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Conformal Compactification of $R^3 \times S^1$

Dedicated to Professor Akio Hattori on his sixtieth birthday

Hiromichi MATSUNAGA

Department of Mathematics, Shimane University, Matsue, Japan
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A conformal compactification of $R^3 \times S^1$ is obtained and we discuss removable singularities.

§ 1. Introduction

In this article a conformal compactification of the space $R^3 \times S^1$ is obtained, (§ 2). In § 3 a decay property of the curvature is given, and in § 4 the maximum principle is applied and we discuss removable singularities. In § 3, 4 we depend heavily on the elaborated works by Uhlenbeck, [2], [4]. The result of this article is used to study symmetry breaking at infinity [3].

§ 2. Compactification

Let B_1^3 be the open kernel of the unit disc B_1^3 in the euclidean space R^3. Denote by $I: (B_1^3 - O) \times S^1 \to (R^3 - B_1^3) \times S^1$ the product of the inversion and the identity mapping. Then $I(x, t) = (x/|x|^2, t)$ for $(x, t) \in (B_1^3 - O) \times S^1$, and

$$I^*(dy_1^2 + dy_2^2 + dy_3^2 + dt^2) = (dx_1^2 + dx_2^2 + dx_3^2)/|x|^4 + dt^2,$$

which is conformally equivalent to the metric $dx_1^2 + dx_2^2 + dx_3^2 + |x|^4 dt^2$. Using the polar coordinates (r, θ, ϕ) in R^3 we have metrics $dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + r^4 dt^2$, and $dr^2/r^2 + d\theta^2 + \sin^2\theta d\phi + r^2 dt^2$. The substitution $r = e^{-\tau}$ gives the coordinates in which the metric is given by $d\tau^2 + d\theta^2 + \sin^2\theta d\phi^2 + e^{-2\tau} dt^2$. Thus the space $(R^3 - B_1^3) \times S^1$ is conformally equivalent to the warped product space $S^2 \times ([0, \infty) \times {}_f S^1)$, where $f(\tau) = e^{-\tau}$ [1]. Denote by $\langle , \rangle$ and $\langle , \rangle_\tau$ the inner products in the space S^1 and $(\tau) \times S^1 \subset S^2 \times ([0, \infty) \times {}_f S^1)$ respectively. Then $\langle \partial/\partial t, \partial/\partial t \rangle_\tau = e^{-\tau} \langle \partial/\partial t, \partial/\partial t \rangle$. Therefore $\langle \partial/\partial t, \partial/\partial t \rangle_\tau$ tends to zero as $\tau \to \infty$ and hence $S^2 \times (\tau) \times {}_f S^1$ tends to the 2-sphere, say S_∞^2. Thus the space $S^2 \times ([0, \infty) \times {}_f S^1) \cup S_\infty^2$ gives a conformal compactification, but the limit set S_∞^2 is possibly singularities. By Mayer-Vietoris exact sequence of homology groups we can see that the compactification is homotopically a 4-sphere.

§3. $\lim_{\tau\to\infty} |F(\theta, \phi, \tau, t)|_f = 0$

Denote by $|\ \ |_f$ the norm in the space $S^2 \times ([0, \infty) \times {}_fS^1)$. Consider a dilation $\pi: r \to r/\sigma$ for $\sigma > 0$, then the metric tensor is a diagonal matrix with entries $(\sigma^{-4}r^4, \sigma^{-2}, \sigma^{-2}, \sigma^{-2})$. Let $F = F_1 + F_2$ be the curvature of a connection A, where $F_1 = \sum a_{0j} dt_\wedge dx_j$ and $F_2 = \sum b_{ij} dx_{i\wedge} dx_j$. Then by the dilation π, their norms and the volume form are transformed as

$$|F_1|_f^2 \longrightarrow \sigma^6 |F_1|_f^2,\ |F_2|_f^2 \longrightarrow \sigma^4 |F_2|_f^2 \quad \text{and} \quad \omega_f \longrightarrow \sigma^{-5}\omega_f.$$

Now we need several lemmas for Coulomb gauge (Hodge gauge). Let $\|F\|_\infty$, $\|A\|_\infty$ denote $\max|F|$, $\max|A|$ respectively.

LEMMA 1 [4]. *Let η be a bundle ober $S^2 \times S^1$ with a covariant derivative D, curvature F. There exists $\gamma_0 > 0$ such that if $\|F\|_\infty < \gamma_0$ then there exists a gauge in which $D = d + A$, $d^*A = 0$, and $\|A\|_\infty < K\|F\|_\infty$.*

PROOF. We have a modified form of Proposition 9.33 in [2], then follow the proof of Theorem 2.5 in [4].

Similarly to Theorem 2.8 in [4] we have

LEMMA 2. *Let D be a covariant derivative in a bundle over $U = \{x \in (B^3 - 0) \times {}_fS^1;\ 1 \leqq r \leqq 2\}$, where the diameter of $B^3 \geqq 2$. There exists $\gamma' > 0$ such that if $\|F\|_\infty \leqq \gamma'$, then there exists a gauge in which $D = d + A$, $d^*A = 0$.*

PROPOSITION 3. *Let D be a connection on B_1 in U, self-dual with respect to a metric and assume $\|F_D\|_{L^2} < \varepsilon$. Then there exists an L_2^2-gauge such that $D = d + A$, A is C^∞ in the half sinzed ball $B_{1/2}$ and the estimate*

$$\|A\|_{C^k(B_{1/2})} \leqq C\|F\|_{L^2(B_1)}.$$

PROOF. By Lemma 2 we have a Coulomb gauge and follow the proof of Proposition 8.3 in [2].

Now we proceed to get our main result in this section. By using the dilation for $0 < \sigma < 1$ we have

$$\sigma^6(|F_1|_f^2 + |F_2|_f^2) \leqq \sigma^6|F_1|_f^2 + \sigma^4|F_2|_f^2 \leqq \int_{\bar\tau-1\leqq\tau\leqq\bar\tau+1} (\sigma|F_1|_f^2 + \sigma^{-1}|F_2|_f^2)\omega_f$$

$$\leqq \sigma^{-1}\int_{\bar\tau-1\leqq\tau\leqq\bar\tau+1} (|F_1|_f^2 + |F_2|_f^2)\omega_f.$$

Then for a sufficiently large $\bar\tau$,

$$|F|_f^2 \leqq (1/\sigma^7)\int_{\bar\tau-1\leqq\tau\leqq\bar\tau+1} |F|_f^2\omega_f \longrightarrow 0 \text{ as } \tau \longrightarrow \infty,$$

where we have assumed that $\int_{R^3 \times S^1} |F|^2 < \infty$ and the connection is self-dual.

Thus we have

THEOREM 4. *If* $\int_{R^3 \times S^1} |F|^2 < \infty$ *and the connection is self-dual, then*

$$\lim_{\tau \to \infty} |F(\theta, \phi, \tau, t)|_f = 0.$$

§ 4. Maximal principle and removable singularities

First we calculate the scalar curvature of a funnel shaped cylinder with metric $d\theta^2 + \sin^2\theta d\phi^2 + d\tau^2 + e^{-2\tau} dt^2$. The curvature of the funnel shaped surface with metric $d\tau^2 + e^{-2\tau} dt^2$ is

$$\{e^{\tau} \cdot e^{-\tau}(1 + e^{-\tau})^{-3/2}\}^{-1} = (1 + e^{-\tau})^{3/2} \equiv 1 \mod e^{-2\tau}.$$

Then the required scalar curvature is $2 \times 1 + 2 \times 1 = 4 \mod e^{-2\tau}$. The space $R^3 \times S^1$ is conformally flat and if the curvature is self-dual, then by Weitzenböck formula, for any $\gamma < 2/\sqrt{3}$

$$|F(\theta, \phi, \tau, t)|_f \leqq \max_{(\theta, \phi, t)} |F(\theta, \phi, \bar{\tau}, t)|_f \; e^{\gamma(\bar{\tau} - \tau)} + \max_{(\theta, \phi, t)} |F(\theta, \phi, \tau_n, t)|_f \; e^{\gamma(\tau - \tau_n)}$$

for $\bar{\tau} \leqq \tau \leqq \tau_n$ (see Appendix D in [2]). On a subspace $S^2 \times (\tau) \times {}_f S^1$ we choose an exponential gauge and a transverse gauge $A_\tau = 0$, then as in Lemma D in [2],

$$|A(\theta, \phi, \tau, t)|_f \leqq Ce^{\gamma(\bar{\tau} - \tau)} \text{ on } \tau \geqq \bar{\tau} \qquad \text{for a sufficiently large } \bar{\tau} \qquad (*).$$

By (*) above if $e^{\gamma\bar{\tau}} \leqq e^{\tau}$, then $|A(x, t)|_f \leqq Cr^{\gamma - 1}$ for $r = |x|$. For $\bar{r} = e^{-\bar{\tau}}$, $1 \geqq \bar{r} \geqq r$ and $(\bar{r})^{-\gamma} \leqq r^{-\gamma} \leqq r^{-2}$, then

$$|F(x, t)|_f \leqq Cr^{\gamma - 2}.$$

The volume element is $\omega_f = r^4 \sin\theta dr d\theta d\phi dt$, then F is bounded in L^p for $p < 5/(2 - \gamma)(> 4)$. Then the assumption in Theorem 4.6 in [4] is satisfied. Using the construction of the broken Hodge gauge we have a Coulomb gauge, and obtain an elliptic system as in the final part of the appendix in [2].

Now we need to define a 'smooth structure' on the limit set S^2_∞. For $y \in S^2_\infty$, the operator $\partial/\partial r$ is defined by

$$\partial/\partial r A(y) = \lim_{r \to 0} \partial/\partial r A(y, r, t) \text{ and similarly for } \partial/\partial\theta, \partial/\partial\phi.$$

These operators are indetendent of t because by the relation $\langle \partial/\partial t, \partial/\partial t \rangle_\tau \to 0$ as τ

$\to \infty$, $(\partial/\partial t)_\tau$ and $\partial/\partial t(A(y, r, t))$ tends to zero. Using the method of Proposition 8.3 in [2] the regularity follows and the extention of the connection to the compactification is obtained.

References

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