(R(X, Y)Z, W) + T(Z, R(X, Y)W)$.

Moreover, it has been shown (cf. Proposition 1.1 in [12]) that the torsion tensor $\tilde{S}$ and the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given, respectively, by

(3.10) $\tilde{S}(X, Y) = S(X, Y) + 2T(X, Y)$,

(3.11) $\tilde{R}(X, Y)Z = R(X, Y)Z - T(S(X, Y), Z) - T(T(X, Y), Z)$.

If μ and $\tilde{\mu}$ are geodesic homogeneous local left loops at the same point, say 1, in an analytic manifold G, then we can apply those results mentioned above in a neighborhood of 1. Especially, we can assert that all equalities (3.5)–(3.11) above are valid at 1. From these facts the following theorem is obtained:

THEOREM 3.2. *Let G be a Lie group with the multiplication μ^0. Assume that, in a neighborhood of the identity* 1, *there is given a local multiplication $\tilde{\mu}$ which is a geodesic homogeneous local left loop at* 1. *If $\tilde{\mu}$ is in projective relation with μ^0, then, there exists a bilinear map T: $T_0(G) \times T_0(G) \to T_0(G)$ such that the values at* 1 *of the torsion $\tilde{S}$ and the curvature $\tilde{R}$ of the canonical connection $\tilde{\nabla}$ of $\tilde{\mu}$ are given respectively by*

(3.12) $\tilde{S}(X, Y) = [X, Y]^0 + 2T(X, Y)$,

(3.13) $\tilde{R}(X, Y)Z = -T([X, Y]^0) - T(T(X, Y), Z)$

for X, Y, Z in the tangent space $T_0(G)$ at 1, *where $[\,,]^0$ denotes the Lie bracket of the Lie algebra of G.*

Moreover, the bilinear map T satisfies the following equalities:

(3.14) $T(X, X) = 0$

(3.15) $T(X, T(Y, Z)) + T(Y, T(Z, X)) + T(Z, T(X, Y)) = O$

(3.16) $T(X, [Y, Z]^0) = [T(X, Y), Z]^0 + [Y, T(X, Z)]^0$.

PROOF. Since the canonical connection ∇^0 of the Lie group μ^0 is reduced to the $(-)$-connection of Cartan, its curvature vanishes identically and the torsion S^0 has its value at 1 as

$$S^0(X, Y) = [X, Y]^0 \qquad \text{for } X, Y \text{ in } T_0(G).$$

The (local) affine homogeneous structure $T = \nabla^0 - \nabla$ should satisfies the equations (3.5)–(3.9) for $S = S^0$ and $R = O$, which are reduced to (3.14)–(3.16). Then, (3.12) and (3.13) follow immediately from (3.10) and (3.11), respectively. q.e.d.

In conclusion of this section, we have the following;

THEOREM 3.3. *Let G be a Lie group with the multiplication μ^0 and the identity element* 1. *Any geodesic homogeneous local left loop $\tilde{\mu}$ at* 1 *in projective relation with μ^0 is given by the homogeneous local left loop constructed in* Theorem 2.1 *for some Lie algebra on the tangent space $T_0(G)$ of G at* 1.

PROOF. Apply Theorem 3.2 to $\tilde{\mu}$. Then, from (3.14) and (3.15), it follows that the bilinear operation T on $T_0(G)$ gives a Lie algebra $\mathfrak{L} = (T_0(G), [\,,])$ with the Lie bracket

$$[X, Y] = T(X, Y).$$

Moreover, the equation (3.16) assures the relation (2.1). Let μ be the geodesic homogeneous local left loop given by Theorem 2.1, associated with this Lie algebra $\mathfrak{L}$. By Proposition 3.1, the tangent Lie triple algebra $\{T_0(G); \langle\,,\rangle \langle\,,\,,\rangle\}$ of μ is given by

$$\langle X, Y\rangle = [X, Y]^0 + 2[X, Y],$$
$$\langle X, Y, Z\rangle = -[[X, Y]^0, Z] - [[X, Y], Z].$$

On the other hand, the equations (3.12) and (3.13) give the tangent Lie triple algebra $\{T_0(G); \langle\langle\,,\rangle\rangle, \langle\langle\,,\,,\rangle\rangle\}$ of $\tilde{\mu}$ by

$$\langle\langle X, Y\rangle\rangle = [X, Y]^0 + 2T(X, Y),$$
$$\langle\langle X, Y, Z\rangle\rangle = -T([X, Y]^0, Z) - T(T(X, Y), Z).$$

That is to say, the geodesic homogeneous local left loops μ and $\tilde{\mu}$ have the same tangent Lie triple algebra. Then, Theorem 1.3 implies that they are locally isomorphic. Since, in this case, the isomorphism Φ in Theorem 1.3 is reduced to the identity map on the tangent space $T_0(G)$, the corresponding local isomorphism (as an affine transformation) must be equal to the identity map in some neighborhood of 1, on which $\mu = \tilde{\mu}$ is valid. q.e.d.

§4. Final remarks

Let G be a Lie group with the multiplication μ^0, and μ a geodesic homogeneous local left loop at the identity which is given by Theorem 2.1, associated with a Lie algebra $\mathfrak{L}$ satisfying (2.1). We have shown in Proposition 2.3 that each 1-parameter subgroup $x(t) = \exp tX$ $(t \in \mathbf{R})$ of the Lie group G is also a 1-parameter subgroup of μ, that is, the local left loop μ is power associative. From a viewpoint of algebraic projectivity of homogeneous left loops introduced in Part I ([14]), we can conclude that μ^0 and μ are in projective relation too, in algebraic sense, as far as the homogeneous system η of μ is well-defined, that is;

PROPOSITION 4.1. *Set $d_x y = \exp A(X)Y$ for $x = \exp X$ and $y = \exp Y$. Then, any operation d_x is a (local) automorphism of the Lie group G and, d satisfies the followings* (cf. Errata for Part I on the last page of this paper):

(i) $d_1 y = y$, *where* 1 *denotes the identity of* G.

(ii) $d_{x(t)}x(s) = x(s)$ *for* $x(t) = \exp tX$, $t \in \mathbf{R}$.

(iii) $d_{x(t)}^{-1} = d_{x(-t)}$.

(iv) $d_x d_y = d_{d_x y} d_x$ *for* $x = \exp X$, $y = \exp Y$.

PROOF. (i) is clear. (ii) and (iii) are obtained from Lemmas 1 and 2 in §2. Also, Lemma 3 in §2 implies

$$A(X)A(Y) = A(A(X)Y)A(X),$$

which proves (iv). From Lemma 5 in §2, it follows that any d_x is a local automorphism of the group G, i.e.,

$$d_x(yz) = (d_x y)(d_x z) \tag{4.1}$$

holds for $x = \exp X$, $y = \exp Y$ and $z = \exp Z$. q.e.d.

The main theorem in Part I ([14]) asserts that any abstract homogeneous left loop (G, μ) which is algebraically in projective relation with the group (G, μ^0) is completely determined by those operations d's which satisfy (4.1) and the algebraic conditions (i)–(iv) in [14], the conditions corresponding to those in Proposition 4.1 above on whole G. In that case, the multiplication μ is given by

$$\mu(x, y) = \mu^0(x, d_x y)$$

which is coincident with (2.2) when $x = \exp X$ and $y = \exp Y$. Thus, we see that Theorem 3.3 means the analytic local version of the main theorem in Part I.

An example of global analytic homogeneous left loop on a Lie group which is in projective relation with the group has been given in [13] when the Lie group is the abelian group $\mathbf{R}^n$. Further examples of geodesic homogeneous local left loops on Lie groups would be presented elsewhere.

Errata for Part I

In the paper "Projectivity of homogeneous left loops on Lie groups I (Algebraic Framework), Mem. Fac. Sci., Shimane Univ. 23 (1989)" ([14]);

p. 19 ↓4 "$d_x x = x$" should read "$d_{x^m} x = x$ for any positive integer m".

p. 19 ↑3 "$\tilde{L}_{x,y} = d^{-1}_{\tilde{\mu}(x,y)} d_y$" should read "$\tilde{L}_{x,y} = d^{-1}_{\tilde{\mu}(x,y)} d_x d_y$".

p. 21 ↓1 "$x^{p+1} y\, y^{-p}$" should read "$x^{p+1} y\, x^{-p}$".

p. 21 ↑16 "$d_X X = X$" should read "$d_{mX} X = X$".

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