Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
40 (2007), pp. 1–5

DIRICHLET PROBLEM AND DIRICHLET SEMI-NORM

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Communicated by J. Sugie

(Received: March 10, 2005)

Abstract. On a finite set of vertices E in a Cartier tree, using the Green's formula, the Dirichlet semi-norm is defined and the Dirichlet solution on E is obtained as the projection on the closed subspace of harmonic functions on E.

1. Introduction

In the context of the classical potential theory in $\mathbf{R}^n$, $n \geq 2$, let Ω be a bounded domain; let P_0 be the class of finite continuous functions on Ω, with a square summable finite continuous gradient. For f, g in P_0, denote by $(f, g) = \int_\Omega (\mathrm{grad} f, \mathrm{grad} g) dx$ the inner product and by $\|f\|_\Omega$ the corresponding Dirichlet semi-norm. Suppose $f \in P_0$ is a continuous function on $\overline{\Omega}$. Then the classical Dirichlet Principle states (see Brelot [3, pp. 122-127]) the generalized Dirichlet solution H_f^Ω is the unique (up to an additive constant) harmonic function in P_0 which minimizes $\|u - f\|_\Omega$, for $u \in P_0$.

In the context of a Cartier tree T [4], we know that if E is a finite set and if f is a real function on ∂E, then the (classical) Dirichlet solution exists on E (see [1]); we also know that the Dirichlet norm can be defined on E (see Yamasaki [7] in the context of an infinite network and Urakawa [6] in the context of an infinite graph).

In this note, we prove: Let E be a finite set of vertices in a Cartier tree. Let f be a finite-valued function on ∂E and h be the Dirichlet solution on E with boundary values f. Then, for any finite-valued function g on E such that $g = f$ on ∂E, we have $\|h\|_E \leq \|g\|_E$ and $\|h\|_E = \|g\|_E$ if and only if $h \equiv g$. (See Murakami and Yamasaki [5] for a version of this Dirichlet Principle in the context of an infinite network.) Conversely, the (classical) Dirichlet solution on E always comes out as a projection.

2. Preliminaries

In a graph, two vertices x and y are said to be neighbours, $x \sim y$, if there exists an edge joining them; a graph is said to be locally finite if any vertex has a finite

2000 *Mathematics Subject Classification.* 31C20.

Key words and phrases. Green's formula, Dirichlet Principle, Cartier tree.

number of neighbours; a graph is said to be connected if any two vertices can be joined by a finite number of edges; a path $\{x = s_0, s_1, \ldots, s_{n-1}, s_n = x\}$ with distinct vertices s_i, $1 \leq i \leq n$, and $n \geq 3$ is called a circuit. A Cartier tree T [4, p. 208] is an infinite graph which is connected and locally finite and has no circuit.

A vertex x_0 in T is said to be terminal, if it has only one neighbour in T. Given a subset E of vertices in T, $x \in E$ is said to be an interior point if x is not terminal and if all the neigbours of x are in E. Let $\overset{o}{E}$ denote the collection of all the interior points of E; let $\partial E = E \setminus \overset{o}{E}$. On a tree T, it is assumed that a transition probability is given: that is, with any two vertices x and y is associated a real number $p(x, y) \geq 0$ such that $p(x, y) > 0$ if and only if $x \sim y$ and $\sum_{y \in T} p(x, y) = 1$ for any $x \in T$.

Fix a vertex e in T. For a vertex x, let $\{e, x_1, \ldots, x_n, x\}$ be a path joining e and x. Write $\phi(x) = \frac{p(e,x_1)p(x_1,x_2)\ldots p(x_n,x)}{p(x,x_n)p(x_n,x_{n-1})\ldots p(x_1,e)}$; take $\phi(e) = 1$. Since there are no circuits in T, it is easy to see that $\phi(x)$ remains the same for any path joining e and x; note $\phi(x)p(x, y) = \phi(y)p(y, x)$ for any pair of vertices x and y. Set $\psi(x, y) = \phi(x)p(x, y)$. Then $\psi(x, y) = \psi(y, x) \geq 0$ and $\psi(x, y) > 0$ if and only if $x \sim y$.

If u is a real-valued function on T, the Laplacian Δu at a vertex x is defined as $\Delta u(x) = \sum_{y \in T} p(x, y)[u(y) - u(x)]$. The function u is said to be harmonic at x if $\Delta u(x) = 0$. Suppose v is a real function defined on a subset E of T. Then we define the inner normal derivative of v at a point $s \in \partial E$ as

$$\frac{\partial v}{\partial n^-}(s) = \sum_{x \in E} p(s, x)[v(x) - v(s)].$$

3. Dirichlet semi-norm

Theorem 1. [1, Theorem 2] *Let f be a real function on the boundary ∂E of a finite subset of vertices E in a Cartier tree. Then there exists unique bounded function u on E such that u is harmonic on $\overset{o}{E}$ and $u = f$ on ∂E.*

Proof. Since ∂E is finite, we can find constants α and β such that $\beta \leq f \leq \alpha$ on ∂E. Since we can consider f^+ and f^- separately, we can assume $\beta \geq 0$. Then

$$v(x) = \begin{cases} \alpha & \text{if } x \in \overset{o}{E} \\ f(x) & \text{if } x \in \partial E \end{cases}$$

is a superharmonic function on E.

Let $\mathcal{F}$ be the family of all subharmonic functions u on $\overset{o}{E}$, such that $u \leq v$ on E. Then as in [1], $h(x) = \sup_{\mathcal{F}} u(x)$ is the Dirichlet solution on E, with boundary values f.

The uniqueness is proved using the minimal principle, similar to the one given in Yamasaki [7, Lemma 2.1] for the case of infinite networks. □

Let E be a (possibly infinite) set of vertices in T. Let u and v be two real functions on E such that $\sum_{x,y \in E} \psi(x, y)u(x)[v(y) - v(x)]$ is absolutely convergent.

Then write

$$(u,v)_E = -\sum_{x,y\in E} \psi(x,y)u(x)[v(y)-v(x)]$$

and note, by rearranging the terms in this double sum, we can write in this case

$$(u,v)_E = \sum_{x,y\in E} \psi(x,y)[u(y)-u(x)][v(y)-v(x)].$$

Remark that if E is finite, the above condition on absolute convergence is redundant; for any two functions u, v on the finite set E, $(u,v)_E$ is always defined.

The following form of the Green's formula is given in [1], which is a variant of the results given in Urakawa [6] and Bendito et al. [2].

Theorem 2. *Let u and v be two real-valued functions on a finite set E of vertices in T. Then*

$$\sum_{x\in \overset{o}{E}} \phi(x)u(x)\Delta v(x) + (u,v)_E = -\sum_{s\in\partial E} \phi(s)u(s)\frac{\partial v}{\partial n^-}(s).$$

Definition. Let u be a real-valued function on a subset of vertices E. Write

$$\|u\|_E^2 = (u,u)_E = \sum_{x,y\in E} \psi(x,y)[u(y)-u(x)]^2,$$

if the double sum is finite.

In the following, we shall consider only connected subsets E with a finite number of vertices. Since $\|u\|_E = 0$ implies that u is a constant on E, $\|u\|_E$ is a semi-norm which is called the *Dirichlet semi-norm* on E. Let $\mathcal{F}$ denote the equivalence classes $\widetilde{f}$ of real-valued functions on E, so that two finite functions on E are in the same class if and only if they differ by a constant. Note that $\mathcal{F}$ is an inner product space; if f and g are any two finite functions on E and if $\widetilde{f}$ and $\widetilde{g}$ are the equivalence classes defined by f and g, then $\|\widetilde{f}\| = \|f\|$ and $\|\widetilde{f}-\widetilde{g}\| = \|f-g\|$.

Let $\mathcal{H}$ denote the subspace of $\mathcal{F}$, determined by the harmonic functions on E. (Recall, h is harmonic on E when h is defined on E and $\Delta h(x) = 0$ at every $x \in \overset{o}{E}$.)

Proposition 3. *$\mathcal{H}$ is a closed subspace of $\mathcal{F}$.*

Proof. Let $\widetilde{h}_n \in \mathcal{H}$ be a Cauchy sequence in $\mathcal{F}$. For each equivalence class $\widetilde{h}_n$, extract a harmonic function h_n from the class $\widetilde{h}_n$ so that $h_n(e) = 0$ where e is a fixed vertex in E. Since $\|h_n - h_m\| = \|\widetilde{h}_n - \widetilde{h}_m\| \to 0$ when $n,m\to\infty$, for any $x\in E$,

$$\psi(x,e)[(h_n-h_m)(e) - (h_n-h_m)(x)]^2 \le \|h_n-h_m\| \to 0$$

when $n,m\to\infty$. Since $(h_n-h_m)(e) = 0$, we deduce that the sequence $h_n(x)$ converges at every $x\sim e$. Then we show that there is convergence at all the neighbours of each $x\sim e$. Thus proceeding, we see that h_n converges on E. If we write $h(x) = \lim_n h_n(x)$, $h(x)$ is harmonic on E and $\|h_n - h\|\to 0$. Hence $\widetilde{h}_n \to \widetilde{h} \in \mathcal{H}$; that is $\mathcal{H}$ is closed in $\mathcal{F}$. □

Consequence For every $\widetilde{f} \in \mathcal{F}$, there exists a unique $\widetilde{h} \in \mathcal{H}$ such that $\|\widetilde{h} - \widetilde{f}\|$ is minimum; $\widetilde{h}$ is the projection of $\widetilde{f}$ on $\mathcal{H}$, so that $\|\widetilde{h}\| \leq \|\widetilde{f}\|$ and $\|\widetilde{h}\| = \|\widetilde{f}\|$ if and only if $\widetilde{f} \in \mathcal{H}$.

Notation Let $\mathcal{F}_0$ be the subspace of $\mathcal{F}$ such that $\widetilde{f} \in \mathcal{F}_0$ if and only if the equivalence class represented by $\widetilde{f}$ contains a function f on E, which is 0 on ∂E.

Theorem 4. *$\mathcal{F}_0$ is the orthogonal complement of $\mathcal{H}$ in $\mathcal{F}$; that is, $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{H}$, and $\mathcal{F}_0 \perp \mathcal{H}$.*

Proof. Remark that if $\widetilde{f_1}$, $\widetilde{f_2} \in \mathcal{F}$, and if we take some f_1 (respectively f_2) from the equivalence class $\widetilde{f_1}$ (respectively $\widetilde{f_2}$), then $(f_1, f_2)_E$ is independent of the choice of the functions f_1 and f_2. We define $(\widetilde{f_1}, \widetilde{f_2})_E = (f_1, f_2)_E$. Now $\mathcal{F}_0 \perp \mathcal{H}$. For let $\widetilde{f} \in \mathcal{F}_0$ and $\widetilde{h} \in \mathcal{H}$. Choose f from the class $\widetilde{f}$ such that $f = 0$ on ∂E; choose a harmonic function h from the class $\widetilde{h}$. Note that $(h, f)_E = 0$ (by taking $v = h$ and $u = f$ in Theorem 2). Hence $(\widetilde{h}, \widetilde{f})_E = 0$.

Suppose $\widetilde{f} \in \mathcal{F}_0 \bigcap \mathcal{H}$. If f is harmonic on E and 0 on ∂E, then $f \equiv 0$, so that $\widetilde{f} = \widetilde{0}$.

Let now $\widetilde{f} \in \mathcal{F}$. Choose some f in the class $\widetilde{f}$. Let h be the Dirichlet solution with boundary values f on ∂E (Theorem 1). Then $\widetilde{f - h} \in \mathcal{F}_0$, $\widetilde{h} \in \mathcal{H}$ and $\widetilde{f} = \widetilde{f - h} + \widetilde{h}$. □

To conclude, we shall reformulate the above result, without any reference to the equivalence classes, to obtain the Dirichlet Principle in the framework of a Cartier tree. For a similar result in an infinite network, see Murakami and Yamasaki [5, Section 2].

Theorem 5. *Let f be a real-valued function on ∂E and h be the Dirichlet solution on E with boundary values f. Then, for any finite-valued function g on E such that $g = f$ on ∂E, we have $\|h\|_E \leq \|g\|_E$ and $\|h\|_E = \|g\|_E$ if and only if $h \equiv g$.*

Proof. From what we have proved, we deduce that $\widetilde{h}$ is the projection of $\widetilde{g} \in \mathcal{F}$ onto $\mathcal{H}$. Hence $\|\widetilde{h}\|_E \leq \|\widetilde{g}\|_E$ which implies that $\|h\|_E \leq \|g\|_E$. Suppose $\|h\|_E = \|g\|_E$; then $\|\widetilde{h}\|_E = \|\widetilde{g}\|_E$, which shows that $\widetilde{h} = \widetilde{g}$ by the property of projection. This means that $h - g$ is a constant on E; but $h = f = g$ on ∂E, so that $h \equiv g$. □

Dirichlet solution as a projection. Let E be a finite set of connected vertices in a Cartier tree. Let f be a finite-valued function on ∂E. Giving arbitrary values at the vertices in $\overset{o}{E}$, we can assume that f is defined on E. Then (by Theorem 4) $\widetilde{f}$ can be written uniquely as $\widetilde{f} = \widetilde{g} + \widetilde{h} \in \mathcal{F}_0 \oplus \mathcal{H}$.

Take a harmonic function H in the equivalence class $\widetilde{h}$. Then (by the definition of $\mathcal{F}_0$), $f - H$ is a function on E taking a constant value c on ∂E. Consequently, $h = H + c$ is a harmonic function on E, with boundary values f on ∂E.

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