Mem. Fac. Sci. Shimane Univ.,
**24**, pp. 21–30 Dec. 24, 1990

# Stability and Oscillation in Delay Differential Equations with Piecewise Constant Arguments

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(Received September 5, 1990)

Asymptotic behaviors of solutions of delay differential equations with piecewise constant arguments are mainly discussed. In particular, we show stability of the zero solution by using Razumikhin-type method, and we generalize two oscillation theorems obtained by Aftabizadeh-Wiener-Xu in [1].

## §1. Introduction

Recently, many results have been obtained for delay differential equations with piecewise constant arguments concerning stability and oscillation of solutions and existence of periodic solutions (for instance, [1, 2, 4, 5] and references cited therein). Though both of linear and nonlinear equations are treated in those investigations, those for nonlinear equations are relatively few. In particular, the studies for nonlinear equations with general forms seem to be quite few.

In this paper, we mainly discuss asymptotic behaviors of solutions of delay differential equations with piecewise constant arguments. Particularly we study stability of solutions of an equation with a general form, and oscillation of solutions of linear equations. In §2, we discuss existence and uniqueness of solutions. In §3, we obtain a few results on stability of solutions by employing a Razumikhin-type method. Finally in §4, we discuss oscillation of solutions of a linear equation which is a generalization of the linear equation treated in [1], and we improve two oscillation theorems obtained by Aftabizadeh-Wiener-Xu in [1].

Let $R^n$ and $|\cdot|$ denote the $n$-dimensional Euclidean space and its norm respectively, and let $I$ denote the interval $0 \leqq t < \infty$. In particular, $R$ is used instead of $R^1$. For any $H$ $(0 < H \leqq \infty)$, let $B_H = \{x \in R^n : |x| < H\}$ and $D = \{(t, x, y_0, \ldots, y_m) : t \in I, x \in B_H, y_i \in B_H\ (0 \leqq i \leqq m)\}$, where $i$ and $m$ are nonnegative integers. For a function $f: D \to R^n$, consider the equation with piecewise constant arguments

$$\dot{x}(t) = f(t, x(t), x([t]), \ldots, x([t-m])), \tag{1}$$

where the superposed dot and $[\cdot]$ designate the derivative and the greatest integer function, respectively, and $f(\cdot, \cdot, y_0, \ldots, y_m): I \times B_H \to R^n$ is a continuous function for any fixed $y_i$ $(0 \leqq i \leqq m)$. Let $\eta = (y_0, \ldots, y_m)$ be an $(m+1)n$-vector.

## §2. Existence and uniqueness of solutions

The initial value problem (IVP in short) for Equation (1) is

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), x([t]),\dots,x([t-m])), \\ x(t_0) &= x_0,\ x([t_0]) = y_0,\dots,x([t_0-m])) = y_m, \end{aligned} \tag{2}$$

where $x_0 = y_0$ if $[t_0] = t_0$. For any $A > 0$, let $I(t_0, A) = \{[t_0 - m],\dots,[t_0]\} \cup [t_0, t_0 + A)$.

DEFINITION 1. A solution of IVP (2) on $[t_0, t_0 + A)$ is a function $x: I(t_0, A) \to B_H$ that satisfies $x(t_0) = x_0$, $x([t_0]) = y_0,\dots,x([t_0 - m]) = y_m$, and the conditions;

(i) $x(t)$ is continuous on $[t_0, t_0 + A)$,

(ii) $\dot{x}(t)$ exists at each point $t \in (t_0, t_0 + A)$, with the possible exception of the points $[t] \in (t_0, t_0 + A)$ where one-sided derivatives exist, and right-hand derivative exists at $t_0$,

(iii) $x(t)$ satisfies Equation (1) on $(t_0, t_0 + A)$, with the possible exception of the points $[t] \in (t_0, t_0 + A)$.

First we consider existence of solutions of IVP (2). Let $k_0 = [t_0] + 1$. On the interval $[t_0, k_0)$, IVP (2) is reduced to IVP

$$\begin{aligned} \dot{x} &= f(t, x, \eta), \\ x(t_0) &= x_0. \end{aligned} \tag{3}$$

Since IVP (3) is an initial value problem for an ordinary differential equation with parameters $\eta = (y_0,\dots,y_m)$, the continuity of $f(\cdot, \cdot, \eta)$ assures local existence of solutions of IVP (3) for $t \geqq t_0$, and consequently, solutions of IVP (2) exist locally for $t \geqq t_0$. We denote this solution by $x(t, t_0, x_0, \eta)$.

Next we consider uniqueness of solutions of Equation (1). If uniqueness of solutions of Equation (1) does not hold, then there exist solutions $x_1(t) = x_1(t, t_0, x_0, \eta)$ and $x_2(t) = x_2(t, t_0, x_0, \eta)$ of IVP (2) such that $x_1(t_1) \neq x_2(t_1)$ for some $t_1 > t_0$. Let $t_2 = \sup\{t: x_1(s) = x_2(s) \text{ for } t_0 \leqq s \leqq t\}$. Then $t_0 \leqq t_2 < t_1$, and $x_1(t_3) \neq x_2(t_3)$ for some $t_3 \in (k_1, k_1 + 1)$, where $k_1 = [t_2]$. Thus in order to consider uniqueness of solutions of Equation (1), it is sufficient to consider uniqueness of solutions on $[k, k + 1)$ with integral endpoints. Here we state a uniqueness theorem. Since the proof is similar to the one of Theorem 1.4 in [6], we omit the proof.

THEOREM 1. Suppose that $f(\cdot, \cdot, \eta)$ of (1) is continuous on $D_1: k \leqq t < k + 1$, $x \in B_H$ for any $y_i \in B_H$ $(0 \leqq i \leqq m)$. In order that every solution of Equation (1) through a point in $D_1$ is unique to the right, it is necessary and sufficient that for any $(\tau_0, \xi_0) \in D_1$ and any $y_i \in B_H$ $(0 \leqq i \leqq m)$, there exists a neighborhood $U_\eta$ of $(\tau_0, \xi_0)$

which has the following property: Let $W_\eta = \{(t, x, \xi): (t, x) \in U_\eta, (t, \xi) \in U_\eta\}$. Then there exists a real-valued continuous function $V^\eta(t, x, \xi)$ on $W_\eta$, which satisfies the conditions (i) $V^\eta(t, x, \xi) \equiv 0$ if $x = \xi$, (ii) $V^\eta(t, x, \xi) > 0$ if $x \neq \xi$, (iii) $V^\eta(t, x, \xi)$ satisfies locally a Lipschitz condition in $x$ and $\xi$, and

$$\dot{V}^\eta_{(1)}(t, x, \xi) = \limsup_{h \to 0+} \frac{1}{h}\{V(t + h, x + hf(t, x, \eta), \xi + hf(t, \xi, \eta)) - V(t, x, \xi)\} \leqq 0.$$

Example 1. If $f(t, x, \xi)$ of (1) satisfies a Lipschitz condition $|f(t, x, \eta) - f(t, \xi, \eta)| \leqq L(\eta)|x - \xi|$, the function $V^\eta(t, x, \xi) = e^{-2L(\eta)t}|x - \xi|^2$ satisfies

$$\dot{V}^\eta_{(1)}(t, x, \xi) \leqq e^{-2L(\eta)t}(-2L(\eta)|x - \xi|^2 + 2|x - \xi|\,|f(t, x, \eta) - f(t, \xi, \eta)|) \leqq 0.$$

Thus $V^\eta(t, x, \xi)$ satisfies the conditions in Theorem 1, and hence, every solution of Equation (1) is unique to the right.

## §3. Stability of solutions

Suppose that $f$ of (1) is identically 0 on $I$ when $x = 0$ and $\eta = (y_0, \dots, y_m) = 0$. Then IVP (2) with $x_0 = 0$ and $\eta = 0$ has the zero solution. A continuous function $V(t, x): J \times B_H \to I$ is called a Liapunov function if $V$ satisfies locally a Lipschitz condition in $x$, where $J = \{-m, \dots, -1\} \cup I$. The derivative $\dot{V}_{(1)}(t, x, \eta)$ is defined by

$$\dot{V}_{(1)}(t, x, \eta) = \limsup_{h \to 0+} \frac{1}{h}\{V(t + h, x + hf(t, x, \eta)) - V(t, x)\}.$$

Definition 2. The zero solution of Equation (1) is stable if for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x_0| < \delta$ and $|y_i| < \delta$ $(0 \leqq i \leqq m)$ imply $|x(t, t_0, x_0, \eta)| < \varepsilon$ for all $t \geqq t_0$.

Definition 3. The zero solution of Equation (1) is uniformly stable if the $\delta$ in Definition 2 is independent of $t_0$.

Definition 4. The zero solution of Equation (1) is uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_0 > 0$ such that for any $\varepsilon > 0$ and any $t_0 \in I$, there exists a $T = T(\varepsilon) > 0$ such that $|x_0| < \delta_0$ and $|y_i| < \delta_0$ $(0 \leqq i \leqq m)$ imply $|x(t, t_0, x_0, \eta)| < \varepsilon$ for all $t \geqq t_0 + T$.

Concerning stability of the zero solution of Equation (1), we obtain a few Razumikhin-type theorems. Though the proofs of them consist of standard arguments of the Razumikhin method (cf. [3, Chapter 5]), here we prove them for the sake of completeness.

Theorem 2. Suppose that there exists a Liapunov function $V(t, x)$ on $J \times B_H$

such that

(i) $V(t, 0) \equiv 0$,

(ii) $V(t, x) \geqq a(|x|)$, where $a(r)$ is continuous, increasing, and positive definite,

(iii) $\dot{V}_{(1)}(t, x, \eta) \leqq 0$ whenever $V(t, x) \geqq V([t-i], y_i)$ $(0 \leqq i \leqq m)$.

Then the zero solution of Equation (1) is stable.

PROOF. For any $\varepsilon \in (0, H)$, $V(t, x) \geqq a(\varepsilon)$ for $t \in J$ and $x$ with $|x| = \varepsilon$. For any $t_0 \in I$, choose a $\delta = \delta(t_0, \varepsilon) > 0$ such that $|x| < \delta$ implies $V(t_0, x) < a(\varepsilon)$ and $V([t_0 - i], x) < a(\varepsilon)$ $(0 \leqq i \leqq m)$. Suppose that a solution $x(t) = x(t, t_0, x_0, \eta)$ of Equation (1) satisfies $|x_0| < \delta$, $|y_i| < \delta$ $(0 \leqq i \leqq m)$, and $|x(t_1)| = \varepsilon$ for some $t_1 > t_0$. Let $S = \{[t_0 - i]: 0 \leqq i \leqq m\} \cup [t_0, t_1]$. From the choice of $x_0$ and $\eta$, there exists a $\tau \in (t_0, t_1)$ such that $V(\tau, x(\tau)) \geqq V(t, x(t))$ for all $t \in S$ with $t \leqq \tau$, and $\dot{V}_{(1)}(\tau, x(\tau), x([\tau]), \ldots, x([\tau - m])) > 0$. On the other hand, (iii) and the choice of $\tau$ imply $\dot{V}_{(1)}(\tau, x(\tau), x([\tau]), \ldots, x([\tau - m])) \leqq 0$, which is a contradiction. Thus the zero solution of Equation (1) is stable.

THEOREM 3. If the condition (ii) in Theorem 2 is replaced by

(ii)′ $a(|x|) \leqq V(t, x) \leqq b(|x|)$, where $a(r)$ and $b(r)$ are continuous, increasing, and positive definite. Then the zero solution of Equation (1) is uniformly stable.

It is easy to prove this theorem. By taking a $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$ and by the same arguments as in the proof of Theorem 2, we can obtain that $|x_0| < \delta$ and $|y_i| < \delta$ $(0 \leqq i \leqq m)$ imply $|x(t, t_0, x_0, \eta)| < \varepsilon$ for all $t \geqq t_0$, and hence, the zero solution of Equation (1) is uniformly stable.

THEOREM 4. In addition to all assumptions of Theorem 3, suppose that there exists a continuous, nondecreasing function $p(r)$ such that $p(r) > r$ for $r > 0$, and

$$\dot{V}_{(1)}(t, x, \eta) \leqq -c(|x|) \text{ whenever } p(V(t, x)) > V([t-i], y_i) \ (0 \leqq i \leqq m), \tag{4}$$

where $c(r)$ is a continuous, increasing, and positive definite function. Then the zero solution of Equation (1) is uniformly asymptotically stable.

PROOF. By Theorem 3, the zero solution of Equation (1) is uniformly stable. Let $\delta > 0$ and $H_1 \in (0, H)$ be numbers with $b(\delta) = a(H_1)$. Then by the similar arguments as in the proof of Theorem 2, it is easily seen that $|x_0| < \delta$ and $|y_i| < \delta$ $(0 \leqq i \leqq m)$ imply $|x(t)| < H_1$ and $V(t) = V(t, x(t)) \leqq b(\delta)$ on $I(t_0, \infty)$, where $x(t) = x(t, t_0, x_0, \eta)$. For any $\varepsilon \in (0, H_1)$, take a $d = d(\varepsilon) > 0$ and an $h = h(\varepsilon) \in (0, H_1)$ such that $p(r) - r > d$ for $a(\varepsilon) \leqq r \leqq b(\delta)$ and $b(h) < a(\varepsilon)$. Let $\gamma = \inf\{c(r): h \leqq r \leqq H_1\}$ and $T = T(\varepsilon) = Nb(\delta)/\gamma + (N-1)(m+1)$, where $N$ is the smallest positive integer with $a(\varepsilon) + Nd \geqq b(\delta)$. To prove

$$V(t) \leqq a(\varepsilon) \qquad \text{for} \quad t \geqq t_0 + T, \tag{5}$$

first we prove

$$V(t_1) \leqq a(\varepsilon) + (N-1)d \qquad \text{for} \quad \text{some } t_1 \leqq t_0 + \frac{b(\delta)}{\gamma}. \tag{6}$$

If $a(\varepsilon) + (N-1)d < V(t)$ for $t_0 \leqq t \leqq t_0 + b(\delta)/\gamma$, which together with the fact that $V(t) \leqq b(\delta)$ on $I(t_0, \infty)$ implies

$$p(V(t)) > V(t) + d > a(\varepsilon) + Nd \geqq b(\delta) > V([t-i], x([t-i])) \ (0 \leqq i \leqq m)$$

for $t_0 \leqq t \leqq t_0 + b(\delta)/\gamma$. Thus from (4) we have

$$\dot{V}_{(1)}(t, x(t), x([t]), \dots, x([t-m])) \leqq -c(|x(t)|) \leqq -\gamma, \ t_0 \leqq t \leqq t_0 + \frac{b(\delta)}{\gamma},$$

which implies

$$V(t) \leqq V(t_0) - (t-t_0)\gamma < b(\delta) - (t-t_0)\gamma, \ t_0 \leqq t \leqq t_0 + \frac{b(\delta)}{\gamma}.$$

But this yields a contradiction $V(t_2) < 0$ for $t_2 = t_0 + b(\delta)/\gamma$. Thus (6) holds.

Next we prove

$$V(t) \leqq a(\varepsilon) + (N-1)d \qquad \text{for} \quad t \geqq t_0 + \frac{b(\delta)}{\gamma}. \tag{7}$$

If (7) is false, then we have $V(t_3) > a(\varepsilon) + (N-1)d$ for some $t_3 \geqq t_0 + b(\delta)/\gamma$. Then from (6), there exists a $t_4 \in (t_1, t_3)$ such that $V(t_4) = a(\varepsilon) + (N-1)d$ and $D^+V(t_4) = \limsup_{\tau \to 0+} \{V(t_4 + \tau) - V(t_4)\}/\tau \geqq 0$. On the other hand, $b(|x(t_4)|) \geqq V(t_4) = a(\varepsilon) + (N-1)d \geqq a(\varepsilon) > b(h)$ implies $|x(t_4)| > h$ and $p(V(t_4)) > V(t_4) + d = a(\varepsilon) + Nd \geqq b(\delta) > V([t_4 - i]) \ (0 \leqq i \leqq m)$. Therefore from (4) we obtain $\dot{V}_{(1)}(t_4, x(t_4), \dots, x([t_4 - m])) \leqq -c(|x(t_4)|) \leqq -\gamma < 0$, which contradicts to $D^+V(t_4) \geqq 0$. Thus (7) holds.

Finally we prove (5). If $N = 1$, then (7) implies (5). Suppose that $N \geqq 2$. By repeating the same arguments as in the proof of (6), from (7) we have $V(t_5) \leqq a(\varepsilon) + (N-2)d$ for some $t_5 \in [t_0 + b(\delta)/\gamma + m + 1, t_0 + 2b(\delta)/\gamma + m + 1]$. By repeating this procedure, we obtain $V(t) \leqq a(\varepsilon) + (N-j)d$ for $t \geqq t_0 + jb(\delta)/\gamma + (j-1)(m+1)$ $(1 \leqq j \leqq N)$, and consequently, we have (5). Thus the zero solution of Equation (1) is uniformly asymptotically stable.

Example 2. Consider a scalar equation

$$\dot{x}(t) = -f(x(t)) + cg(x([t])), \tag{8}$$

where $f, g: R \to R$ are continuous and increasing functions, $xf(x) > 0$ for $x \neq 0$, $|g(x)| \leqq \min\{|f(x)|, |f(-x)|\}$ on $R$, and $c$ is a constant. For a Liapunov function $V(x) = x^2/2$ on $R$, we have

$$\dot{V}_{(8)}(x, y) = -(f(x) - cg(y))x,$$

which together with Theorems 2 and 3 implies:

COROLLARY 1. Under the above assumptions, the zero solution of Equation (8) is

(i) uniformly stable if $|c| \leqq 1$, and

(ii) uniformly asymptotically stable if $|c| < 1$.

## §4. Oscillation of solutions of linear equations

In [1], oscillation of solutions of the linear delay differential equation with a piecwise constant argument

$$\dot{x}(t) + a(t)x(t) + b(t)x([t-1]) = 0 \tag{9}$$

is discussed, where $a(t), b(t): I \to R$ are continuous functions. For any positive integer $p$, let $D_p = \int_p^{p+1} b(t)\exp\left(\int_{p-1}^{t} a(s)ds\right)dt$.

DEFINITION 5. A function $x(t): [t_0, \infty) \to R$ is oscillatory if $x(t)$ has an arbitrary large zero point.

In [1], the following two theorems are proved. These theorems give sufficient conditions for oscillation of all solutions of Equation (9).

THEOREM 5 (Aftabizadeh-Wiener-Xu). Suppose that $b(t) > 0$ on $I$ and

$$\limsup_{p\to\infty} D_p > 1. \tag{10}$$

Then Equation (9) has oscillatory solutions only.

THEOREM 6 (Aftabizadeh-Wiener-Xu). Suppose that

$$\liminf_{p\to\infty}\left(\exp\left(\int_p^{p+1} a(s)ds\right)\right)\cdot \liminf_{p\to\infty}\int_p^{p+1} b(t)\exp\left(\int_p^t a(s)ds\right)dt > \frac{1}{4}. \tag{11}$$

Then Equation (9) has oscillatory solutions only.

For Equation (9), consider a linear equation with a more general form

$$\dot{x}(t) + a(t)x(t) + \sum_{i=1}^{m} b_i(t)x([t-i]) = 0, \tag{12}$$

where $a(t)$, $b_i(t)$ $(1 \leqq i \leqq m)$: $I \to R$ are continuous functions, and $m \geqq 2$ is an integer.

For any integer $i$ $(1 \leqq i \leqq m)$ and $p \geqq m$, let $D_{i,p} = \int_p^{p+1} b_i(t) \exp\left(\int_{p-1}^t a(s)ds\right)dt$. Corresponding to Theorem 5, first we obtain:

THEOREM 7. Suppose that for any integer $i$ $(1 \leqq i \leqq m)$, $D_{i,p} \geqq 0$ for all sufficiently large $p$ and that

$$\liminf_{p\to\infty} D_{1,p} + \limsup_{p\to\infty} \sum_{i=1}^{m} D_{i,p} > 1. \tag{13}$$

Then Equation (12) has oscillatory solutions only.

Proof. For any integer $k \geqq 0$, Equation (12) reduces to $\dot{x}(t) + a(t)x(t) + \sum_{i=1}^{m} b_i(t)x(k-i) = 0$ for $t \in [k, k+1)$, which gives

$$x(k+1)\exp\left(\int_k^{k+1} a(s)ds\right) = x(k) - \sum_{i=1}^{m} x(k-i)\int_k^{k+1} b_i(t)\exp\left(\int_k^t a(s)ds\right)dt. \tag{14}$$

Suppose that Equation (12) has an eventually positive solution $x(t)$. Then for some $T$, $x(t) > 0$ for $t > T$ and for any integer $i$ $(1 \leqq i \leqq m)$, $D_{i,p} \geqq 0$ for $p > T$. For any integer $p \geqq [T] + 2m + 1$, let $y_p = x(p)\exp\left(\int_{p-1}^{p} a(s)ds\right)/x(p-1)$. Now the facts that $x(p-j-1) > 0$ $(2 \leqq j \leqq 2m)$ and $D_{i,p} \geqq 0$ $(1 \leqq i \leqq m)$ and (14) with $k = p - m, \ldots, p-1$ imply

$$x(p)\exp\left(\int_{p-i}^{p} a(s)ds\right) \leqq x(p-i), \qquad 1 \leqq i \leqq m.$$

From this and (14) with $k = p$, we have $y_{p+1} + \sum_{i=1}^{m} D_{i,p} \leqq 1$. This implies

$$y_* + \limsup_{p\to\infty} \sum_{i=1}^{m} D_{i,p} \leqq 1, \tag{15}$$

where $y_* = \liminf_{p\to\infty} y_p$. On the other hand, (14) with $k = p$ and the fact that $D_{i,p} \geqq 0$ $(2 \leqq i \leqq m)$ give $D_{1,p} \leqq y_p$. From this, we obtain $\liminf_{p\to\infty} D_{1,p} \leqq y_*$, which together with (15) imply

$$\liminf_{p\to\infty} D_{1,p} + \limsup_{p\to\infty} \sum_{i=1}^{m} D_{i,p} \leqq 1.$$

This contradicts to Assumption (13). Thus Equation (12) cannot have an eventually positive solution. It is easy to see that the case of eventually negative solution is reduced to the above case. Hence Equation (12) has oscillatory solutions only.

If $b_i(t) \equiv 0$ for any integer $i$ with $2 \leqq i \leqq m$, then Equation (9) becomes a special case of Equation (12) by replacing $b_1(t)$ by $b(t)$. From Theorem 7, for Equation (9) we have:

COROLLARY 2. Suppose that $D_p \geqq 0$ for all sufficiently large $p$ and

$$\liminf_{p\to\infty} D_p + \limsup_{p\to\infty} D_p > 1. \tag{16}$$

Then Equation (9) has oscillatory solutions only.

REMARK. Under the assumptions in Theorem 5, Condition (10) implies Condition (16). Actually, it is easy to show that Corollary 2 is an improvement of Theorem 5. Let $\alpha$ and $\beta$ be numbers such that $0 < \alpha \leqq 1/4 < \beta \leqq 1$ and $\alpha + \beta > 1$, and let $a(t)$ and $b(t)$ be periodic continuous functions on I with periods 1 and 2 respectively, and $D_1 = \alpha$ and $D_2 = \beta$. Then we have $\liminf_{p\to\infty} D_p = \alpha$ and $\limsup_{p\to\infty} D_p = \beta$. Thus $a(t)$ and $b(t)$ do not satisfy Condition (10), though they satisfy Condition (16). Moreover, since all assumptions in Corollary 2 are satisfied under the assumptions in Theorem 5, Corollary 2 is an improvement of Theorem 5.

Next, corresponding to Theorem 6 we have:

THEOREM 8. Suppose that for any integer $i$ $(2 \leqq i \leqq m)$, $D_{i,p} \geqq 0$ for all sufficiently large $p$ and that

$$\liminf_{p\to\infty} D_{1,p} > \frac{1}{4}. \tag{17}$$

Then Equation (12) has oscillatory solutions only.

PROOF. Suppose that Equation (12) has an eventually positive solution $x(t)$, and that for some $T$, $x(t) > 0$ for $t > T$ and for any integer $i$ $(2 \leqq i \leqq m)$, $D_{i,p} \geqq 0$ for $p > T$. For any integer $p \geqq [T] + m + 1$, let $y_p = x(p)\exp\left(\int_{p-1}^{p} a(s)\,ds\right)/x(p-1)$. Then from (14) with $k = p$, we obtain

$$D_{1,p} + y_p y_{p+1} \leqq y_p. \tag{18}$$

Since we have $D_{1,p} > 0$ for all sufficiently large $p$ from Assumption (17), (18) implies $y_{p+1} < 1$, and hence, we obtain $y_* = \liminf_{p\to\infty} y_p \leqq 1$. Again from (18), we have

$$\liminf_{p\to\infty} D_{1,p} + y_*^2 \leqq y_*,$$

which contradicts to Assumption (17). Thus Equation (12) cannot have an eventually positive solution. The case of an eventually negative solution is reduced to the above case. Hence Equation (12) has oscillatory solutions only.

By taking $b(t) \equiv b_1(t)$ and $b_i(t) \equiv 0$ $(2 \leqq i \leqq m)$, from Theorem 8 we have:

COROLLARY 3. Suppose that

$$\liminf_{p \to \infty} D_p > \frac{1}{4}. \tag{19}$$

Then Equation (9) has oscillatory solutions only.

Condition (11) in Theorem 6 is a pretty sharp condition. In fact, it is shown in [1] that if $a(t)$ and $b(t)$ are constant functions, then Condition (11) is necessary and sufficient in order that all solutions of Equation (9) are oscillatory. On the other hand, Condition (11) implies Condition (19). Actually, it is easy to show that Corollary 3 is an improvement of Theorem 6. Let $\alpha$ and $\beta$ be numbers with $0 < \alpha \leqq 1/2$ and $1/4 < \beta \leqq 1/2$, and let $a(t)$ and $b(t)$ be periodic continuous functions on $I$ with periods 2, $\int_0^1 a(s)ds = 0$, $\exp\left(\int_1^2 a(s)ds\right) = \alpha$, and $D_p = \beta$ for any positive integer $p$. Then we have $\liminf_{p \to \infty} \exp\left(\int_p^{p+1} a(s)ds\right) = \alpha$ and $\liminf_{p \to \infty} \int_p^{p+1} b(t)\exp\left(\int_p^t a(s)ds\right)dt = \liminf_{p \to \infty} D_p = \beta$. Thus $a(t)$ and $b(t)$ do not satisfy Condition (11), though they satisfy Condition (19). Hence Corollary 3 is an improvement of Theorem 6.

Finally we show the independence of Theorems 7 and 8. For $a(t)$ and $b(t)$ in Remark, take $b_1(t) \equiv b(t)$ and $b_i(t)$ $(2 \leqq i \leqq m)$ such that for any integer $i$ $(2 \leqq i \leqq m)$, $D_{i,p} = 0$ for all sufficiently large $p$. Then $a(t)$ and $b_i(t)$ $(1 \leqq i \leqq m)$ satisfy Condition (13), while they do not satisfy Condition (17), and hence, Theorem 7 is applicable to Equation (12) with these $a(t)$ and $b_i(t)$ $(1 \leqq i \leqq m)$, while Theorem 8 is not. On the other hand, let $a(t)$ and $b(t)$ be periodic continuous functions on $I$ with periods 1 and $1/4 < D_1 \leqq 1/2$. Take $b_1(t) \equiv b(t)$ and $b_i(t)$ $(2 \leqq i \leqq m)$ such that for any integer $i$ $(2 \leqq i \leqq m)$, $D_{i,p} = 0$ for all sufficiently large $p$. Then these $a(t)$ and $b_i(t)$ $(1 \leqq i \leqq m)$ satisfy Condition (17), while they do not satisfy Condition (13). Thus Theorem 8 is applicable to Equation (12) with these $a(t)$ and $b_i(t)$ $(1 \leqq i \leqq m)$, while Theorem 7 is not. The independence of Corollaries 2 and 3 is obvious from the above examples constructed to show the independence of Theorems 7 and 8.

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