Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
40 (2007), pp. 7–13

EQUILIBRIUM MEASURES ON AN INFINITE NETWORK OR A TREE

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Communicated by M. Kimura

(Received: December 6, 2005)

Abstract. The equilibrium problem and the condenser problem have been solved in finite networks. This note considers such problems in the context of infinite networks and trees. The basic tool employed is the solution of a Dirichlet problem for an arbitrary subset of an infinite network.

1. Introduction

An electrical network (V, E, c) is a simple finite connected graph, with vertex set V and edge set E in which each edge is assigned a conductance $c(x, y) > 0$. The Laplacian L of the network is a matrix of finite order whose entries are $L(x, y) = -c(x, y)$ if $x \neq y$ and $L(x, x) = c(x) = \sum_{y \in V} c(x, y)$. For a function $u(x)$ on V, define $Lu(x) = \sum_{y \in V} c(x, y)[u(x) - u(y)]$.

Considering the Laplacian as a kernel on the vertex set and using the energy principle and the maximum principle for L, Bendito et al. [3] show that L satisfies the equilibrium principle for the kernel L: For every proper set $F \subset V$, there exists a unique $u \geq 0$ on V (called the equilibrium measure for F) such that $u > 0$ on F, $u = 0$ on $V \setminus F$ and $Lu = 1$ on F.

As a consequence, the Green function of the Dirichlet problem and the Poisson problems and the solution of the condenser problem are obtained solely in terms of the equilibrium measures for suitable subsets.

Our intention is to consider these questions in the context of a (Yamasaki) infinite network [8] or a (Cartier) tree [5]. We first obtain the solution for a version of the Dirichlet problem which, along with the results in Yamasaki [8], is useful in constructing the equilibrium measures of subsets that are not necessarily finite. Also, keeping this Dirichlet solution as the basis, we extend the results proved for the finite case in Bendito et al. [3] to the infinite network. Two of these results are the condenser principle for finite or infinite subsets and the construction of a

2000 *Mathematics Subject Classification.* 31C20.
Key words and phrases. Dirichlet problem and equilibrium measures in infinite networks.

function $v_{x,y}(z)$ which, in the context of random walks, gives the probability that if the walk starts at z it will reach x before it reaches y.

2. Preliminaries

Let $N = \{X, Y, k, r\}$ be an infinite network in the sense of Yamasaki [8], that is connected, locally finite and has no self-loop. Two vertices x and a of N are said to be neighbours, $x \sim a$, if there is an arc in Y connecting x and a. For a vertex x, denote $W(x) = \{a : a \sim x\}$, $x \in W(x)$. For each pair of vertices x and a in X, $t(x, a) \geq 0$ is defined by that $t(x, a) = 0$ if and only if x and a are not neighbours, also $t(x, a) = t(a, x)$ and $t(a, a) = 0$. Then, for a real-valued function $u(x)$ defined on X, the Laplacian of u is defined as $\Delta u(x) = \sum_{a\in X} t(x, a)[u(a) - u(x)]$.

Suppose that u is a real-valued function defined on a subset A of X. Let us say that a is an interior point of A, $a \in \overset{o}{A}$, if and only if $x \in A$ for every $x \sim a$. Write $\partial A = A \setminus \overset{o}{A}$. Then u is said to be harmonic (respectively superharmonic) on A if and only if $\Delta u(a) = 0$ (respectively $\Delta u(a) \leq 0$) for all $a \in \overset{o}{A}$.

A Cartier tree T is an infinite connected graph, locally finite and without any circuits. (A circuit means a path $\{x, x_1, \ldots, x_n, x\}$ connecting a vertex x to itself, with $n \geq 2$.) A vertex x is said to be terminal if it has only one neighbour in T. A transition probability is given to T: that is, with any two vertices x and y is associated a real number $p(x, y) \geq 0$ such that $p(x, y) > 0$ if and only if $x \sim y$ and $\sum_{y\in T} p(x, y) = 1$ for any $x \in T$. Note that $p(x, y)$ and $p(y, x)$ may not be the same. For a real function $u(x)$ on T, the Laplacian of u is defined as $\Delta u(x) = \sum_{y\in T} p(x, y)[u(y) - u(x)]$.

On a tree T, fix a vertex e. For any $x \in T$, define

$$\phi(x) = \frac{p(e, x_1)p(x_1, x_2)\ldots p(x_n, x)}{p(x, x_n)p(x_n, x_{n-1})\ldots p(x_1, e)},$$

where $\{e, x_1, \ldots, x_n, x\}$ is a path connecting e and x. Note that $\phi(x)$ is independent of the path chosen and depends only on x; also note that for any pair of vertices x and y, $\phi(x)p(x, y) = \phi(y)p(y, x)$. Now, define the conductance $t(x, y) = \phi(x)p(x, y)$ on T. With this conductance, T becomes a connected infinite network in the framework of Yamasaki [8], but without circuits.

Let Δ_T and Δ_N denote the Laplacian on T, when T is considered as a tree and when T is considered as an infinite network respectively. Then for any $u(x)$ on T,

$$\begin{aligned} \Delta_N u(x) &= \sum t(x, y)[u(y) - u(x)] \\ &= \phi(x) \sum p(x, y)[u(y) - u(x)] \\ &= \phi(x)\Delta_T u(x). \end{aligned}$$

Note that $\phi(x) > 0$ for every $x \in T$ (take $\phi(e) = 1$). Hence, the fact that a function $u(x)$ is harmonic or superharmonic on T does not depend on whether T is considered as a tree or as a network.

In a network N, the existence of a positive superharmonic function that is not harmonic implies that given any $y \in N$, there exists the Green potential $G_y(x)$ on N such that $\Delta G_y(x) = -\delta_y(x)$, $G_y(x) \leq G_y(y)$ for every $x \in N$.

Let u be a real-valued function on a finite set F in a network. The inner normal derivative of u with respect to F at a vertex $s \in \partial F$ is defined [1] as

$$\frac{\partial u}{\partial n^-}(s) = \sum_{x \in F} t(s,x)[u(x) - u(s)].$$

Theorem 1 (**Green's formula:** Bendito et al. [2], Urakawa [7]). *Let u and v be real-valued functions on a finite set F in a network. We set $(u,v)_F = \frac{1}{2}\sum_{x,y \in F} t(x,y)[u(y) - u(x)][v(y) - v(x)]$. Then*

$$\sum_{x \in \overset{o}{F}} u(x)\Delta v(x) + (u,v)_F = -\sum_{s \in \partial F} u(s)\frac{\partial v}{\partial n^-}(s). \tag{1}$$

$$\sum_{x \in \overset{o}{F}} [u(x)\Delta v(x) - v(x)\Delta u(x)] = -\sum_{s \in \partial F}\left[u(s)\frac{\partial v}{\partial n^-}(s) - v(s)\frac{\partial u}{\partial n^-}(s)\right]. \tag{2}$$

Usually, Equation 1 is called Green's formula I and Equation 2 is called Green's formula II.

Theorem 2 (**Minimum principle).** *Let F be a finite set in a network N. Let u be superharmonic on F. Then $\inf_{z \in \partial F} u(z) = \inf_{x \in F} u(x)$.*

Proof. Let $\inf_{x \in F} u(x) = \beta$, and $\inf_{z \in \partial F} u(z) = \alpha$. Suppose that $\beta < \alpha$. Then at some point $x \in \overset{o}{F}$, $u(x) = \beta$.

Let $z \notin F$. Connect x and z by an arc $\{x, x_1, \ldots, x_n, z\}$. (It is possible, since N is connected.) Let i be the smallest index such that $x_i \in \overset{o}{F}$ and $x_{i+1} \notin \overset{o}{F}$. Since $x_i \in \overset{o}{F}$ and $x_{i+1} \sim x_i$, we deduce that $x_{i+1} \in F$. This implies that $x_{i+1} \in \partial F$. Now, since $\Delta u(x) \leq 0$ and since $u(x) = \beta$ is a minimum value, $u \equiv \beta$ on $W(x)$, which is the set of all neighbours of x including x. This implies that $u(x_1) = \beta$. Proceeding thus, we show that $u(x_i) = \beta$. But $u(x_{i+1}) \geq \alpha$. This is a contradiction, since $\Delta u(x_i) \leq 0$ and $u(x_i) = \beta$, so that $u(x_{i+1})$ should be β; that is, $\alpha \leq \beta$, a contradiction. □

3. A version of the Dirichlet solution

We start with a generalized version of Theorem 2.2 in [1].

Theorem 3. *Let E be a (finite or infinite) set of vertices in a network N. Let $F \subset \overset{o}{E}$. Let $f \geq 0$ be a function on $E \setminus F$. Suppose that there exists a function $s \geq 0$ on E such that $s \geq f$ on $E \setminus F$and $\Delta s \leq 0$ on F. Then there exists a function $g \geq 0$ on E such that*

i): *$\Delta g \equiv 0$ on F, and*

ii): *$g = f$ on $E \setminus F$.*

Moreover, g is the smallest of the all non-negative functions satisfying the conditions (i) and (ii). (We denote g by $H_f^{E,F}$)

Proof. Let $\mathcal{F}$ be the family of non-negative functions u on E such that $u \equiv f$ on $E \setminus F$ and u is superharmonic on F. Note that if

$$u_0 = \begin{cases} f & \text{on} \quad E \setminus F \\ s & \text{on} \quad F, \end{cases}$$

then $u_0 \in \mathcal{F}$. Let $y \in F$ and $u \in \mathcal{F}$. Then $t(y)u(y) \geq \sum t(y,z)u(z)$.

Define

$$u_1(x) = \begin{cases} u(x) & \text{if} \quad x \in E \setminus \{y\} \\ \frac{1}{t(y)} \sum t(y,z)u(z) & \text{if} \quad x = y. \end{cases}$$

Then $u_1 \in \mathcal{F}$, $u_1 \leq u$ on E and $\Delta u_1(y) = 0$. Taking into account the arbitrariness of y in F and u in $\mathcal{F}$, if we define $g(x) = \inf_{u \in \mathcal{F}}$, then we can see that $\Delta g \equiv 0$ on F and $g = f$ on $E \setminus F$. If p is another such positive function satisfying i) and ii), then $p \in \mathcal{F}$ and hence $g \leq p$. □

Corollary 4 (Dirichlet solution)**.** *Let E be a finite set. Given a function f on ∂E, there exists a unique function g on E such that $g = f$ on ∂E and $\Delta g = 0$ on $\overset{o}{E}$.*

Proof. Since we can treat f^+ and f^- separately, we can assume that $f \geq 0$. Since E is finite, for some $\alpha > 0$, $f \leq \alpha$ on E and $\Delta \alpha = 0$ on $\overset{o}{E}$. Hence by the theorem above, there exists a function $g \geq 0$ on E such that $g = f$ on ∂E and g is harmonic on $\overset{o}{E}$.

The uniqueness of the solution follows from the minimum principle. □

Corollary 5 (Reduced functions)**.** *Let $s \geq 0$ be a superharmonic function on N. Let A be a finite or infinite set. Then there exists a superharmonic function $R_s^A \geq 0$ on N such that $R_s^A = s$ on A, R_s^A is harmonic on $(N \overset{o}{\setminus} A)$ and $R_s^A \leq s$ on N.*

Proof. Let $E = N \setminus A$. Define the function

$$R_s^A = \begin{cases} s & \text{on} \quad A \\ H_s^{E,\overset{o}{E}} & \text{on} \quad E. \end{cases}$$

Then R_s^A has all the properties stated in the corollary. □

Remarks. 1. R_s^A is the smallest positive superharmonic function on N that majorizes s on A.

2. In the classical potential theory in $\mathbf{R}^n$, a function similar to R_s^A is called the reduced function of s on A (See Brelot [4, p.33]).

From now on, let us assume that there exist Green potentials on the network N. Then, as shown in Yamasaki [8], for any vertex a, there exists a unique Green potential $G_a(x) > 0$ such that $\Delta G_a(x) = -\delta_a(x)$; also $G_a(x) \leq G_a(a)$ for any $x \in N$.

We say that a real-valued function u defined on a subset A of X is a δ-superharmonic function on A if and only if there exist two superharmonic functions s_1 and s_2 on A such that $u = s_1 - s_2$ on A.

Theorem 6. *Let F be a set of vertices in a network N such that $\sum_{a\in F} G_a(a) < \infty$. Let f be a bounded function on F. Then there exists a function $u(x)$ on N such that u is δ-superharmonic on F, $\Delta u = -f$ on F and $u \equiv 0$ on $N \setminus F$. Moreover $u \geq 0$ if $f \geq 0$.*

Proof. Let $E = W(F) = \bigcup_{x\in F} W(x)$. For every $a \in F$, let $h_a(x)$ be the extended Dirichlet solution (as in Theorem 3) with values $G_a(x)$ on $E \setminus F$ and harmonic on F. Then $g_a(x) = G_a(x) - h_a(x)$ in E is such that

$$\begin{aligned} g_a(x) &= |g_a(x)| \\ &\leq |G_a(x)| + |h_a(x)| \\ &\leq G_a(a) + G_a(a). \end{aligned}$$

(Remark that for a, b in F, $g_a(b) = g_b(a)$ by using Green's formula II above.)

Given the bounded function f on F, write $f = f^+ - f^-$. Let

$$u_1(x) = \begin{cases} \sum_{a\in F} f^+(a) g_a(x) & \text{if} \quad x \in F \\ 0 & \text{if} \quad x \in F^c, \end{cases}$$

and

$$u_2(x) = \begin{cases} \sum_{a\in F} f^-(a) g_a(x) & \text{if} \quad x \in F \\ 0 & \text{if} \quad x \in F^c. \end{cases}$$

Then, $u_1(x) \geq 0$ on N and u_1 is superharmonic on F; similarly for u_2 also.

Let $u(x) = u_1(x) - u_2(x)$ on N. Then $u = 0$ on $N \setminus F$ and u is δ-superharmonic on F such that $\Delta u = -f$ on F. □

Remark. In contrast to the above theorem, we can prove for a Cartier tree (with or without Green potentials) the following : Given an arbitrary real function f on an arbitrary subset F of T, there exists a δ-superharmonic u on T such that $\Delta u = -f$ on F and $\Delta u = 0$ on $T \setminus F$.

Corollary 7. *Let F be a finite set of vertices in a network N. Then given a real function f on F there exists a unique function $u(x)$ on N such that $\Delta u = -f$ on F and $u = 0$ on F^c. Further $u \geq 0$ if $f \geq 0$, and if $f > 0$ on F, then $u > 0$ on F.*

Proof. Take $E = W(F) = \cup_{x\in F} W(x)$, which is the smallest set containing F and all the neighbours of the vertices in F. Remark that $F \subset \overset{o}{E}$. Then the above theorem establishes the existence of u. For the uniqueness, we proceed as follows: Suppose that v is another such function. Let $h = u - v$. Then $\Delta h = 0$ on F and $h = 0$ on F^c.

i) Suppose that $\overset{o}{E} = F$. Then h is a harmonic function on the finite set $\overset{o}{E}$, such that $h = 0$ on ∂E. Hence by the minimum principle (Theorem 2), $h \equiv 0$.

ii) Suppose that F is a proper subset of $\overset{o}{E}$, in which case a direct application of Theorem 2 is not possible. Write $F = \cup F_i$ where each F_i is connected and $F_i \cap F_j = \phi$. Then $W(F) = \cup W(F_i)$. Write $E_i = W(F_i)$. If $\overset{o}{E_i} = F_i$, then by (i), $h \equiv 0$ on F_i. Let F_i be a proper subset of $\overset{o}{E_i}$. Suppose that h takes a positive value on F_i. Then $h \equiv M$ on F_i, where $M = \max_{F_i} h$. Let $z \in \overset{o}{E_i} \setminus F_i$ and $z \sim y$, where $y \in F_i$. Then $h(y) = M$, $h(z) = 0$ (since $z \in E \setminus F$) and $\Delta h(y) = 0$, a contradiction. Hence h does not take positive values on F_i; similarly h does not take negative values on F_i. Hence $h \equiv 0$ on F_i. (See Yamasaki [8, Lemma 2.1].)

Finally, by the construction of u (as in the proof of Theorem 6), we note that $u \geq 0$ if $f \geq 0$. Suppose that $f > 0$ on F, and $u(a) = 0$ for some $a \in F$. Then,

$$\begin{aligned} -f(a) = \Delta u(a) &= \sum t(a,x)[u(x) - u(a)] \\ &= \sum t(a,x)u(x) \geq 0, \end{aligned}$$

a contradiction. Consequently, if $f > 0$ on F, then $u > 0$ on F. □

Note. In particular, the above corollary contains the following equilibrium principle with respect to the Laplacian kernel [3, Proposition 2.3]: Let F be an arbitrary finite set on a network N. Then there exists a unique function $u_F \geq 0$ on N such that $u_F > 0$ on F, $u_F = 0$ on $N \setminus F$ and $\Delta u_F = -1$ on F. u_F is called the equilibrium measure for F.

Combining the above results we have the following Dirichlet-Poisson solution. (See [3, Section 3] on Dirichlet and Poisson problems for finite networks.)

Theorem 8. *Let F be a subset of vertices in an infinite network N, such that $\sum_{a\in F} G_a(a) < \infty$. Let f and g be real-valued functions on N, g bounded on F and f bounded on $W(F)$. Then there exists a function u on N such that $\Delta u = -g$ on F and $u = f$ on F^c.*

Proof. Let $E = W(F)$. Then $F \subset \overset{o}{E}$. Since f is bounded on E, there exists (Theorem 3) a function s on E such that $\Delta s = 0$ on F and $s = f$ on $E \setminus F$. Extend s by f outside F.

Since g is bounded on F, from Theorem 6 there exists a function t on N such that $\Delta t = -g$ on F and $t = 0$ on F^c. Write $u = s + t$. Then u is a real-valued function on N such that $\Delta u = -g$ on F and $u = f$ on F^c. □

In Bendito et al. [3, Proposition 3.3] prove the condenser principle in a finite network, using the Poisson kernel. In an infinite network, this principle can be stated as follows:

Theorem 9. *Let A and B be arbitrary disjoint subsets of an infinite network N. Then, there exists a function ϕ on N such that (i) $0 \leq \phi \leq 1$ on N and ϕ is harmonic on $(N \setminus (A \cup B))^o$, (ii) on A, $\phi = 1$ and $\Delta\phi \leq 0$, and (iii) on B, $\phi = 0$ and $\Delta\phi \geq 0$. Moreover, if u is any other function satisfying i), ii) and iii), then $\phi \leq u$ on N.*

Proof. Take $f = \begin{cases} 0 & \text{on} \quad B \\ 1 & \text{on} \quad A. \end{cases}$ Let $E = N \setminus (A \cup B)$. Then $\partial E \subset A \cup B$. By Theorem 3, there exists a function g on E such that g is harmonic on $\overset{o}{E}$ and $g = f$ on $\partial E = E \setminus \overset{o}{E}$. Define

$$\phi(x) = \begin{cases} g(x) & \text{if} \quad x \in E \\ f(x) & \text{if} \quad x \in A \cup B. \end{cases}$$

Then, $0 \leq \phi \leq 1$ on N, $\phi(x)$ is harmonic on $\overset{o}{E}$, $\Delta\phi(x) \leq 0$ on A and $\Delta\phi(x) \geq 0$ on B. (With these properties, ϕ is uniquely determined if E is a finite set.) In any event, if u is any other function satisfying i) , ii) and iii), by the construction $\phi \leq u$ on N. $\square$

Remarks. 1) In the above theorem, if $B = \emptyset$ and A is finite, the solution ϕ is the same as the reduced function R_1^A (See Corollary 5), which defines the capacitary potential of A; and the capacity of A is defined as $-\sum_{x\in N} \Delta R_1^A(x)$. The capacity of A can be defined (using Green's formula I, see Section 2) also as $\sum_{s\in\partial F} \frac{\partial R_1^A}{\partial n^-}(s)$ where F is any finite set such that $\overset{o}{F} \supset A$. (See Kellogg [6, p. 330] and Brelot [4, p. 52] for similar notions in the classical potential theory.)

2) In the above theorem, if $A = \{y\}$ and $B = \{z\}$ such that $G_z(y) < G_z(z)$, then on N, $\phi(x) \leq \frac{G_z(x)-G_z(z)}{G_z(y)-G_z(z)}$, where $G_z(x)$ is the Green function with pole at $\{z\}$. For, in Theorem 3, take $E = N$. Then $\overset{o}{E} = N$. Let $F = N \setminus \{y, z\}$. Let $f(y) = 0$ and $f(z) = 1$. Define $u(x) = \frac{G_z(z)-G_z(x)}{G_z(z)-G_z(y)}$. Since $G_z(z) \geq G_z(x)$ for all $x \in N$, we note that $u \geq 0$, $u = f$ on $E \setminus F$ and $\Delta u = 0$ on F. Then $u(x) \geq \phi(x)$ since $\phi(x)$ is the smallest such function. Clearly $0 \leq \phi(x) \leq 1$. Such a function $\phi(x)$ is useful in the context of random walks.

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