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Nonlinear Poisson Equations on an Infinite Network

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On a locally finite infinite network, the existence of a solution of a nonlinear Poisson equation is discussed with the aid of a flow problem on the network.

§1. Introduction

Let $N = \{X, Y, K, r\}$ be an infinite network which is locally finite and has no self-loop. Denote by $L(X)$ the set of all real functions on X and by $L_0(X)$ the set of all $u \in L(X)$ with finite support. Let p and q be positive numbers such that $1 < p < \infty$ and $1/p + 1/q = 1$. Let $\varphi_p(t)$ be the real function on R defined by

$$\varphi_p(t) = |t|^{p-1}\operatorname{sign}(t),$$

where $\operatorname{sign}(t) = 1$ if $t \geq 0$ and $\operatorname{sign}(t) = -1$ if $t < 0$.

For $u \in L(X)$, its p-Laplacian $\Delta_p u \in L(X)$ is defined by

$$\Delta_p u(x) = \sum_{y \in Y} K(x, y)\varphi_p(du(y)),$$

where du is the discrete derivative of u, i.e.,

$$du(y) = -r(y)^{-1}\sum_{x \in X} K(x, y)u(x).$$

Given a function $\mu \in L(X)$, we study the problem of finding a solution of the following nonlinear Poisson equation:

$$\Delta_p u(x) = \mu(x) \quad \text{on } X. \tag{1.1}$$

Since $\varphi_2(t) = t$, $\Delta_2 u$ is the usual discrete Laplacian of u and Δ_2 is a linear operator on $L(X)$. Note that $\Delta_p u$ is nonlinear in u unless $p = 2$.

This problem has been investigated by many mathematicians in case $p = 2$. For instance, R. J. Duffin [1] studied this problem on the lattice domain of the 3-dimensional Euclid space by using Fourier transforms. T. Kayano and M. Yamasaki [3] studied this problem on a locally finite infinite network by using a flow problem as in [2].

In the present paper, we shall prove the existence of a Dirichlet potential which satisfies the nonlinear Poisson equation (1.1) by using a flow problem as in [3].

For notation and terminology, we mainly follow [3] and [5].

§2. Preliminaries

To state our problem more precisely, we recall some fundamental notion. For $w \in L(Y)$, the energy $H_p(w)$ of w of order p is defined by

$$H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p.$$

For $w, w' \in L(Y)$, we define the mutual energy $\langle w, w' \rangle$ of w and w' by

$$\langle w, w' \rangle = \sum_{y \in Y} r(y) w(y) w'(y)$$

if the sum is well-defined. Denote by $L_p(Y; r)$ the set of all $w \in L(Y)$ such that $H_p(w) < \infty$. Clearly $L_0(Y) \subset L_p(Y; r)$. The mutual energy is well-defined for the pair of elements in $L_p(Y; r)$ and $L_q(Y; r)$.

For $u \in L(X)$, its Dirichlet integral $D_p(u)$ of order p is defined by

$$D_p(u) = H_p(du) = \sum_{y \in Y} r(y)|du(y)|^p.$$

Denote by $\boldsymbol{D}^{(p)}(N)$ the set of all Dirichlet functions u on X, i.e., $D_p(u) < \infty$ and by $\boldsymbol{D}_0^{(p)}(N)$ the set of all Dirichlet potentials of order p. Namely, $\boldsymbol{D}_0^{(p)}(N)$ is the closure of $L_0(X)$ in $\boldsymbol{D}^{(p)}(N)$ with respect to the norm:

$$\|u\|_p = [D_p(u) + |u(x_0)|^p]^{1/p},$$

where x_0 is a fixed node.

We proved in [3; Theorem 4.3]

PROPOSITION 2.1. *If* $\mu \in L_0(X)$ *and* $\sum_{x \in X} \mu(x) = 0$, *then there exists* $u \in \boldsymbol{D}^{(2)}(N)$ *such that* $\Delta_2 u(x) = \mu(x)$ *on* X.

We say that N is of parabolic type of order p if the value of the following extremum problem vanishes for some nonempty finite subset A of X:

(2.1) $$d_p(A, \infty) = \inf\{D_p(u) \colon u \in L_0(X) \text{ and } u = 1 \text{ on } A\}.$$

We also say that N is of hyperbolic type of order p if it is not of parabolic type of order p.

For a nonempty finite subset A of X, denote by $F(A, \infty)$ the set of all flows $w \in L(Y)$ from A to the ideal boundary ∞, i.e.,

(2.2) $$\sum_{y \in Y} K(x, y) w(y) = 0 \quad \text{on } X - A.$$

The strength $I(w)$ of $w \in F(A, \infty)$ is defined by

$$I(w) = -\sum_{x \in A} \sum_{y \in Y} K(x, y) w(y).$$

We recall some criteria for the parabolicity of N (cf. [4]):

PROPOSITION 2.2. *An infinite network* N *is of hyperbolic type of order* p *if and only if any one of the following conditions is fulfilled:*

(a) $1 \notin \boldsymbol{D}_0^{(p)}(N)$;

(b) $\boldsymbol{D}^{(p)}(N) \neq \boldsymbol{D}_0^{(p)}(N)$;

(c) *For every nonempty finite subset A of X, there exists $w \in F(A, \infty)$ such that $H_q(w) < \infty$ and $I(w) = 1$.*

In case N is of hyperbolic type of order p, note that

$$d_p(\{a\}, \infty) = \inf\{D_p(u);\ u \in \boldsymbol{D}_0^{(p)}(N) \text{ and } u(a) = 1\} > 0.$$

With the aid of the optimal solution of this problem, we can prove that there exists a function $g_a^{(p)} \in L(X)$ with the following properties:

(2.3) $$g_a^{(p)} \in \boldsymbol{D}_0^{(p)}(N) \text{ and } \Delta_p g_a^{(p)}(x) = -\varepsilon_a(x) \text{ on } X.$$

For $\mu \in L_0(X)$, let us put

$$G^{(p)}\mu(x) = -\sum_{x \in X} g_a^{(p)}(x)\mu(x).$$

Note that $g_a^{(2)}$ is the Green function of N with pole at a and that $G^{(2)}\mu$ is a solution of the Poisson equation: $\Delta_2 u(x) = \mu(x)$, since Δ_2 is a linear operator. However we can not expect that $G^{(p)}\mu$ is a solution of (1.1) unless $p = 2$.

Denote by $\boldsymbol{H}^{(p)}(N)$ the set of all p-harmonic functions u on X, i.e., $\Delta_p u(x) = 0$ on X and by $\boldsymbol{HD}^{(p)}(N)$ the set of all Dirichlet finite p-harmonic functions on X, i.e.,

$$\boldsymbol{HD}^{(p)}(N) = \boldsymbol{D}^{(p)}(N) \cap \boldsymbol{H}^{(p)}(N).$$

For each $u \in \boldsymbol{D}^{(p)}(N)$, we have

(2.4) $$D_p(u) = \langle \varphi_p(du), du \rangle = H_q(\varphi_p(du)),$$

since $|\varphi_p(t)|^q = |t|^{q(p-1)} = |t|^p$.

For $u \in L(X)$ and $f \in L_0(X)$, we obtain the following equality by interchanging the order of summation:

(2.5) $$\langle \varphi_p(du), df \rangle = -\sum_{x \in X} [\Delta_p u(x)] f(x).$$

We have

LEMMA 2.1. *Let $u \in \boldsymbol{D}^{(p)}(N)$ and $v \in \boldsymbol{D}_0^{(p)}(N)$. If $\{f_n\}$ is a sequence in $L_0(X)$ such that $\|v - f_n\|_p \to 0$ as $n \to \infty$, then*

$$\langle \varphi_p(du), dv \rangle = \lim_{n \to \infty} \langle \varphi_p(du), df_n \rangle.$$

PROOF. By Hölder's inequality and (2.4),

$$\begin{aligned} |\langle \varphi_p(du), dv - df_n \rangle| &\le H_q(\varphi_p(du))^{1/q} H_p(dv - df_n)^{1/p} \\ &\le D_p(u)^{1/q} D_p(v - f_n)^{1/p} \\ &\le D_p(u)^{1/q} \|v - f_n\|_p \longrightarrow 0 \end{aligned}$$

as $n \to \infty$.

COROLLARY. *Let* $h \in \boldsymbol{HD}^{(p)}(N)$. *Then* $\langle \varphi_p(dh), dv \rangle = 0$ *for every* $v \in \boldsymbol{D}_0^{(p)}(N)$.

We need the following discrete analogue of Royden's decomposition of a Dirichlet function (cf. [5]):

PROPOSITION 2.3. *Assume that N is of hyperbolic type of order p. Then every* $u \in \boldsymbol{D}^{(p)}(N)$ *can be decomposed uniquely in the form:* $u = v + h$, *where* $v \in \boldsymbol{D}_0^{(p)}(N)$ *and* $h \in \boldsymbol{HD}^{(p)}(N)$.

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LEMMA 2.2. *Assume that N is of hyperbolic type of order p and let* $u \in \boldsymbol{D}^{(p)}(N)$. *If* $\langle \varphi_p(dh), du \rangle = 0$ *for every* $h \in \boldsymbol{HD}^{(p)}(N)$, *then there exists a constant c such that* $u - c \in \boldsymbol{D}_0^{(p)}(N)$.

PROOF. By Proposition 2.3, u can be decomposed in the form: $u = v + f$ with $v \in \boldsymbol{D}_0^{(p)}(N)$ and $f \in \boldsymbol{HD}^{(p)}(N)$. It follows from the corollary of Lemma 2.1 and our assumption that

$$D_p(f) = \langle \varphi_p(df), df \rangle = \langle \varphi_p(df), du - dv \rangle = 0,$$

so that $f(x) = c$ on X. Therefore $u - c = v \in \boldsymbol{D}_0^{(p)}(N)$.

We have by [5; Lemma 2.1]

LEMMA 2.3. $\langle \varphi_p(w_1) - \varphi_p(w_2), w_1 - w_2 \rangle \geq 0$ *for every* $w_1, w_2 \in L_p(Y; r)$. *The equality holds only if* $w_1 = w_2$.

§3. Main results

For $\mu \in L(X)$, denote by $\boldsymbol{PSD}^{(p)}(\mu)$ the set of all Dirichlet finite solutions of the nonlinear Poisson equation (1.1) of order p, i.e.,

$$\boldsymbol{PSD}^{(p)}(\mu) = \{u \in \boldsymbol{D}^{(p)}(N); \Delta_p u = \mu\},$$

and put

$$\boldsymbol{PSD}_0^{(p)}(\mu) = \boldsymbol{PSD}^{(p)}(\mu) \cap \boldsymbol{D}_0^{(p)}(N).$$

Our problem is to study when $\boldsymbol{PSD}_0^{(p)}(\mu)$ or $\boldsymbol{PSD}^{(p)}(\mu)$ is nonempty.

For $w \in L(Y)$, define its nodal current excess $\partial w \in L(X)$ by

$$\partial w(x) = \sum_{y \in Y} K(x, y) w(y).$$

Denote by $KL_q(N)$ the image of $L_q(Y; r)$ under the mapping ∂ (i.e., the linear transformation associated with the incidence matrix K):

$$KL_q(N) = \{\partial w; w \in L_q(Y; r)\}.$$

For $w \in L(Y)$ and $f \in L_0(X)$, we have the following fundamental relation by interchanging the order of summation:

(3.1) $$\langle w, df\rangle = -\sum_{x\in X} f(x)[\partial w(x)].$$

Let us consider the following flow problem on N:

$(\mathrm{FP}(\mu))_q$ Given $\mu\in L(X)$, find a function (called a flow) $w\in L(Y)$ which satisfies $w\in L_q(Y; r)$ and $\partial w = -\mu$, i.e.,

$$\sum_{y\in Y} K(x, y)w(y) = -\mu(x) \quad \text{on } X.$$

Clearly, $(\mathrm{FP}(\mu))_q$ has a slution if and only if $-\mu\in KL_q(N)$.

We say that $\omega\in L(Y)$ is a cycle if $\partial\omega = 0$, i.e.,

(3.2) $$\sum_{y\in Y} K(x, y)\omega(y) = 0 \text{ on } X.$$

Denote by $C_q(Y)$(resp. $C_0(Y)$) the set of all cycles ω such that $\omega\in L_q(Y; r)$ (resp. $L_0(Y)$).

We shall prove

THEOREM 3.1. $\{\Delta_p u; u\in \boldsymbol{D}^{(p)}(N)\} \subset KL_q(N)$.

PROOF. Let $u\in \boldsymbol{D}^{(p)}(N)$ and put $w = \varphi_p(du)$. Since $H_q(w) = D_p(u) < \infty$ by (2.4), $w\in L_q(Y; r)$. By definition,

$$\partial w(x) = \sum_{y\in Y} K(x, y)\varphi_p(du(y)) = \Delta_p u(x).$$

Hence $\Delta_p u\in KL_q(N)$.

THEOREM 3.2. *If problem* $(\mathrm{FP}(\mu))_q$ *has a solution, then there exists a Dirichlet potential u of order p which satisfies the nonlinear Poisson equation* (1.1), i.e., $\boldsymbol{PSD}_0^{(p)}(\mu) \neq \emptyset$.

PROOF. Consider the following extremum problem:

(3.3) Minimize $H_q(w)$ subject to

$$w\in L_q(Y; r) \text{ and } \sum_{y\in Y} K(x, y)w(y) = -\mu(x) \quad \text{on } X.$$

Let α be the value of this problem and $\{w_n\}$ be a sequence of feasible solutions such that $H_q(w_n)\to\alpha$ as $n\to\infty$. Recall the following Clarkson's inequality (cf. [5]):

(1) $H_q(w + w') + H_q(w - w') \leq 2^{q-1}[H_q(w) + H_q(w')]$ in case $q\geq 2$;

(2) $[H_q(w + w')]^{p-1} + [H_q(w - w')]^{p-1} \leq 2[H_q(w) + H_q(w')]^{p-1}$ in case $1 < q \leq 2$.

It follows from Clarkson's inequality that $H_q(w_n - w_m)\to 0$ as $n, m\to\infty$ (cf. the proof of Theorem 2.1 in [5]). Thus there exists $w^*\in L_q(Y; r)$ such that $H_q(w_n - w^*)\to 0$ as $n\to\infty$. Since N is locally finite, we see that w^* is an optimal solution of problem (3.3). Let $\omega\in C_q(Y)$. For any $t\in R$, $w^* + t\omega$ is a feasible solution of problem (3.3), so that $H_q(w^*) \leq H_q(w^* + t\omega)$. Therefore the derivative of $H_q(w^* + t\omega)$ with respect to t vanishes at $t = 0$. It follows that

(3.4) $$\sum_{y\in Y} r(y)\varphi_q(w^*(y))\omega(y) = 0$$

for every $\omega\in C_q(Y)$. Let $x_0\in X$ be fixed. For any $x\neq x_0$, let P_1 and P_2 be paths from x_0 to x and p_1 and p_2 be path indices of P_1 and P_2 respectively. Then $\omega = p_1 - p_2 \in C_0(Y) \subset C_q(Y)$, and hence

$$\sum_{y \in Y} r(y)p_1(y)\varphi_q(w^*(y)) = \sum_{y \in Y} r(y)p_2(y)\varphi_q(w^*(y))$$

by (3.4). Namely, the above sum does not depend on the choice of paths from x_0 to x. Thus we can define $u^* \in L(X)$ by

$$u^*(x_0) = 0 \text{ and } u^*(x) = \sum_{y \in Y} r(y)p(y)\varphi_q(w^*(y)) \text{ for } x \neq x_0,$$

where p is the path index of a path P from x_0 to x. Now we show the equality:

$$du^*(y) = -\varphi_q(w^*(y)) \text{ on } Y. \tag{3.5}$$

Let $y' \in Y$ with $e(y') = \{a, b\}$ and let P' be a path from x_0 to b such that

$$C_X(P') = \{x_0, x_1, \cdots, x_n\} \text{ with } x_{n-1} = a \text{ and } x_n = b,$$
$$C_Y(P') = \{y_1, \cdots, y_n\} \text{ with } y_n = y'.$$

Furthermore, let P'' be the subpath of P' from x_0 to x_{n-1} and let p' and p'' be the path indices of P' and P'' respectively. Then

$$\begin{aligned} u^*(b) &= \sum_{y \in Y} r(y)p'(y)\varphi_q(w^*(y)) \\ &= \sum_{y \in Y} r(y)p''(y)\varphi_q(w^*(y)) + r(y')p'(y')\varphi_q(w^*(y')) \\ &= u^*(a) + r(y')[-K(a, y')]\varphi_q(w^*(y')), \end{aligned}$$

so that

$$\begin{aligned} du^*(y') &= -r(y')^{-1}[K(a, y')u^*(a) + K(b, y')u^*(b)] \\ &= -\varphi_q(w^*(y')), \end{aligned}$$

since $K(a, y') + K(b, y') = 0$. This shows (3.5). Noting that the inverse function of $\varphi_q(t)$ is equal to $\varphi_p(t)$, we have by (3.5)

$$w^*(y) = \varphi_p(-du^*(y)) = -\varphi_p(du^*(y)). \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned} \Delta_p u^*(x) &= \sum_{y \in Y} K(x, y)\varphi_p(du^*(y)) \\ &= -\sum_{y \in Y} K(x, y)w^*(y) = \mu(x), \\ D_p(u^*) &= H_p(du^*) = H_p(\varphi_q(w^*)) = H_q(w^*) < \infty. \end{aligned}$$

Namely $u^* \in \boldsymbol{PSD}^{(p)}(\mu)$. By (3.4) and (3.5), we have

$$\langle du^*, \omega \rangle = 0 \tag{3.7}$$

for every $\omega \in C_q(Y)$. Now we show that there exists a constant c such that $u^* - c \in \boldsymbol{D}_0^{(p)}(N)$. Let $h \in \boldsymbol{HD}^{(p)}(N)$ and $\omega_h(y) = \varphi_p(dh(y))$. Then $H_q(\omega_h) = D_p(h) < \infty$ by (2.4) and

$$\sum_{y\in Y} K(x, y)\omega_h(y) = \sum_{y\in Y} K(x, y)\varphi_p(dh(y)) = \Delta_p h(x) = 0,$$

namely $\omega_h \in C_q(Y)$. By (3.7), we have

$$\langle \varphi_p(dh), du^* \rangle = \langle \omega_h, du^* \rangle = 0.$$

On account of Lemma 2.2, there exists a constant c such that $v^* = u^* - c \in \boldsymbol{D}_0^{(p)}(N)$. Since $dv^* = du^*$, we see that $v^* \in \boldsymbol{PSD}_0^{(p)}(\mu)$.

As for the uniqueness of the solution of the nonlinear Poisson equation, we have

THEOREM 3.3. *Assume that N is of hyperbolic type of order p. If u_1 and u_2 belong to $\boldsymbol{PSD}_0^{(p)}(\mu)$, then $u_1 = u_2$.*

PROOF. By our assumption, $u_1, u_2 \in \boldsymbol{D}_0^{(p)}(N)$ and $\Delta_p u_1(x) = \Delta_p u_2(x) = \mu(x)$ on X. For any $v \in L_0(X)$, we have by (2.5)

$$\begin{aligned} \langle \varphi_p(du_1), dv \rangle &= -\sum_{x\in X} [\Delta_p u_1(x)] v(x) \\ &= -\sum_{x\in X} [\Delta_p u_2(x)] v(x) = \langle \varphi_p(du_2), dv \rangle. \end{aligned}$$

By Lemma 2.1,

$$\langle \varphi_p(du_1), dv \rangle = \langle \varphi_p(du_2), dv \rangle$$

for every $v \in \boldsymbol{D}_0^{(p)}(N)$. Since $v = u_1 - u_2 \in \boldsymbol{D}_0^{(p)}(N)$, we have

$$\langle \varphi_p(du_1) - \varphi_p(du_2), du_1 - du_1 \rangle = 0,$$

and hence $du_1 = du_2$ by Lemma 2.3. It follows that $u_1 - u_2 = c$ for some constant c. By Proposition 2.2, $c = 0$. Therefore $u_1 = u_2$.

§4. Sufficient conditions

Now we discuss the feasibility of the flow problem $(\mathrm{FP}(\mu))_q$, or equivalently, sufficient conditions which assure $\boldsymbol{PSD}_0^{(p)}(\mu) \neq \emptyset$.

THEOREM 4.1. *Assume that N is of hyperbolic type of order p. Then $L_0(X) \subset KL_q(N)$ and $\boldsymbol{PSD}_0^{(p)}(\mu) \neq \emptyset$ for every $\mu \in L_0(X)$.*

PROOF. Let $\mu \in L_0(X)$ and let a be any node of X. Since N is of hyperbolic type of order p, there exists $w_a \in L_q(Y; r)$ by Proposition 2.2 which satisfies the conditions: $w_a \in F(\{a\}, \infty)$ and $I(w_a) = 1$, or equivalently,

$$\sum_{y\in Y} K(x, y) w_a(y) = -\varepsilon_a(x) \quad \text{on } X.$$

Let us put

$$w(y) = \sum_{a\in X} \mu(a) w_a(y).$$

Then $w \in L_q(Y; r)$ and

$$\sum_{y\in Y} K(x, y)w(y) = \sum_{a\in X}\mu(a)\sum_{y\in Y} K(x, y)w_a(y) = -\mu(x)$$

for each $x\in X$. Therefore $-\mu\in KL_q(N)$ and $PSD_0^{(p)}(\mu)\neq\emptyset$ by Theorem 3.2. Since $KL_q(N)$ is a linear space, $\mu\in KL_q(N)$.

As a generalization of [3; Lemma 3.1], we have

THEOREM 4.2. *Let N be of parabolic type of order p and let $u\in D^{(p)}(N)$. If $\sum_{x\in X}|\Delta_p u(x)|<\infty$, then $\sum_{x\in X}\Delta_p u(x)=0$.*

PROOF. Since $1\in D_0^{(p)}(N)$, there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $0\le f_n\le 1$ on X and $\|f_n-1\|_p\to 0$ as $n\to\infty$. Put $w=\varphi_p(du)$. Since $\{df_n\}$ converges weakly to 0 in $L_p(Y; r)$ and $w\in L_q(Y; r)$, $\langle w, df_n\rangle\to 0$ as $n\to\infty$. We may assume that $c=\sum_{x\in X}|\Delta_p u(x)|>0$. For any $\varepsilon>0$, there exists a finite subset X' of X such that $\sum_{x\in X-X'}|\Delta_p u(x)|<\varepsilon$. Since $\{f_n\}$ converges pointwise to 1, we can find n_0 such that $|f_n(x)-1|<\varepsilon/c$ on X' for all $n\ge n_0$. It follows that

$$\begin{aligned}|\textstyle\sum_{x\in X}\Delta_p u(x)+\langle w, df_n\rangle| &= |\textstyle\sum_{x\in X}[\Delta_p u(x)][1-f_n(x)]|\\ &\le \textstyle\sum_{x\in X'}|\Delta_p u(x)||1-f_n(x)|+\sum_{x\in X-X'}|\Delta_p u(x)|\\ &\le \textstyle\sum_{x\in X'}|\Delta_p u(x)|\varepsilon/c+\varepsilon<2\varepsilon\end{aligned}$$

for all $n\ge n_0$. Therefore $\sum_{x\in X}\Delta_p u(x)=0$.

COROLLARY. *Let N be of parabolic type of order p. Then $PSD^{(p)}(\mu)=\emptyset$ for every nonzero $\mu\in L^+(X)$.*

As a generalization of Proposition 2.1, we have

THEOREM 4.3. *If $\mu\in L_0(X)$ satisfies $\sum_{x\in X}\mu(x)=0$, then $PSD_0^{(p)}(\mu)\neq\emptyset$. Therefore, $KL_q(N)\supset\{\mu\in L_0(X); \sum_{x\in X}\mu(x)=0\}$.*

PROOF. Let $\mu\in L_0(X)$ and $A=\{x\in X; \mu(x)\neq 0\}$ and take $b\notin A$. Define $\dot{w}_b\in L(Y)$ by

$$\dot{w}_b(y) = -\sum_{x\in A}\mu(x)p_x(y),$$

where p_x is the path index of a path P_x from b to x $(x\neq b)$. Observing that p_x is a flow from b to x with unit strength, i.e.,

$$\sum_{y\in Y} K(z, y)p_x(y) = -\varepsilon_b(z)+\varepsilon_x(z) \text{ on } X,$$

we have

$$\sum_{y\in Y} K(z, y)\dot{w}_b(y) = -\sum_{x\in A}\mu(x)\sum_{y\in Y} K(z, y)p_x(y) = -\mu(z).$$

Since $H_q(p_x)=\sum_{P_x} r(y)<\infty$ and A is a finite set, we conclude that $\dot{w}_b$ is a solution of $(\mathrm{FP}(\mu))_q$ and $PSD_0^{(p)}(\mu)\neq\emptyset$ by Theorem 3.2.

THEOREM 4.4. *Let μ, ν, $\sigma\in L(X)$. If μ, $\nu\in KL_q(N)$ and if $\mu\le\sigma\le\nu$ on X, then $\sigma\in KL_q(N)$.*

PROOF. There exist $\tilde{w}_\mu$, $\tilde{w}_\nu \in L_q(Y; r)$ such that $\partial\tilde{w}_\mu = \mu$ and $\partial\tilde{w}_\nu = \nu$. For every nonnegative $f \in L_0(X)$, we have by (3.1)

$$\begin{aligned}\textstyle\sum_{x\in X} f(x)\sigma(x) &\le \textstyle\sum_{x\in X} f(x)\nu(x) \\ &= \textstyle\sum_{x\in X} f(x)[\partial w_\nu(x)] = -\langle \tilde{w}_\nu, df\rangle, \\ \textstyle\sum_{x\in X} f(x)\sigma(x) &\ge \textstyle\sum_{x\in X} f(x)\mu(x) = -\langle \tilde{w}_\mu, df\rangle.\end{aligned}$$

For any $f \in L_0(X)$, we have $f = f^+ - f^-$ with $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$, so that

$$\begin{aligned}\textstyle\sum_{x\in X} f(x)\sigma(x) &= \textstyle\sum_{x\in X} f^+(x)\sigma(x) - \sum_{x\in X} f^-(x)\sigma(x) \\ &\le -\langle \tilde{w}_\nu, df^+\rangle + \langle \tilde{w}_\mu, df^-\rangle \\ &\le [H_q(\tilde{w}_\nu)]^{1/q}[D_p(f^+)]^{1/p} + [H_q(\tilde{w}_\mu)]^{1/q}[D_p(f^-)]^{1/p}.\end{aligned}$$

Since $D_p(f^+) \le D_p(f)$ and $D_p(f^-) \le D_p(f)$, we have

$$|\textstyle\sum_{x\in X} f(x)\sigma(x)| \le M[D_p(f)]^{1/p} = M[H_p(df)]^{1/p},$$

where $M = [H_q(\tilde{w}_\nu)]^{1/q} + [H_q(\tilde{w}_\mu)]^{1/q}$. Therefore, the linear functional Φ on the linear subspace $dL_0(X) = \{df; f \in L_0(X)\}$ of $L_p(Y; r)$ defined by

$$\Phi(df) = \textstyle\sum_{x\in X} f(x)\sigma(x)$$

is continuous. Here we note that d is a one-to-one mapping from $L_0(X)$ to $dL_0(X)$. By the well-known Hahn-Banach's theorem, there exists a continuous linear functional $\tilde{\Phi}$ on $L_p(Y; r)$ such that $\tilde{\Phi}(df) = \Phi(df)$ for all $df \in dL_0(X)$. The dual space of $L_p(Y; r)$ is isometric to $L_q(Y; r)$, so there exists $\hat{w} \in L_q(Y; r)$ such that $\tilde{\Phi}(w) = \langle \hat{w}, w\rangle$ for every $w \in L_p(Y; r)$. It follows that

$$-\textstyle\sum_{x\in X} f(x)[\partial\hat{w}(x)] = \langle \hat{w}, df\rangle = \sum_{x\in X} f(x)\sigma(x)$$

for every $f \in L_0(X)$, and hence $\partial\hat{w} = -\sigma$. Put $\tilde{w}_\sigma = -\hat{w}$. Then $\tilde{w}_\sigma \in L_q(Y; r)$ and $\partial\tilde{w}_\sigma = \sigma$, and hence $\sigma \in KL_q(N)$.

COROLLARY. *Assume that* $\boldsymbol{PSD}_0^{(p)}(\mu) \neq \emptyset$ *and* $\boldsymbol{PSD}_0^{(p)}(\nu) \neq \emptyset$. *If* $\sigma \in L(X)$ *and if* $\mu \le \sigma \le \nu$ *on* X, *then* $\boldsymbol{PSD}_0^{(p)}(\sigma) \neq \emptyset$.

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