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Note on Invariant Self-Dual Connections

Dedicated to Professor Miyuki Yamada on his 60th birthday

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It is shown that S^1-equivariant connections in the previous paper [6] are S^1-invariant connections due to Jaffe-Taubes [4] and vice versa. Further an existence theorem is proved.

§1. Introduction

In the previous paper [6], the author has introduced the notion of Γ-principal bundle of diagonal type and given a Γ-action on the space of connections, where Γ is a compact Lie group. Here again we mention them and use in this article.

Let $p: P \to M$ be a Γ-principal bundle with structure group G over a Γ-manifold M.

DEFINITION. The bundle $p: P \to M$ is *of diagonal type* if and only if there exist a covering $M = \cup U_i$ consisting of Γ-invariant open sets, homomorphisms $\{\alpha_i: \Gamma \to G\}$ and equivariant local trivialities $\{\phi_i: U_i \times G \to p^{-1}(U_i)\}$ with the property

$$\phi_i(\gamma x, \alpha_i(\gamma)h) = \gamma(\phi_i(x, h)) \text{ for } \gamma \in \Gamma,\ (x, h) \in U_i \times G.$$

Now we consider the case $\Gamma = S^1$, the circle group. Let $\{A_U\}$ be a connection of an S^1-principal bundle $P \to M$ of diagonal type with structure group G, where M is an S^1-manifold. By Proposition 1 in [6], an S^1-acton on the space of connections is given by

$$(\gamma A_U)(x) = \mathrm{Ad}(\alpha_U(\gamma))A_U(\gamma^{-1}x) \text{ for } \gamma \in \Gamma,\ x \in U.$$

DEFINITION. A connection $\{A_U(x)\}$ is *S^1-equivariant* if and only if

$$\gamma A_U = A_U \text{ for each } U \text{ and } \gamma \in S^1.$$

In §2 we prove that any S^1-equivariant connection is an invariant connection due to Jaffe-Taubes and vice versa. In §3, by making use of the method in [2], we prove an existence theorem for invariant self-dual connections on an S^1 4-manifold with non empty fixed point set and a positive definite intersection form.

§2. Equivalence of two concepts

Let $p: P \to M$ be a Γ-principal bundle of diagonal type over a smooth Γ-

manifold M, where the structure group G acts on the space P on the right. Let $\sigma_U: U \to p^{-1}(U)$ is the local section given by

$$\sigma_U(x) = \phi_U(x, e) \text{ for } x \in U, \text{ where } e \text{ denotes the unit of } G.$$

Denote by $\mathfrak{g}$ the Lie algebra of G, i.e. left invariant vector fields on G. We have the relation

$$\gamma^{-1}(\sigma_U^*(Y)) = R_{\alpha(\gamma)^{-1}}\sigma_U^*(\gamma^{-1}Y) \quad \text{for } \gamma \in \Gamma,\ Y \in T_x(M),$$

where $T_x(M)$ denotes the tangent space at $x \in M$, and $\alpha(\gamma) = \alpha_U(\gamma)$ for $\gamma \in \Gamma$. For a connection $\{A_U\}$, the corresponding Ehresmann connection on P is given by

$$\bar{A}_U(u)(\sigma_U^*(Y) + X^{\#}) = A_U(x) \quad \text{for } x = p(u),$$

where $X^{\#}$ denotes the fundamental vector field corresponding to $X \in \mathfrak{g}$. Denote by $H_{\bar{A}}$ the horizontal subspace.

DEFINITION ([4]). A connection $\bar{A}$ is *invariant* with respect to the Γ-action on P if and only if $\gamma H_{\bar{A}} \subset H_{\bar{A}}$ for all $\gamma \in \Gamma$.

Now we prove

THEOREM 1. *A connection $\{A_U\}$ is Γ-equivariant if and only if the connection $\bar{A}$ is Γ-invariant.*

PROOF. Suppose that a connection $\{A_U\}$ is Γ-equivariant and that $\sigma_U^*(Y) + X^{\#} \in H_{\bar{A}}$. Since a fundamental vector field $X^{\#}$ is left invariant,

$$\begin{aligned}
&\bar{A}(\gamma^{-1}u\alpha(\gamma)^{-1})(\gamma^{-1}\{\sigma_U^*(Y) + X^{\#}\}) \\
&= \bar{A}(\gamma^{-1}u\alpha(\gamma)^{-1})(R_{\alpha(\gamma)^{-1}}\sigma_U^*(\gamma^{-1}Y) + X^{\#}) \\
&= \mathrm{Ad}(\alpha(\gamma))\bar{A}(\gamma^{-1}u)(\sigma_U^*(\gamma^{-1}Y)) + X \\
&= \mathrm{Ad}(\alpha(\gamma))A(\gamma^{-1}x)(\gamma^{-1}Y) + X \\
&= A(x)(Y) + X, \text{ by the assumption,} \\
&= \bar{A}(u)(\sigma_U^*(Y) + X^{\#}) = 0, \text{ where } \alpha(\gamma) \text{ denotes } \alpha_U(\gamma) \text{ for } \gamma \in \Gamma.
\end{aligned}$$

Conversely assume that $\gamma H_{\bar{A}} \subset H_{\bar{A}}$ for all $\gamma \in \Gamma$. For any $Y \in T_x(M)$, there exists $X \in \mathfrak{g}$ such that $A(x)(Y) + X = 0$. Since $\bar{A}(u)(\sigma_U^*(Y) + X^{\#}) = 0$,

$$\begin{aligned}
&\bar{A}(\gamma^{-1}u\alpha_U(\gamma)^{-1})(\gamma^{-1}\{\sigma_U^*(Y) + X^{\#}\}) = 0 \\
&= \mathrm{Ad}(\alpha_U(\gamma))A(\gamma^{-1}x)(\gamma^{-1}Y) + X.
\end{aligned}$$

Therefore we have $\gamma A = A$ for each $\gamma \in \Gamma$.

§3. An existence theorem

First we prove a proposition which is intrinsically due to [1].

PROPOSITION. *Let P be an S^1-principal bundle of diagonal type with structure* *group $SU(2)$ over a smooth S^1-manifold M. Let $\{A_U\}$ be an irreducible connection such that γA_U is gauge equivalent to A_U for each $\gamma \in S^1$. Then there exists a lifting action of diagonal type which fixes the connection.*

PROOF. Put $S^1 = \Gamma$ and let $\mathrm{Aut}^\Gamma(P)$ be all bundle automorphisms of P which cover the S^1-action on M, and $\mathrm{Aut}_A^\Gamma(P)$ be the elements of $\mathrm{Aut}^\Gamma(P)$ which fix the connection. Denote by $\mathscr{G}$ the gauge group. The homomorphisms $\{\alpha_U: S^1 \to G\}$ determine the lifting action $\bar{\alpha}(\gamma): P \to P$ for each $\gamma \in S^1$. Then we have an exact sequence

$$e \longrightarrow \mathscr{G} \xrightarrow{\ i\ } \mathrm{Aut}^\Gamma(P) \xrightarrow{\ j\ } S^1 \longrightarrow e,$$

where $i(g) = \bar{\alpha}(\gamma)\cdot g$ and $j(\bar{\alpha}(\gamma)\cdot g) = \gamma$ for $g \in \mathscr{G}$ and $\gamma \in S^1$. By assumption, for some $g \in \mathscr{G}$,

$$(\gamma\cdot A)(x) = g(x, \gamma)^{-1}A(x)g(x, \gamma) + g(x, \gamma)^{-1}dg(x, \gamma) \text{ for } x \in U.$$

Then $(g(x, \gamma)^{-1})^*(\gamma\cdot A(x)) = A(x)$ and $(g(x, \gamma)^{-1}\cdot\bar{\alpha}(\gamma))^{-1} \in \mathrm{Aut}_A^\Gamma(P)$ for each $\gamma \in S^1$. Since the connection $\{A_U\}$ is irreducible, we obtain an exact sequence,

$$e \longrightarrow Z_2 \longrightarrow \mathrm{Aut}_A^\Gamma(P) \longrightarrow S^1 \longrightarrow e \qquad (*),$$

then $\mathrm{Aut}_A^\Gamma(P) =$ the double cover of S^1, or $Z_2 \times S^1$. Now the relation

$$\mathrm{Ad}(\alpha_U(\gamma))A_U(\gamma^{-1}x) = g_U(x, \gamma)^* A_U(x),$$

shows that for each $x \in U$, the map $g_U(x,): S^1 \to \mathscr{G}$ is a continuous map for $\gamma \in S^1$. Then the map $\bar{\alpha}(\gamma)^{-1}\cdot g(x,)^{-1}: S^1 \to \mathrm{Aut}_A^\Gamma(P)$ gives a section in the exact sequence $(*)$, and also the desired lifting action.

Now we prove an existence theorem for invariant connections by the method in sections 8 and 9 [2]. In our proof, the same notations to the reference are used.

THEOREM 2. *Let M be a compact, simply connected, oriented smooth S^1 4-manifold whose intersection form is positive definite, and P be a principal S^1-bundle of diagonal type with structure group $SU(2)$ whose index* is 1. *Suppose that the fixed point set M^Γ is not empty. Then the bundle P admits a lifting action which fixes an irreducible self-dual connection.*

PROOF. Let $y \in M$ be a fixed point. As in the section 8 [2], we choose invariant balls B_2, B_4 with the center y, and C_k^Γ be the set of invariant C_k-metrics on B_4. For $\omega \in \mathscr{F} = \left\{\omega \in L^1(B_4; \Lambda^4 R^4);\ \omega \geqq 0,\ \int_{B_2} \omega \geqq 4\pi^2,\ \int_{B_4} \omega \leqq 8\pi^2\right\}$. define

$$(\gamma\omega)(z) = \omega(\gamma^{-1}z) \text{ for } \gamma \in S^1,\ z \in B_4.$$

Then

$$(\gamma\omega(D))(z) = \omega(D)(\gamma^{-1}z) = -\mathrm{tr}(F_{\gamma D} \wedge *F_{\gamma D})(z) = \omega(\gamma D)(z),$$

where $D = D_A$ is the covariant derivative of a connection A. For the smooth function R in the section 8 [2],

$$R\colon (0, 2) \times B_2 \times \mathscr{F} \times C_k^\Gamma \longrightarrow R, \ R(\lambda, x, \omega, g) = \int_{B_4} \beta\left[\frac{\rho_g(x, z)}{\lambda}\right]\omega(z),$$

we have

$$R(\lambda, \gamma^{-1}x, \gamma\omega, g) = R(\lambda, x, \omega, g) \qquad (**).$$

Let $(**)$ be equal to $4\pi^2$, then by Theorem 8.28 [2],

$$x(\gamma\omega(D), g) = \gamma^{-1}x(\omega(D), g) \quad \text{for each } \gamma \in S^1.$$

Hence by Theorem 9.1 [2], there exists a class of an irreducible self-dual connection which corresponds to some $\langle \lambda, y \rangle$. Thus by Proposition above, the theorem is obtained.

Lastly we give an

EXAMPLE. On the complex projective plane $P_2(C)$, we define an S^1-action by

$$\gamma[z_1, z_2, z_3] = [z_1, z_2, \gamma z_3] \text{ for } \gamma \in S^1, \ [z_1, z_2, z_3] \in P_2(C).$$

Then the fixed point set is the set $P_1(C) \cup [0, 0, 1]$, where the projective line $P_1(C)$ is given by $z_3 = 0$, and the action is semi-free. Let $P \to P_2(C)$ is a principal S^1-bundle with structure group $SU(2)$ of index 1. Then by Proposition 4 [6], the bundle is of diagonal type. Hence by Theorem 2 above we obtain a lifting action which fixes an irreducible self-dual connection.

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