Mem. Fac. Sci. Shimane Univ.,
23, pp. 57–71 Dec. 20, 1989

On the delay-differential equation $\dot{N}(t) = N(t)(a - bN(t-1) - cN(t-2))$

Dedicated to Professor Miyuki Yamada on his 60th birthday

Tetsuo FURUMOCHI

Department of Mathematics, Shimane University, Matsue, Japan
(Received September 6, 1989)

Boundedness and asymptotic behavior of solutions and existence of periodic solutions of the scalar generalized logistic equation $\dot{N}(t) = N(t)(a - bN(t-1) - cN(t-2))$ are discussed. In particular, we show partial global uniform asymptotic stability of the constant solution $N(t) = a/(b+c)$, and existence of nontrivial periodic solutions by using a Hopf bifurcation and a fixed point theorem for a closed convex set.

§1. Introduction

Recently, Seifert [8] has obtained certain results concerning boundedness and asymptotic behavior of solutions and existence of nontrivial periodic solutions of the scalar generalized logistic equation

$$\dot{N}(t) = N(t)(a - bN(t) - N(t-1)), \tag{1}$$

which arises in population dynamics. Here the superposed dot denotes the right-hand derivative, a and b are positive constants. In [8], concerning existence of periodic solutions, it is shown that (1) has nontrivial periodic solutions for a fixed $b\,(0 < b < 1)$ and a near $a_0(b)\ (= \sqrt{(1+b)/(1-b)}\,\mathrm{Cos}^{-1}(-b))$ by using a Hopf bifurcation. Moreover in [3], the author has shown that if $a > a_0(b)$, then (1) has nontrivial periodic solutions by using fixed point techniques established and developed by Jones [4], Nussbaum [7], Chow and Hale [2], and others.

On the other hand, for differential equations with two time delays, corresponding results seem to be comparatively few in spite of their similarities to (1). In fact, concerning existence of nontrivial periodic solutions of the scalar equations

$$\dot{x}(t) = -(1 + x(t))(bx(t-1) + cx(t-2)), \tag{2}$$

$$\dot{x}(t) = -(1 - x^2(t))(bx(t-1) + cx(t-2)), \tag{3}$$

corresponding results are almost nonexistent for (2), though a few results are obtained in [4, 6] for (3) with $b = c$.

In this paper, for the several results in [3, 8] on boundedness and asymptotic

This paper contains partly the results presented at the US-Japan Seminar on Dynamical System, Kyoto, July 1989.

behavior of solutions and existence of nontrivial periodic solutions of (1), we obtain certain corresponding results for the scalar equation

$$\dot{N}(t) = N(t)(a - bN(t-1) - cN(t-2)), \tag{4}$$

where a, b, and c are positive constants, and we are concerned with solutions of (4) such that their initial functions $N_0(t)$ are defined and positive continuous on $[-2, 0]$. If we put $x(t) = N(t) - a/(b+c)$ for a solution $N(t)$ of (4), we obtain from (4) the equation equivalent to (4):

$$\dot{x}(t) = -\left(x(t) + \frac{a}{b+c}\right)(bx(t-1) + cx(t-2)). \tag{5}$$

The zero solution of (5) corresponds to the constant solution $N(t) = a/(b+c)$ of (4). Here we note that (2) is a special case of (5) with $a = b + c$.

There are various methods and many results for boundedness ans asymptotic behavior of solutions and existence of periodic solutions of functional differential equations [cf. 1–9]. In §2, we obtain some results on boundedness and asympotic behavior of solutions of (4). In particular, we show partial global uniform asymptotic stability of the constant solution $N(t) = a/(b+c)$ of (4) by employing a few results obtained by Yoneyama and Sugie [9]. In §3, we show existence of nontrivial periodic solutions by using a Hopf bifurcation and a fixed point theorem for a closed convex set.

Let R denote the interval $-\infty < t < \infty$, and let C be the Banach space of continuous functions $\phi: [-2, 0] \to R$ with the uniform norm $|\phi| = \sup_{-2<\theta<0} |\phi(\theta)|$. For any $\gamma > 0$, let $C(\gamma) = \{\phi \in C : |\phi| < \gamma\}$ and $S_\gamma = \{\phi \in C : |\phi| = \gamma\}$. For any continuous function $x(s)$ defined on $-2 \leqq s < T (0 < T \leqq \infty)$, and any fixed $t (0 \leqq t < T)$, $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$, $-2 \leqq \theta \leqq 0$.

§2. Boundedness and asymptotic behavior of solutions

In this section, we discuss boundedness and asymptotic behavior of solutions of (4). For given positive continuous functions $\psi_1(t)$ and $\psi_2(t)$ on $[0, 1]$ with $\psi_1(1) = \psi_2(0)$, define

$$\psi_{k+2}(t) = \psi_{k+1}(1)\exp(at - \int_0^t (b\psi_{k+1}(s) + c\psi_k(s))ds), \ 0 \leqq t \leqq 1,$$

$k = 1, 2, \cdots$. Then the solution $N(t)$ of (4) for $t \geqq -2$ such that $N(t) = \psi_1(t+2)$, $-2 \leqq t \leqq -1$ and $N(t) = \psi_2(t+1)$, $-1 \leqq t \leqq 0$ is given by

$$N(t + k - 3) = \psi_k(t), \ 0 \leqq t \leqq 1, \quad k = 1, 2, \cdots. \tag{6}$$

Corresponding to Theorems 1–3 in [8], we obtain the following theorem on boundedness of solutions of (4).

THEOREM 1. *Let $N(t)$ be any solution of* (4) *with $N(t) = N_0(t) > 0$, $-2 \leqq t \leqq 0$. Then*

(i) $0 < N(t) < max\{N(0),\ a/(b+c)\}\, e^{2a}$ *for* $t \geqq 0$,

(ii) *for any $\alpha > a/(b+c)$, there exists $T = T(\alpha) \geqq 0$ such that $N_0(t) \leqq \alpha$, $-2 \leqq t \leqq 0$ implies $0 < N(t) < ae^{2a}/(b+c)$ for $t \geqq T(\alpha)$.*

PROOF. It is clear that $N(t) > 0$ for $t \geqq 0$. First of all, we show that

$$N(t_0) = \frac{a}{b+c} \text{ implies } 0 < N(t) < \frac{ae^{2a}}{b+c}, \quad t \geqq t_0, \tag{7}$$

where $t_0 \geqq 0$. Suppose that $N(t_1) = ae^{2a}/(b+c)$ for some $t_1 > t_0$ and $N(t) < ae^{2a}/(b+c)$ on $[t_0, t_1)$. First we show that $t_1 - t_2 \leqq 2$ for $t_2 = \sup\{t \in [t_0, t_1] : N(t) = a/(b+c)\}$. If $t_1 - t_2 > 2$, then we have $\dot{N}(t) < aN(t)$ for $t_2 \leqq t \leqq t_2 + 2$, which implies

$$N(t_2 + 2) < N(t_2)e^{2a} = \frac{ae^{2a}}{b+c}.$$

Moreover, $N(t) \geqq a/(b+c)$ for $t_2 \leqq t \leqq t_1$ implies $\dot{N}(t) \leqq 0$ for $t_2 + 2 \leqq t \leqq t_1$. Thus we obtain

$$N(t_1) \leqq N(t_2 + 2) < \frac{ae^{2a}}{b+c},$$

which contradicts the choice of t_1. Hence we have $t_1 - t_2 \leqq 2$ and

$$N(t_1) < N(t_2)\exp((t_1 - t_2)a) \leqq \frac{ae^{2a}}{b+c},$$

which contradicts the choice of t_1 again. Thus (7) holds.

(i) First we consider the case $N(0) \leqq a/(b+c)$. Suppose that $N(t_3) = a/(b+c)$ for some $t_3 \geqq 0$, and let $t_4 = \inf\{t \geqq 0 : N(t) = a/(b+c)\}$. Then (7) with $t_0 = t_4$ implies (i). Next in the case $N(0) > a/(b+c)$, it is shown similarly as in the proof of (7) that (i) holds.

(ii) It is sufficient to prove in the case $\alpha > ae^{2a}/(b+c)$. Suppose that the conclusion is false. Then there exists $\alpha_0 > ae^{2a}/(b+c)$, and for $T = 4 + (\ln((b+c)\alpha_0/a))/a(e^{2a} - 1)$, there exist a solution $N_*(t)$ of (4) and $t_* \geqq T$ such that $0 < N_*(t) \leqq \alpha_0$ for $-2 \leqq t \leqq 0$ and $N_*(t_*) = ae^{2a}/(b+c)$. From (7), we have $N_*(t) > a/(b+c)$ for $0 \leqq t < t_*$. Thus we obtain $\dot{N}_*(t) < 0$ for $t \geqq 2$, which implies $N_*(t) > ae^{2a}/(b+c)$ for $2 \leqq t < t_*$. Hence we have $\dot{N}_*(t) < (1 - e^{2a})aN_*(t)$ for $4 \leqq t < t_*$. Morever, since $N_*(4) < \alpha_0 e^{2a}$ from (i), we obtain

$$\left.\begin{aligned} N_*(t) &< N_*(4)\exp(a(1 - e^{2a})(t-4)) \\ &< \alpha_0 e^{2a}\exp(a(1 - e^{2a})(t-4)), \end{aligned}\right\} \quad 4 \leqq t \leqq t_*,$$

which yields a contradiction such that $N_*(t_*) < \alpha_0 e^{2a} \exp(a(1 - e^{2a})(t_* - 4)) \leqq ae^{2a}/(b + c) = N_*(t_*)$. Thus (ii) holds.

The next theorem is related to Theorem 4 in [8]. This result means that we can make the value of $N(2)$ arbitrarily small by taking the initial function sufficiently large.

THEOREM 2. *For any $\varepsilon > 0$ and any solution $N(t)$ of* (4) *such that $N(t) = N_0(t) > 0, \ -2 \leqq t \leqq 0$,*

$$b\int_{-1}^{0} N_0(s)\,ds + c\int_{-2}^{0} N_0(s)\,ds \geqq 2a + \ln\left(\frac{N_0(0)}{\varepsilon}\right) \tag{8}$$

implies $0 < N(2) < \varepsilon$.

PROOF. By using (6) with $k = 3$ for $\psi_1(t) = N_0(t-2)$, $\psi_2(t) = N_0(t-1)$, $0 \leqq t \leqq 1$, we have

$$N(1) = \psi_3(1) = N_0(0)\exp\left(a - \int_0^1 (b\psi_2(s) + c\psi_1(s))\,ds\right).$$

From this, (6) with $k = 4$, and (8), we obtain

$$\begin{aligned} N(2) = \psi_4(1) &= \psi_3(1)\exp\left(a - \int_0^1 (b\psi_3(s) + c\psi_2(s))\,ds\right) \\ &< N_0(0)\exp\left(2a - b\int_0^1 \psi_2(s)\,ds - c\int_0^1 (\psi_1(s) + \psi_2(s))\,ds\right) \leqq \varepsilon \end{aligned}$$

which proves the theorem.

Now we discuss asymptotic behavior of solutions of (4). First we prove the following theorem, which corresponds to Theorem 3 in [3].

THEOREM 3. *Let α and δ be any numbers with $0 < \delta < a/(b+c) < \alpha$. Any solution $N(t)$ of* (4) *with $\delta \leqq N(t) = N_0(t) \leqq \alpha$, $-2 \leqq t \leqq 0$, satisfies*

$$N(t) > \delta\exp(2(a - (b+c)\alpha e^{2a})), \ t \geqq 0. \tag{9}$$

PROOF. Let $\eta = \delta\exp(2(a - (b+c)\alpha e^{2a}))$. If (9) does not hold, then there exist a solution $N_*(t)$ of (4) and $t_1 > 0$ such that $\delta \leqq N_*(t) \leqq \alpha$ for $-2 \leqq t \leqq 0$, $N_*(t_1) = \eta$, and $N_*(t) > \eta$ for $0 \leqq t < t_1$, which together with Theorem 1(i) imply

$$\eta < N_*(t) < \alpha e^{2a}, \ -2 \leqq t < t_1. \tag{10}$$

Since $\dot{N}_*(t) > N_*(t)(a - (b+c)\alpha e^{2a})$ for $0 \leqq t < t_1$ from (10), we have

$$N_*(t) > N_*(0)\exp((a - (b+c)\alpha e^{2a})t) \geqq \eta, \ 0 \leqq t \leqq \min\{2, t_1\},$$

and consequently, t_1 must be greater than 2.

If $N_*(t) < a/(b+c)$ for $0 \leqq t < t_1$, then $\dot{N}_*(t) > 0$ for $2 < t < t_1$, which yields a contradiction that $N_*(t_1) > N_*(2) > \eta$. Thus $N_*(t_2) = a/(b+c)$ for some $t_2 \in [0, t_1)$. If $t_1 - t_3 > 2$ for $t_3 = \sup\{t < t_1 : N_*(t) = a/(b+c)\}$, then we have $\dot{N}_*(t) > 0$ for $t_3 + 2 \leqq t \leqq t_1$, which contradicts the choice of t_1. Hence $t_1 - t_3$ must be not greater than 2. From (10) we obtain $\dot{N}_*(t) > N_*(t)(a - (b+c)\alpha e^{2a})$ for $t_3 \leqq t \leqq t_1$. This implies

$$N_*(t_1) > N_*(t_3)\exp((a - (b+c)\alpha e^{2a})(t_1 - t_3)) > \eta,$$

Which contradicts the choice of t_1 again. Thus (9) holds.

Now we discuss partial global uniform asymptotic stability of the constant solution $N(t) = a/(b+c)$ of (4). Results on stability of solutions of equations with two time delays such as (4) seem to be not so many. In [1], stability and instability of solutions of linear equations with two time delays are discussed. Moreover in [9], results on uniform asymptotic stability of solutions of nonlinear equations with two time delays are obtained. Here we show partial global uniform asymptotic stability of the constant solution $N(t) = a/(b+c)$ of (4) by applying a few results in [9].

First we state a few known results in [9]. Consider the equation

$$\dot{x}(t) = F(t, x_t) + G(t, x_t), \tag{11}$$

where $F, G : [0, \infty) \times C(\gamma) \to R$ are continuous for a positive constant γ. Moreover suppose that there exist constants $\mu \geqq 0$, $\nu \geqq 0$, and $1 \leqq q \leqq 2$ such that for $t \geqq 0$ and $\phi \in C(\gamma)$, F and G satisfy Yorke conditions

$$-\mu M_1(\phi) \leqq F(t, \phi) \leqq \mu M_1(-\phi), \tag{12}$$

$$-\nu M_q(\phi) \leqq G(t, \phi) \leqq \nu M_q(-\phi), \tag{13}$$

where $M_p(\phi) = \max\{0, \sup_{-p<s<0} \phi(s)\}$. Clearly (12) and (13) imply $F(t, 0) \equiv G(t, 0) \equiv 0$. We denote by $x(t, t_0, \phi)$ a solution of (11) such that $x_{t_0} = \phi$.

DEFINITION 1. *The zero solution of* (11) *is uniformly stable if, for any* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for any* $t_0 \geqq 0$ *and* $\phi \in C(\delta)$, *any solution* $x(t, t_0, \phi)$ *of* (11) *is defined for all* $t \geqq t_0$ *and* $|x(t, t_0, \phi)| < \varepsilon$ *for all* $t \geqq t_0 - q$.

DEFINITION 2. *The zero solution of* (11) *is uniformly asymptotically stable if it is uniformly stable and if there exists* $\delta_0 > 0$ *such that for any* $\varepsilon > 0$, *there exists* $T = T(\varepsilon) > 0$ *such that* $t_0 \geqq 0$ *and* $\phi \in C(\delta_0)$ *imply* $|x(t, t_0, \phi)| < \varepsilon$ *for all* $t \geqq t_0 + T$.

The next result is obtained by Yoneyama and Sugie [9, Theorem 3.2].

THEOREM 4. *Suppose that* (12) *and* (13) *hold, and one of the following conditions is satisfied*:

$$\mu + \nu < \frac{1}{q},$$

$$\frac{1}{q} \leqq \mu + \nu \text{ and } \mu(\mu + \nu) + 2\nu q - \frac{\nu}{\mu + \nu} < 2,$$

$$1 \leqq \mu + \nu \text{ and } \mu + \nu q < \frac{3}{2}$$

and further suppose that for all sequences $t_n \to \infty$ *and* $\phi_n \in C(\gamma)$ *converging to a nonzero constant function in* $C(\gamma)$, $F(t_n, \phi_n) + G(t_n, \phi_n)$ *does not converge to* 0. *Then the zero solution of* (11) *is uniformly asymptotically stable.*

The next lemma is prepared for the proof of Theorem 4 (cf. [9, Lemma 3.2]).

LEMMA 1. *Suppose that* (12) *and* (13) *hold, and* μ, ν *satisfy the conditions in Theorem* 4. *If* $x(t)$ *is a solution of* (11) *on* $[t_1 - 2q, T]$ *such that* $T > t_1 + q$ *and* $x(t_1) = 0$, *then*

$$|x(t)| \leqq \theta \sup_{t_1 - 2q < s < t_1} |x(s)| \text{ for all } t \in [t_1, T],$$

where $0 \leqq \theta = \theta(\mu, \nu) < 1$.

We are now ready to prove the following theorem.

THEOREM 5. *Suppose that one of the following conditions is satisfied*:

$$ae^{2a} < \frac{1}{2},$$

$$ae^{2a} \geqq \frac{1}{2} \text{ and } ae^{2a}(abe^{2a} + 4c) < 2b + 3c,$$

$$ae^{2a} \geqq 1 \text{ and } 2ae^{2a}(b + 2c) < 3(b + c).$$

Then for any $\alpha > \delta > 0$ *and* $\varepsilon > 0$, *there exists* $T = T(\alpha, \delta, \varepsilon) > 0$ *such that for any solution* $N(t)$ *of* (4) *with* $\delta \leqq N(t) = N_0(t) \leqq \alpha$, $-2 \leqq t \leqq 0$, $|N(t) - a/(b + c)| < \varepsilon$ *for all* $t \geqq T$.

PROOF. It is sufficient to prove that for any $\alpha > a/(b + c) > \delta$ and $\varepsilon > 0$, there exists $T = T(\alpha, \delta, \varepsilon) > 0$ such that for any solution $x(t)$ of (5) with $\delta \leqq x(t) + a/(b + c) \leqq \alpha$, $-2 \leqq t \leqq 0$, $|x(t)| < \varepsilon$ for all $t \geqq T$. Equation (5) can be rewritten in the form

$$\dot{x}(t) = f(x_t) + g(x_t),$$

where $f, g: C \to R$ are defined by $f(\phi) = -(\phi(0) + a/(b + c))b\phi(-1)$ and $g(\phi) = -(\phi(0) + a/(b + c))c\phi(-2)$. For $\eta = \delta\exp(2(a - (b + c)\alpha e^{2a}))$, consider the equation

$$\dot{x}(t) = f_\eta(x_t) + g_\eta(x_t), \tag{14}$$

where $f_\eta, g_\eta : C \to R$ is defined by

$$f_\eta(\phi) = \begin{cases} -abe^{2a}\phi(-1)/(b+c), & \phi(0) > (e^{2a}-1)a/(b+c), \\ f(\phi), & \eta - a/(b+c) \leqq \phi(0) \leqq (e^{2a}-1)a/(b+c), \\ -b\eta\phi(-1), & \phi(0) < \eta - a/(b+c), \end{cases}$$

$$g_\eta(\phi) = \begin{cases} -abe^{2a}\phi(-2)/(b+c), & \phi(0) > (e^{2a}-1)a/(b+c), \\ g(\phi), & \eta - a/(b+c) \leqq \phi(0) \leqq (e^{2a}-1)a/(b+c), \\ -c\eta\phi(-2), & \phi(0) < \eta - a/(b+c). \end{cases}$$

By the choice of η, Theorem 1(ii) and Theorem 3, there exists $T_0 = T_0(\alpha) \geqq 0$ such that for any solution $x(t)$ of (5) with $\delta \leqq x(t) + a/(b+c) \leqq \alpha$, $-2 \leqq t \leqq 0$, we have

$$\eta - \frac{a}{b+c} < x(t) < \frac{(e^{2a}-1)a}{b+c}, \quad t \geqq T_0, \tag{15}$$

and consequently, $x(t)$ is a solution of (14) for $t \geqq T_0$.

Next we show uniform asymptotic stability of the zero solution of (14). It is clear that for $\mu = abe^{2a}/(b+c)$ and $\nu = ace^{2a}/(b+c)$, $F(t, \phi) = f_\eta(\phi)$ and $G(t, \phi) = g_\eta(\phi)$ satisfy (12) and (13) with $q = 2$, respectively. Thus μ and ν satisfy the conditions in Theorem 4 with $\gamma = \max\{a/(b+c) - \eta, (e^{2a}-1)a/(b+c)\}$, under the assumptions of this theorem. Further for all sequences $t_n \to \infty$ and $\phi_n \in C(\gamma)$ converging to a nonzero constant function in $C(\gamma)$, it is clear that $F(t_n, \phi_n) + G(t_n, \phi_n)$ does not converge to 0. Since Theorem 4 implies uniform asymptotic stability of the zero solution of (14), we have that there exist $\delta_0 = \delta_0(\alpha, \delta) > 0$ and $T_1 = T_1(\alpha, \delta, \varepsilon) > 0$ such that $|\phi| < \delta_0$ implies

$$|\xi(t, \phi)| < \min\left\{\frac{a}{b+c} - \eta, \frac{(e^{2a}-1)a}{b+c}\right\} \text{ for all } t \geqq 0,$$
$$|\xi(t, \phi)| < \varepsilon \text{ for all } t \geqq T_1,$$

where $\xi(t, \phi)$ denotes a solution of (15) with $\xi_0 = \phi$. This implies that if there exists $T_2 = T_2(\alpha, \delta) \geqq 2$ such that any solution $x(t)$ of (5) with $\delta \leqq x(t) + a/(b+c) \leqq \alpha$, $-2 \leqq t \leqq 0$, satisfies $|x_{t_0}| < \delta_0$ for some $t_0 \in [T_0 + 2, T_0 + T_2]$, then we obtain that for $T = T(\alpha, \delta, \varepsilon) = T_0(\alpha) + T_1(\alpha, \delta, \varepsilon) + T_2(\alpha, \delta)$, $|x(t)| < \varepsilon$ for all $t \geqq T$.

Hence we prove finally that there exists $T_2 = T_2(\alpha, \delta) \geqq 2$ such that any solution $x(t)$ of (5) with $\delta \leqq x(t) + a/(b+c) \leqq \alpha$, $-2 \leqq t \leqq 0$, satisfies $|x_{t_0}| < 0$ for some $t_0 \in [T_0 + 2, T_0 + T_2]$. Let $\tau_+ = \tau_+(\alpha, \delta)$ and $\tau_- = \tau_-(\alpha, \delta)$ be numbers such that

$$\tau_+ = \frac{(e^{2a}-1)a/(b+c) - \delta_0}{(b+c)\delta_0\eta}, \tag{16}$$

$$\tau_- = \frac{a/(b+c) - \eta - \delta_0}{(b+c)\delta_0\eta}, \tag{17}$$

and let k be a positive integer with $\gamma\theta^k < \delta_0$. Define $T_2 = T_2(\alpha, \delta) = k\tau$ for $\tau = \tau(\alpha, \delta) = \max\{\tau_+, \tau_-\} + 8$. First we prove that if $|x(t)| > 0$ on any interval $[s, \sigma]$ $(\sigma - s = \tau - 2)$ in $[T_0, T_0 + T_2]$, then we have $|x_\sigma| < \delta_0$. Since $|x(t)|$ is decreasing on $[s + 2, \sigma]$, we need only to prove that $|x(\sigma - 2)| < \delta_0$. In the case that $x(t) > \delta_0$ on $[s + 2, \sigma - 2]$, (15) implies for $s + 4 \leqq t \leqq \sigma - 2$

$$\dot{x}(t) < -\left(x(t) + \frac{a}{b + c}\right)(b + c)\delta_0 < -(b + c)\delta_0\eta,$$

and hence, we obtain from (16)

$$x(\sigma - 2) < \frac{(e^{2a} - 1)a}{b + c} - (b + c)\delta_0\eta\tau_+ = \delta_0,$$

which is a contradiction. Thus we have $x(\sigma - 2) < \delta_0$. We can similarly prove that $x(\sigma - 2) > -\delta_0$ holds from (17) in the case that $x(t) < 0$ on $[s, \sigma]$. Next divide the interval $[T_0\ T_0 + T_2]$ into k subintervals as $[T_0, T_0 + T_2] = [T_0, T_0 + \tau] \cup \cdots \cup [T_0 + (k - 1)\tau, T_0 + k\tau]$. If $|x(t)| > 0$ on $[T_0 + (j - 1)\tau, T_0 + j\tau - 2]$ for some $j (1 \leqq j \leqq k)$, then the above argument implies $|x_{T_0 + j\tau - 2}| < \delta_0$. On the other hand, if $x(t)$ has its zero point in each subinterval $[T_0 + (j - 1)\tau, T_0 + j\tau - 2]$ $(1 \leqq j \leqq k)$, then from Lemma 1 we obtain

$$|x_{T_0 + k\tau}| \leqq \gamma\theta^k < \delta_0,$$

which completes the proof.

§3. Existence of nontrivial periodic solutions

The linear part of (5) is

$$\dot{x}(t) = -\frac{a}{b + c}(bx(t - 1) + cx(t - 2)), \tag{18}$$

and the characteristic equation for (18) is

$$\frac{b + c}{a}\lambda + be^{-\lambda} + ce^{-2\lambda} = 0. \tag{19}$$

Clearly all real characteristic roots of (19) are negative. Let $a_0(b, c)$ be a function defined for $b > 0$ and $c > 0$ by

$$a_0(b, c) = \frac{(b + c)\beta}{(b + 2c\cos\beta)\sin\beta},$$

where $\pi/4 < \beta = \mathrm{Cos}^{-1}((\sqrt{b^2 + 8c^2} - b)/4c) < \pi/2$. Concerning complex characteristic roots of (19), we have:

LEMMA 2. (i) *If a satisfies*

$$a_0(b, c) < a \leqq \frac{(b+c)\pi}{2b},$$

then (19) *has a simple characteristic root* $\lambda = \alpha + i\beta$ *with* $0 < \alpha < a$ *and* $\pi/4 < \beta < \pi/2$.

(ii) *If* $a = a_0(b, c)$, *purely imaginary characteristic roots of* (19) *are* $\lambda = \pm i\beta_0$ $(\pi/4 < \beta_0 = \mathrm{Cos}^{-1}((\sqrt{b^2 + 8c^2} - b)/4c) < \pi/2)$ *and both are simple. For fixed* b *and* c, *if* $\lambda = \lambda(a) = \alpha(a) + i\beta(a)$ *is a characteristic root of* (19) *with* $\lambda(a_0(b, c)) = i\beta_0$, *then* $(d\alpha/da)(a_0(b, c)) > 0$.

(iii) *If* $a < a_0(b, c)$, *all characteristic roots of* (19) *have negative real parts.*

PROOF. (i) From the definition of $a_0(b, c)$, it is easily seen that $a_0(b, c) < (b+c)\pi/2b$. Let $h(\lambda) = (b+c)\lambda/a + be^{-\lambda} + ce^{-2\lambda}$, $\lambda = \alpha + i\beta$, and let Γ be a closed curve consisting of $\Gamma_1 : \lambda = t + i\pi/4$ $(0 \leqq t \leqq a)$, $\Gamma_2 : \lambda = a + i\theta (\pi/4 \leqq \theta \leqq \pi/2)$, Γ_3: $\lambda = t + i\pi/2$ $(0 \leqq t \leqq a)$, and $\Gamma_4 : \lambda = i\theta$ $(\pi/4 \leqq \theta \leqq \pi/2)$. Clearly $\lambda \in \Gamma_1 \cup \Gamma_2$ implies $\mathrm{Re}\, h(\lambda) > 0$. From the assumption $a \leqq (b+c)\pi/2b$, $\lambda \in \Gamma_3$ implies $h(\lambda) \neq 0$. If $\mathrm{Re}\, h(\lambda) = 0$ for some $\lambda \in \Gamma_4$, then $a > a_0(b, c)$ implies $\mathrm{Im}\, h(\lambda) < 0$, and consequently, Γ contains no zero points of $h(\lambda)$. Thus for $\omega = \arg h$, we have

$$\tan \omega = \frac{(b+c)\beta/a - be^{-\alpha}\sin\beta - ce^{-2\alpha}\sin 2\beta}{(b+c)\alpha/a - be^{-\alpha}\cos\beta + ce^{-2\alpha}\cos 2\beta}.$$

When $\lambda = t + i\pi/2$ moves on Γ_3 from $a + i\pi/2$ to $i\pi/2$, both of $\mathrm{Re}\, h(t + i\pi/2)$ and $\mathrm{Im}\, h(t + i\pi/2)$ are decreasing, $\mathrm{Re}\, h(a + i\pi/2) > 0$, $\mathrm{Im}\, h(a + i\pi/2) > 0$, $\mathrm{Re}\, h(i\pi/2) < 0$, and $\mathrm{Im}\, h(i\pi/2) \geqq 0$. These imply that there exists only one $t_0 \in (0, a)$ with $\lim_{t \to t_0 + 0} \tan \omega = \infty$. Moreover, when $\lambda = i\theta$ moves on Γ_4 from $i\pi/2$ to $i\pi/4$, $\mathrm{Re}\, h(i\theta)$ is increasing, $\mathrm{Re}\, h(i\pi/2) < 0$, and $\mathrm{Re}\, h(i\pi/4) > 0$. Thus there exists only one $\theta_0 \in (\pi/4, \pi/2)$ with $\mathrm{Re}\, h(i\theta_0) = 0$. Since $\theta_0 = \mathrm{Cos}^{-1}((\sqrt{b^2 + 8c^2} - b)/4c)$, $a > a_0(b, c)$ implies $\mathrm{Im}\, h(i\theta_0) < 0$, and hence, $\lim_{\theta \to \theta_0 + 0} \tan \omega = \infty$. Finally the argument principle implies that (19) has a simple characteristic root $\lambda = \alpha + i\beta$ with $0 < \alpha < a$ and $\pi/4 < \beta < \pi/2$.

(ii) Suppose $\lambda = i\beta$, $\beta > 0$, solves (19). Then

$$b\cos\beta + c\cos 2\beta = 0 \text{ and } \frac{b+c}{a} = \frac{\sin\beta}{\beta}(b + 2c\cos\beta). \tag{20}$$

Thus $i\beta_0$ is a root of (20) from the definitions of $a_0(b, c)$ and β_0. For any $\beta > \beta_0$ with $b\cos\beta + c\cos 2\beta = 0$, and for $a = a_\beta(b, c) = (b+c)\beta/(b + 2c\cos\beta)\sin\beta$, $i\beta$ solves (20), and $a_\beta(b, c) > a_0(b, c)$. These imply that for $a = a_0(b, c)$, $i\beta_0$ is the unique root of (20) with zero real part and positive imaginary part. The simplicity of $i\beta_0$ can be easily seen as in (i) by applying the argument principle for $\arg h$ along a closed curve Γ^ε consisting of $\Gamma_1^\varepsilon : \lambda = t + i\pi/4$ $(-\varepsilon \leqq t \leqq a)$, $\Gamma_2^\varepsilon : \lambda = a + i\theta$ $(\pi/4 \leqq \theta \leqq \pi/2)$, $\Gamma_3^\varepsilon : \lambda = t + i\pi/2$ $(-\varepsilon \leqq t \leqq a)$, and $\Gamma_4^\varepsilon : \lambda = -\varepsilon + i\theta$ $(\pi/4 \leqq \theta \leqq \pi/2)$ for a sufficiently small $\varepsilon > 0$.

Next for fixed b and c, let $\lambda = \alpha(a) + i\beta(a)$ be a root of (19). Then a straightforward calculation yields

$$\frac{d\alpha(a)}{da} = \frac{H(a)}{D(a)},$$

where

$$H(a) = \frac{b+c}{a^2}\left(\frac{b+c}{a}\alpha - b\alpha e^{-\alpha}\cos\beta - 2c\alpha e^{-2\alpha}\cos 2\beta + b\beta e^{-\alpha}\sin\beta - 2c\beta e^{-2\alpha}\sin 2\beta\right),$$

$$D(a) = \left(\frac{b+c}{a} - be^{-\alpha}\cos\beta - 2ce^{-2\alpha}\cos 2\beta\right)^2 + (be^{-\alpha}\sin\beta + 2ce^{-2\alpha}\sin 2\beta)^2 > 0.$$

Since $a = a_0(b, c)$ implies $\alpha = 0$ and $\pi/4 < \beta = \mathrm{Cos}^{-1}((\sqrt{b^2 + 8c^2} - b)/4c) < \pi/2$, we have

$$\frac{d\alpha(a_0(b, c))}{da} = \frac{b+c}{a^2}(b + 4c\cos\beta)\,\beta\sin\beta/ \left\{\left(\frac{b+c}{a} - b\cos\beta - 2c\cos 2\beta\right)^2 + \left(b\sin\beta + 2c\sin 2\beta\right)^2\right\} > 0.$$

(iii) If $a < a_0(b, c)$, from the definition of $a_0(b, c)$, $(b + c)/a > (b + 2c\cos\beta)(\sin\beta)/\beta$ for any $\beta > 0$ with $b\cos\beta + c\cos 2\beta = 0$. Thus if $\lambda = \alpha + i\beta$ solves (19), α must be negative from $(b + c)/a = (be^{-\alpha} + 2ce^{-2\alpha})(\sin\beta)/\beta$.

From (ii) of this lemma, a Hopf bifurcation exists. For fixed b, c, and a near $a_0(b, c)$, (5) and consequently (4), has a nontrivial periodic solution with period near $2\pi/\mathrm{Cos}^{-1}((\sqrt{b^2 + 8c^2} - b)/4c)$. Since $\pi/4 < \mathrm{Cos}^{-1}((\sqrt{b^2 + 8c^2} - b)/4c) < \pi/2$, the periods of such solutions are between 4 and 8 (see [5, pp. 245–249]). Moreover, (iii) of this lemma implies that if $0 < a < a_0(b, c)$, the solution $N(t) = a/(b + c)$ is exponentially asymptotically stable (cf. [5] Corollary 2.2 on p. 213). These facts and the proof of Theorem 5 imply the following corollary.

COROLLARY. *Suppose that a satisfies one of the conditions in Theorem 5. Then $0 < a < a_0(b, c)$ and there exists a positive constant m such that for any $\alpha > \delta > 0$, there exists $M = M(\alpha, \delta) > 0$ such that for any solution $N(t)$ of (4) with $\delta \leqq N(t) \leqq \alpha$, $-2 \leqq t \leqq 0$, $|N(t) - a/(b + c)| \leqq Me^{-mt} \sup_{-2<\theta<0} |N(\theta) - a/(b + c)|$ for all $t \geqq 0$.*

Now we discuss existence of nontrivial periodic solutions of (5) by using a fixed point theorem for a closed convex set. For any $k > 0$, the set $K(k)$ is defined by

$$K(k) = \left\{ \phi \in C : \begin{array}{l} \phi(-2) = 0,\ 0 \leqq \phi(\theta) \leqq k \text{ and } |\phi(\theta_1) - \phi(\theta_2)| \\ \leqq L|\theta_1 - \theta_2| \text{ for } -2 \leqq \theta,\ \theta_1,\ \theta_2 \leqq 0 \end{array} \right\}$$

where $L = a^2 e^{2a} \max\{1, e^{2a} - 1\}/(b+c)$. Then $K(k)$ is a compact convex set in C, and we have:

LEMMA 3. *If $1 < a < \ln(2 + c/b)$, then there exist positive constants $k_0 = k_0(k)$ and k_1 such that if $\phi \in K(k) \backslash \{0\}$, then*

(i) $x(t) = x(t, \phi) = 0$ *for some* $t \in [0, 3]$.

(ii) $x(t) \geqq -k_0$ *as long as* $\sup_{-2<s<t} x(s) \leqq k$ *for* $t \geqq 0$.

(iii) *There exists a finite $\tau(\phi) > 4$ such that*

$$x_{\tau(\phi)}(\phi) \in K(k_1),$$

where the set $\{\tau(\phi) \colon \phi \in K(k) \backslash \{0\}\}$ is bounded.

PROOF. (i) Suppose that $x(t) > 0$ on $[0, 3]$. Since $x(t)$ is nonincreasing as long as $x(t) > 0$ for $t \geqq 0$, we obtain

$$\dot{x}(t) \leqq -ax(t-1) \text{ as long as } x(t) > 0 \text{ for } t \geqq 2.$$

Thus we have

$$x(3) \leqq x(2) - a \int_2^3 x(s-1)ds \leqq (1-a)x(2) < 0.$$

This contradiction shows that $x(t) = 0$ for some $t \in [0, 3]$.

(ii) It is clear that

$$\dot{x}(t) \geqq -((b+c)x(t) + a)k \tag{21}$$

holds as long as $-a/(b+c) < x(s) \leqq k$, $-2 \leqq s \leqq t$, for $t \geqq 0$. Let $x_0(t)$ be the solution of the equation $\dot{x} = -((b+c)x + a)k$ through $(0, 0)$. Then $x_0(t)$ is decreasing on $[0, 2]$ and $-a/(b+c) < x_0(t) \leqq 0$. Now we show that

$$x(t) \geqq -k_0 \text{ as long as } \sup_{-2<s<t} x(s) \leqq k \text{ for } t \geqq 0, \tag{22}$$

where $k_0 = k_0(k)$ is a number with $-x_0(2) \leqq k_0 < a/(b+c)$.

Suppose that for some $t_0 > 0$, $x(t_0) < -k_0$, $x(t_0) < x(t) \leqq k$ on $[-2, t_0)$. First we show that $t_0 - t_1 \leqq 2$ for $t_1 = \sup\{t \in [0, t_0) \colon x(t) = 0\}$. If $t_0 - t_1 > 2$, then we obtain $x(t_1 + 2) \geqq x_0(2) \geqq -k_0$ from (21). Since we have $x(t) \leqq 0$ on $[t_1, t_0]$ and $\dot{x}(t) \geqq 0$ on $[t_1 + 2, t_0]$, we obtain $x(t_0) \geqq x(t_1 + 2) \geqq -k_0$, which contradicts the choice of t_0. Thus we have $t_0 - t_1 \leqq 2$ and $x(t_0) \geqq x_0(t_0 - t_1) \geqq -k_0$, which contradicts the choice of t_0 again. Hence (22) holds.

(iii) First we show that $x(t) < 0$ for some $t > 0$. Suppose that $x(t) \geqq 0$ for $t \geqq 0$. Then $x(t)$ is nonincreasing for $t \geqq 0$, and it follows from (i) that $x(t) \equiv 0$ for $t \geqq 3$, and consequently, $x(t) \equiv 0$ for $t \geqq -2$. But this contradicts the fact that

$\phi \neq 0$. Next we show that

$$x(t) < 0, \ \tau_0 < t \leqq \tau_0 + 2, \tag{23}$$

where $\tau_0 = \inf\{t > 0 : x(t) < 0\}$. If we put $\xi = x_{\tau_0}(\phi)$, then

$$x(t) = \frac{a}{b+c}\left(\exp\left(- \int_{\tau_0}^{t} (b\xi(s - \tau_0 - 1) + c\xi(s - \tau_0 - 2))ds\right) - 1\right), \quad \tau_0 \leqq t \leqq \tau_0 + 1. \tag{24}$$

Since $x(t)$ is nonincreasing on $[\tau_0, \tau_0 + 1]$, we obtain on $[\tau_0 + 1, \tau_0 + 2]$, $\dot{x}(t) \leqq -\left(x(t) + \frac{a}{b+c}\right)(bx(\tau_0 + 1) + c\xi(t - \tau_0 - 2))$, which together with (24) imply

$$\begin{aligned} x(t) &\leqq -\frac{a}{b+c} + \left(x(\tau_0 + 1) + \frac{a}{b+c}\right)\exp\left(- bx(\tau_0 + 1) - c\int_{\tau_0+1}^{t} \xi(s - \tau_0 - 2)ds\right) \\ &\leqq \frac{a}{b+c}\left(\exp\left(- c\int_{-2}^{-1} \xi(s)ds - b\int_{-1}^{0} \xi(s)ds - bx(\tau_0 + 1)\right) - 1\right). \end{aligned}$$

Put $p = c\int_{-2}^{-1} \xi(s)ds + b\int_{-1}^{0} \xi(s)ds$. Since $p > 0$ and $a < \ln(2 + c/b) < 1 + c/b$, we have

$$c\int_{-2}^{-1} \xi(s)ds + b\int_{-1}^{0} \xi(s)ds + bx(\tau_0 + 1) = p + \frac{ab}{b+c}(e^{-p} - 1) > 0,$$

which implies $x(t) < 0$ on $[\tau_0 + 1, \tau_0 + 2]$, and consequently, (23) holds.

Define numbers α, β, and γ by $\alpha = (a - 1)/(b + c)$, $\beta = (a - (b + c)k_0)\alpha$, and $\gamma = (\alpha - k_0)/\beta$. First we show that $x(t) = -\alpha$ for some $t \in [\tau_0, \tau_0 + \gamma + 4]$ if $x(t_2) < -\alpha$ for some $t_2 \in (\tau_0, \tau_0 + 2)$. If $x(\tau_0 + 4) < -\alpha$, then we have $\dot{x}(t) > \beta$ as long as $x(t) < -\alpha$ for $t \geqq \tau_0 + 4$. Thus it is easy to see that $x(t) = -\alpha$ for some $t \in [\tau_0, \tau_0 + \gamma + 4]$.

Next let $t_3 \in [\tau_0 + 2, \tau_0 + \gamma + 4]$ be a number with $-\alpha \leqq x(t_3) < 0$. If $x(t) < 0$ on $[t_3, t_3 + 3]$, then $x(t)$ is increasing on $[t_3, t_3 + 3]$ and we obtain $\dot{x}(t) \geqq -(a - (b + c)\alpha)x(t - 1) \geqq -x(t - 1)$ on $[t_3 + 2, t_3 + 3]$, which implies

$$x(t_3 + 3) \geqq x(t_3 + 2) - \int_{t_3+2}^{t_3+3} x(s - 1)ds \geqq x(t_3 + 2) - x(t_3 + 2) = 0.$$

Since this is a contradiction, $x(t) = 0$ for some $t \in [\tau_0, \tau_0 + \gamma + 7]$.

Now we show that

$$x(t) > 0, \ \tau_1 < t \leqq \tau_1 + 2, \tag{25}$$

where $\tau_1 = \inf\{t > \tau_0 : x(t) = 0\}$. By a similar argument as in the proof of (24), we have on $[\tau_1 + 1, \tau_1 + 2]$

$$x(t) \geqq \frac{a}{b+c}\left(\exp\left(-c\int_{-2}^{-1}\zeta(s)ds - b\int_{-1}^{0}\zeta(s)ds - bx(\tau_1+1)\right)-1\right),$$

where $\zeta = x_{\tau_1}(\phi)$. Put $q = c\int_{-2}^{-1}\zeta(s)ds + b\int_{-1}^{0}\zeta(s)ds$. Since $q > -a$, and since $x < ab(1-e^{-x})/(b+c)$ holds for $x = -a$ from $a < \ln(2+c/b)$, we obtain

$$c\int_{-2}^{-1}\zeta(s)ds + b\int_{-1}^{0}\zeta(s)ds + bx(\tau_1+1) = q + \frac{ab}{b+c}(e^{-q}-1) < 0,$$

which implies $x(t) > 0$ on $[\tau_1+1, \tau_1+2]$, and consequently, (25) holds.

Since we have $x(t) > -a/(b+c)$ on $[-2, \tau_1]$, we obtain $\dot{x}(t) < a(x(t)+a/(b+c))$ on $[\tau_1, \tau_1+2]$, and hence, we have

$$0 \leqq x(t) < k_1, \quad \tau_1 \leqq t \leqq \tau(\phi),$$

where $k_1 = a(e^{2a}-1)/(b+c)$ and $\tau(\phi) = \tau_1 + 2$. Moreover we obtain on $[\tau_1, \tau(\phi)]$ that $|\dot{x}(t)| \leqq a^2e^{2a}/(b+c)$ if $x(t-1)x(t-2) > 0$ and that $|\dot{x}(t)| \leqq ae^{2a}\max\{bx(t-1), -cx(t-2)\}/(b+c)$ if $x(t-1)x(t-2) \leqq 0$, which imply $|\dot{x}(t)| \leqq a^2e^{2a}\max\{1, e^{2a}-1\}/(b+c) = L$. Thus it follows that $x_{\tau(\phi)}(\phi) \in K(k_1)$. Finally $4 < \tau(\phi) < \gamma + 13$ implies that the set $\{\tau(\phi) \in K(k)\backslash\{0\}\}$ is bounded.

Remark. Since we have $q > e^{-a} - e^{-2a} - a$, the condition $a < \ln(2+c/b)$ can be replaced by a weaker condition that $x \leqq ab(1-e^{-x})/(b+c)$ holds for $x = e^{-a} - e^{-2a} - a$.

Now we state a known result for (18). For any characteristic root λ of (19), there exists a decomposition of C as $C = P_\lambda \oplus Q_\lambda$, where P_λ and Q_λ are invariant under the solution operator $T(t)$ of (18), $T(t)\phi = x_t(\phi)$, $\phi \in C$, and P_λ is finite dimensional. Let the projection operators defined by the above decomposition of C be π_λ and $I - \pi_\lambda$, where I denotes the identity operator and the range of π_λ is P_λ.

For $k > k_1$, let $K = K(k)$. For $\phi \in K\backslash\{0\}$, define the mapping A by

$$A\phi = x_{\tau(\phi)}(\phi).$$

Since we have $x(t) < 0$ on $[\tau_1 - 2, \tau_1)$ from (23) and the definition of τ_1, we obtain $\dot{x}(\tau_1) > 0$. Thus by the continuity of $x(t, \phi)$ in t and ϕ, $\tau(\phi)$ is continuous on $K\backslash\{0\}$, and hence, $\tau: K\backslash\{0\} \to [4, \infty)$ is completely continuous from Lemma 3(iii). On the other hand, A is continuous and $A\phi \in K(k_1) \subset K$ on $K\backslash\{0\}$. Thus A takes $K\backslash\{0\}$ into K and is completely continuous. Moreover we have:

Lemma 4. (i) *Let λ be the characteristic root of* (19) *given in Lemma* 2(i). *Then there exists a $\delta > 0$ such that*

$$\inf\{|\pi_\lambda\phi| : \phi \in K \cap S_\delta\} > 0. \tag{26}$$

(ii) *There exists $M > 0$ such that $A\phi = \mu\phi$, $\phi \in K \cap S_M$ implies $\mu < 1$.*

PROOF. (i) For the characteristic root $\lambda = \alpha + i\beta$ of (19) given in Lemma 2(i), let $\xi_1^*(\theta) = e^{\alpha\theta}\cos\beta\theta$, $\xi_2^*(\theta) = e^{\alpha\theta}\sin\beta\theta$, $-2 \leqq \theta \leqq 0$, and $\eta_1(s) = e^{-\alpha s}\cos\beta s$, $\eta_2(s) = e^{-\alpha s}\sin\beta s$, $0 \leqq s \leqq 2$. The adjoint equation of (18) is

$$\dot{z}(t) = \frac{a}{b+c}(bz(t+1) + cz(t+2))$$

and the bilinear form is given by

$$(\eta, \xi) = \eta(0)\xi(0) - \frac{a}{b+c}\left(c\int_{-2}^{0} \eta(\theta+2)\xi(\theta)d\theta + b\int_{-1}^{0} \eta(\theta+1)\xi(\theta)d\theta\right).$$

Define ξ_1 and ξ_2 by

$$\left.\begin{aligned} \xi_1(\theta) &= \frac{1}{\Delta}((\eta_2, \xi_2^*)\xi_1^* - (\eta_2, \xi_1^*)\xi_2^*), \\ \xi_2(\theta) &= \frac{1}{\Delta}((\eta_1, \xi_1^*)\xi_2^* - (\eta_1, \xi_2^*)\xi_1^*), \end{aligned}\right\} \quad -2 \leqq \theta \leqq 0,$$

where $\Delta = (\eta_1, \xi_1^*)(\eta_2, \xi_2^*) - (\eta_1, \xi_2^*)(\eta_2, \xi_1^*) \neq 0$. Then it is easy to see that $(\eta_1, \xi_1) = (\eta_2, \xi_2) = 1$ and $(\eta_1, \xi_2) = (\eta_2, \xi_1) = 0$. Therefore for any $\phi \in C$, $\pi_\lambda\phi = (\eta_1, \phi)\xi_1 + (\eta_2, \phi)\xi_2$ (cf. [5] Lemma 3.4 on p. 177).

Let δ be a number with $0 < \delta < k$. If (26) does not hold, then $\pi_\lambda\phi = 0$ for some $\phi \in K \cap S_\delta$, since $|\pi_\lambda\phi|$ is a continuous function in ϕ on the compact set $K \cap S_\delta$. Thus we have $(\eta_1, \phi) = (\eta_2, \phi) = 0$. On the other hand, (η_2, ϕ) is given by

$$\begin{aligned} (\eta_2, \phi) = -\frac{a}{b+c}\Big(c\int_{-2}^{0} & e^{-\alpha(\theta+2)}\sin\beta(\theta+2)\phi(\theta)d\theta \\ & + b\int_{-1}^{0} e^{-\alpha(\theta+1)}\sin\beta(\theta+1)\phi(\theta)d\theta\Big). \end{aligned}$$

Since $\pi/4 < \beta < \pi/2$, we have $\sin\beta(\theta+2) > 0$ for $-2 < \theta < 0$ and $\sin\beta(\theta+1) > 0$ for $-1 < \theta < 0$, and hence, $\phi \in K \cap S_\delta$ implies $(\eta_2, \phi) < 0$. But this contradicts the fact that $(\eta_2, \phi) = 0$.

(ii) For M with $k_1 < M < k$, where k_1 is given in Lemma 3, $A\phi = \mu\phi \in K \cap S_M$ implies $\mu < 1$ from Lemma 3(iii).

We are now ready to prove existence of a nontrivial periodic solution of (5) by using the following theorem, which can be found in [5].

THEOREM 6. *Suppose that the following conditions are satisfied*:

(i) *There exists a characteristic root* λ *of* (19) *with* $\mathrm{Re}\lambda > 0$.

(ii) *There exists a closed convex set* $K \subset C$, $0 \in K$, *and* $\delta > 0$, *such that*

$$\inf\{|\pi_\lambda\phi| : \phi \in K \cap S_\delta\} > 0.$$

(iii) *There exists a completely continuous function* $\tau: K\backslash\{0\} \to [\varepsilon, \infty)$, $\varepsilon \geqq 0$ *such that the mapping defined by*

$$A\phi = x_{\tau(\phi)}(\phi), \quad \phi \in K\backslash\{0\}$$

takes $K\backslash\{0\}$ *into* K *and is completely continuous.*

(iv) *There exists* $M > 0$ *such that* $A\phi = \mu\phi$, $\phi \in K \cap S_M$ *implies* $\mu < 1$.

Then there exists a nontrivial periodic solution of (5) *with initial function in* $K\backslash\{0\}$.

Among the assumptions of Theorem 6, (i) holds from Lemma 2(i), (ii) and (iv) hold from Lemma 4, and (iii) holds for $\varepsilon = 4$ by Lemma 3 and the continuity of $\tau(\phi)$, under the conditions in Lemma 2(i) and Lemma 3. Hence we have:

THEOREM 7. *If* $\max\{1, a_0(b, c)\} < a < \ln(2 + c/b)$, *then there exists a nontrivial periodic solution* $x(t)$ *of* (5) *with* $-k_0 < x(t) < k_1$, *its period is between* 4 *and* $\gamma + 13$, $x(t)$ *has at most one zero point in any interval of length* 2, $x(t)$ *crosses the t-axis at its zero point, and any half-open interval of length* ω *contains two zero points of* $x(t)$, *where* $\omega > 0$ *is the smallest period of* $x(t)$.

References

[1] Barnea D. I., A method and new results of stability and instability of autonomous functional differential equations, SIAM J. Appl. Math., **17** (1969), 681–697.

[2] Chow S. N. and Hale J. K., Periodic solutions of autonomous equtions, J. Math. Anal. Appl. **66** (1978), 495–506.

[3] Furumochi T., Periodic solutions of the equation $\dot{x}(t) = -f(x(t))(g(x(t)) + h(x(t-1)))$, Mem. Fac. Sci. Shimane Univ. **22** (1988), 1–9.

[4] Jones G. S., Periodic motions in Banach space and applications to functional differential equations, Contrib. Differential Equations **3** (1964), 75–106.

[5] Hale J. K., Theory of Functional Differential Equations, Springer-Verlag (1977).

[6] Kaplan J. L. and Yorke J. A., Ordinary differential equations which yield periodic solutions of differential delay equations, J. Math. Anal. Appl. **48** (1974), 317–324.

[7] Nussbaum R. D., Periodic solutions of some nonlinear autonomous functional differential equations, Ann. Math. Pura. Appl. **101** (1974), 263–306.

[8] Seifert G., On a delay-differential equation for single specie population variations, Nonlinear Analysis **11** (1987), 1051–1059

[9] Yoneyama T. and Sugie J., On the stability region of differsential equations with two delays, Funkcial. Ekvac. **31** (1988), 233–240.