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ANR of σ-Metric Stratifiable Spaces

Bao-Lin GUO and Takuo MIWA

Department of Mathematics, Shimane University, Matsue, Japan
Department of Mathematics, Northeast Normal University,
ChangChun, Jilin, China
and
Department of Mathematics, Shimane University, Matsue, Japan

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For a real vector space E, the second author introduced the locally convex topology $\mathscr{T}$ in [15] such that $(E, \mathscr{T})$ is the strongest locally convex topology contained in the finite topology. In this paper, we shall prove the following:

(1) $(E, \mathscr{T})$ is a σ-metric stratifiable space.

(2) For any σ-metric stratifiable space X, X can be embedded in a AR(σ-metric stratifiable)-space as a closed subset.

(3) For each natural number n, the fundamental subspace E_n of $(E, \mathscr{T})$ is AE(stratifiable).

(4) For any σ-metric stratifiable space X, X is AR(σ-metric stratifiable)(resp. ANR) if and only if X is AE(σ-metric stratifiable)(resp. ANE).

§1. Introduction

In [18], K. Nagami called a topological space *σ-metric* if the space is the countable union of closed metric subsets. (Gruenhage called it F_σ-metrizable in [7].) K. Nagami introduced the notion of σ-metric spaces for the purpose of investigations of dimension theory, and dimension theory of σ-metric spaces was studied in [18], [19], [17] etc.

On the other hand, many examples of stratifiable spaces seem to have the σ-metric type. For example, every CW-complex is σ-metric, and even every chunk complex [5] is also σ-metric. Further every Hyman's M-space is also of this type (cf. [10], [20]).

In this paper, we study ANR of σ-metric stratifiable spaces. In section 3, we prove that the space $|E|_C$ is σ-metric, where $|E|_C$ is the linear space E equipped with the locally convex topology (cf. [15]). Furthermore, we show that each σ-metric stratifiable space X can be embedded into the AR(σ-metric stratifiable)-space $E(X)$ as a closed subset (for $E(X)$, see [14]). In section 4, we prove that, for each natural number n, the fundamental subspace E_n of $|E|_C$ is hyperconnected, accordingly it is AE(stratifiable). In section 5, we shall give some considerations for adjunction spaces and some generalizations of the Borsuk-Whitehead-Hanner's theorem.

Throughout this paper, we assume that all spaces are regular and all maps are continuous. The letters N and R denote the set of all natural numbers and all real numbers, respectively. For M_1-spaces and stratifiable spaces, see [5] and [1]. For AR, AE, ANR and ANE, see [9]. Every terminology should be referred to [6], [9] and [11], unless otherwise stated.

§2. Preliminaries

In this paper, we exclusively use the notation which we state in this section. E is a real vector space with a Hamel basis $\mathscr{B} = \{u_\alpha: \alpha \in \Lambda\}$. Let $\mathscr{E}_n$ be all n-dimensional linear subspaces of E generated by n elements of $\mathscr{B}$ (i.e. $\mathscr{E}_n = \{\langle u_{\alpha_1}, \dots, u_{\alpha_n}\rangle : \alpha_i \in \Lambda$, for $i = 1, \dots, n\})$.

Now, we restate the construction of the locally convex topology in a real vector space ([15; Construction 2.1]).

CONSTRUCTION 2.1. Let E be a real vector space with a Hamel basis $\mathscr{B} = \{u_\alpha: \alpha \in \Lambda\}$, and $\mathscr{E}_n$ all n-dimensional linear subspaces of E generated by n elements of $\mathscr{B}$. For each $\alpha \in \Lambda$, pick up $n_\alpha \in N$. Let $U_1 = \cup\{\{tu_\alpha: |t| < 1/n_\alpha\}: \alpha \in \Lambda\}$. By using induction, if U_{n-1} has been defined for $n \geq 2$, let $U_n = \cup \{\text{conv}(F \cap U_{n-1}): F \in \mathscr{E}_n\}$, where conv A is the convex hull of A. Let $U(n_\alpha: \alpha \in \Lambda) = \cup\{U_n: n \in N\}$ and $\mathscr{U}$ be all $U(n_\alpha: \alpha \in \Lambda)$.

By [15; Lemma 2.2], $\mathscr{U}$ satisfies the local base condition. Therefore by [11; Theorem 5.1], $\mathscr{T} = \{W \subset E$: For each $x \in W$, there is $U \in \mathscr{U}$ with $x + U \subset W\}$ is a vector topology (i.e. $(E, \mathscr{T})$ is a linear topological space) and $\mathscr{U}$ is a local base for $\mathscr{T}$. We denote the space E equipped with this topology $\mathscr{T}$ by $|E|_C$, and we call it the *locally convex topology*.

For a full simplicial complex K, we embed K in a suitable vector space E with the locally convex topology so that its vertices are at the unit points of E. In this case, we say that K has the *locally convex topology*, and we denote the space K with this topology by $|K|_C$. (Note that the original definition of the locally convex topology of K [13] coincides with the above definition.) For some investigations of $|E|_C$ and $|K|_C$, see [13], [15] and [16].

For a space X, we restate the construction of $E(X)$([14; Construction 3.1]).

CONSTRUCTION 2.2. Let X be a space. $A(X)$ denotes the full simplicial complex with the locally convex topology which has all points of X as the set of vectices. Let i be the canonical bijection from the 0-skeleton A^0 of $A(X)$ onto X. Then $E(X)$ is the set $A(X)$ equipped with the topology generated by sets U such that

(C1) U is open in $A(X)$ and $i(U \cap X)$ is open in X,

(C2) U is convex in $A(X)$.

It is clear from (C1) that X is closed in $E(X)$. By (C2), it is clear that $E(X)$ is locally convex. For some considertion of $E(X)$, see [14].

§3. Embeddings to AR spaces

For a real vector space E, we first prove the following:

THEOREM 3.1. $|E|_C$ *is* σ-*metric.*

PROOF. For each $n\in N$ and each $F\in\mathscr{E}_n$, since F is homeomorphic to the n-dimensional Euclidean space, we can suppose that d is the Euclidean metric function on F. For $x, y\in F$, we define a metric function d_F on F as follows:

$$d_F(x, y) = \min\{1, d(x, y)\}.$$

For any $F\in\mathscr{E}_1$ and each $m\in N$, let $F^m = \{x\in F: d_F(x, 0)\geq 1/m\}$, where 0 is the origin of E. For any $F = \langle u_{\alpha_1}, u_{\alpha_2}\rangle\in\mathscr{E}_2$ and each $m\in N$, let $F^m = \{x\in F: d_F(x, \langle u_{\alpha_i}\rangle)\geq 1/m, i = 1,2\}$. In general, for any $F = \langle u_{\alpha_1},\dots, u_{\alpha_n}\rangle\in\mathscr{E}_n$ and each $m\in N$, let $F^m = \{x\in F: d_F(x, \langle u_{\alpha_1},\dots,\hat{u}_{\alpha_j},\dots, u_{\alpha_n}\rangle)\geq 1/m, j = 1,\dots, n\}$, where $\langle u_{\alpha_1},\dots,\hat{u}_{\alpha_j},\dots, u_{\alpha_n}\rangle = \langle u_{\alpha_1},\dots, u_{\alpha_{j-1}}, u_{\alpha_{j+1}},\dots, u_{\alpha_n}\rangle$.

Now, we construct a countable cover of $|E|_C$. Let $A_0 = \{0\}$. For each $m\in N$ and $n\in N$, let $A_n^m = \cup\{F^m: F\in\mathscr{E}_n\}$. Then it is clear that $\{A_0\}\cup\{A_n^m: m, n\in N\}$ is a countable cover of $|E|_C$. Next, we shall prove the following:

(1) A_n^m is closed in $|E|_C$ for each $m\in N$ and $n\in N$.

(2) A_n^m is metrizable for each $m\in N$ and $n\in N$.

Proof of (1): Let $x\notin A_n^m$. If $x = 0$, for each $\alpha\in\Lambda$ we can pick up some $n_\alpha\in N$ such that $1/n_\alpha < 1/m$. Then $U(n_\alpha: \alpha\in\Lambda)$ is a neighborhood of $x = 0$, and $U(n_\alpha: \alpha\in\Lambda)\cap F^m = \varnothing$ for each $F\in\mathscr{E}_n$. Therefore $U(n_\alpha: \alpha\in\Lambda)\cap A_n^m = \varnothing$. Next, if $x\neq 0$, there is $G = \langle u_{\alpha_1},\dots, u_{\alpha_k}\rangle\in\mathscr{E}_k$ such that $x\in G-\cup\{\langle u_{\alpha_1},\dots,\hat{u}_{\alpha_j},\dots, u_{\alpha_k}\rangle: j = 1,\dots, k\}$. In case $k < n$, for each $\alpha\in\Lambda$, there is $n_\alpha\in N$ such that $1/n_\alpha < 1/m$. Then $W = x + U(n_\alpha: \alpha\in\Lambda)$ is a neighborhood of x, and it is easily seen that $W\cap F^m = \varnothing$ for each $F\in\mathscr{E}_n$. Thus $W\cap A_n^m = \varnothing$. In case $k = n$, let $\varepsilon = d_G(x, G^m)$. For each $\alpha_i(i = 1,\dots, k)$, there is $n_{\alpha_i}\in N$ such that $1/n_{\alpha_i} < \varepsilon/k$. For each $\beta\in\Lambda$—$\{\alpha_1,\dots,\alpha_k\}$, there is $n_\beta\in N$ such that $1/n_\beta < 1/m$. For these $n_\alpha(\alpha\in\Lambda)$, $W = x + U(n_\alpha: \alpha\in\Lambda)$ is a neighborhood of x, and it is easily seen that $W\cap F^m = \varnothing$ for each $F\in\mathscr{E}_n$. Thus $W\cap A_n^m = \varnothing$. In case $k > n$, if $x = a_{\alpha_1}u_{\alpha_1} + \cdots + a_{\alpha_k}u_{\alpha_k}$, let $\varepsilon = \min\{|a_{\alpha_i}|: i = 1,\dots, k\}$. For each $\alpha_i(i = 1,\dots, k)$, there is $n_{\alpha_i}\in N$ such that $1/n_{\alpha_i} < \varepsilon$. For each $\beta\in\Lambda-\{\alpha_1,\dots,\alpha_k\}$, there is $n_\beta\in N$ such that $1/n_\beta < 1/m$. For these $n_\alpha(\alpha\in\Lambda)$, $W = x + U(n_\alpha: \alpha\in\Lambda)$ is a neighborhood of x, and it is easily verified that $W\cap F^m = \varnothing$ for each $F\in\mathscr{E}_n$. Thus $W\cap A_n^m = \varnothing$. For all cases, there is a neighborhood W of x such that $W\cap A_n^m = \varnothing$. This proves that A_n^m is closed in $|E|_C$.

Proof of (2): We define a metric function on A_n^m as follows: For each $x, y\in A_n^m$,

$$d(x, y) = \begin{cases} d_F(x, y) & (\text{if } x, y\in F^m \text{ for some } F\in\mathscr{E}_n) \\ 1 & (\text{if } x\in F^m,\ y\in G^m,\ F\neq G, \text{ for some } F, G\in\mathscr{E}_n). \end{cases}$$

It is easy to see that d is a metric function on A_n^m. Further, the relative topology of A_n^m coincides with the topology induced by d. In fact, for any point $x \in A_n^m$, $\{(x + U(n_\alpha: \alpha \in \Lambda)) \cap A_n^m: U(n_\alpha: \alpha \in \Lambda) \in \mathscr{U}\}$ and $\{B(x; \varepsilon): \varepsilon > 0\}$ (where $B(x; \varepsilon) = \{y \in A_n^m: d(x, y) < \varepsilon\}$) are equivalent local bases of x in A_n^m. Thus the proof is completed.

The following corollary is trivial.

COROLLARY 3.2. *Every subspace of* $|E|_C$ *is* σ*-metric. In particular, for a simplicial complex* K, $|K|_C$ *is* σ*-metric.*

We obtain the next theorem as a by-product of the proof of Theorem 3.1. In fact, each A_n^m does not contain any open subset of $|E|_C$.

THEOREM 3.3. $|E|_C$ *is not a Baire space.*

In conclusion of this section, we prove the closed embedding theorem of σ-metric stratifiable spaces.

THEOREM 3.4. *If* X *is a* σ*-metric stratifiable space, then* $E(X)$ *is an* AR(σ-metric stratifiable)-*space containing* X *as a closed subset.*

PROOF. We use the notation of Construction 2.2. First since X is σ-metric space, let $X = \cup\{A_n: n \in N\}$, where A_n is closed in X for each $n \in N$. Then since X is closed in $E(X)$, each A_n is closed in $E(X)$. Next, since $E(X)$ is stratifiable by [14; Theorem 3.3], X is a G_δ-subset of $E(X)$. There is a countable open family $\{U_n: n \in N\}$ of $E(X)$ such that $\cap\{U_n: n \in N\} = X$. Since $E(X) - U_n$ is a closed subset of $A(X)$, by Corollary 3.2 there is a countable closed family $\{B_{nk}: k \in N\}$ of $E(X) - U_n$ such that $E(X) - U_n = \cup\{B_{nk}: k \in N\}$ and each B_{nk} is metrizable. Therefore $E(X) = (\cup\{A_n: n \in N\}) \cup (\cup\{B_{nk}: n \in N, k \in N\})$. Thus $E(X)$ is σ-metric. By [14; Theorem 3.4], since $E(X)$ is hyperconnected, $E(X)$ is AR(σ-metric stratifiable). Thus the proof is completed.

§4. The fundamental subspaces E_n *of* $|E|_C$

Let $E_n = \cup \mathscr{E}_n = \cup\{F: F \in \mathscr{E}_n\}$. We call each E_n the *fundamental subspace* of $|E|_C$. In this section, we prove that each E_n is AE(stratifiable). Before proving this theorem, we state the definition of hyperconnectedness (cf. [12] or [2]). Throughout this section, let P_{n-1} denote the unit simplex in the n-dimensional Euclidean space R^n(i.e., $P_{n-1} = \{t \in R^n: \sum_{i=1}^{n} t_i = 1$ and each $t_i \geq 0\}$), and A^n the n-fold cartesian product of any set A. Furthermore, let $\delta_i: A^n \to A^{n-1}$ be the function defined by

$$\delta_i(a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

for $i = 1, \ldots, n$.

DEFINITION 4.1. A space L will be called *hyperconnected* if there exist functions $h_i: L^i \times P_{i-1} \to L$ for each $i \in N$, such that they satisfy conditions (a), (b), (c):

(a) $t \in P_{n-1}$ and $t_i = 0$ implies $h_n(x, t) = h_{n-1}(\delta_i x, \delta_i t)$ for each $x \in L^n$ and $n \in N-\{1\}$,

(b) for each $x \in L^n$, the function $t \to h_n(x, t)$, from P_{n-1} to L, is continuous,

(c) for each $x \in L$ and neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and

$$\cup\{h_i(V^i \times P_{i-1}) : i \in N\} \subset U.$$

Now, we begin to prove the following lemmas.

LEMMA 4.2. *E_1 is hyperconnected.*

PROOF. $h_1: E_1 \times P_0 \to E_1$ is defined by $h_1(x, \{1\}) = x$.

In case $i = 2$, let $x = (x_1, x_2) \in (E_1)^2$ and $t = (t_1, t_2) \in P_1$. First, we consider the case that there is $F \in \mathscr{E}_1$ such that $x_1, x_2 \in F$. Then $h_2: (E_1)^2 \times P_1 \to E_1$ is defined by

$$h_2(x, t) = t_1 x_1 + t_2 x_2.$$

Next, in case that there are $F_i \in \mathscr{E}_1 (i = 1, 2)$ such that $x_i \in F_i (i = 1, 2)$, $x_1 = x_{1\beta} u_\beta$ and $x_2 = x_{2\gamma} u_\gamma$. If $x_{1\beta} x_{2\gamma} > 0$, the segment $[x_1, x_2] (= \{s_1 x_1 + s_2 x_2 : s_1 + s_2 = 1,\ s_1, s_2 \geq 0\})$ and the line $\langle u_\beta + u_\gamma \rangle$ cross at $\bar{t}_1 x_1 + (1 - \bar{t}_1) x_2$, where $\bar{t}_1 = \dfrac{x_{2\gamma}}{x_{1\beta} + x_{2\gamma}}$. If $x_{1\beta} x_{2\gamma} < 0$, the segment $[x_1, x_2]$ and the line $\langle u_\beta - u_\gamma \rangle$ cross at $\bar{t}_1 x_1 + (1 - \bar{t}_1) x_2$, where $\bar{t}_1 = \dfrac{x_{2\gamma}}{x_{2\gamma} - x_{1\beta}}$. Then h_2 is defined by

$$h_2(x, t) = \begin{cases} \dfrac{\bar{t}_1 - t_1}{\bar{t}_1} x_2 & (\text{if } 0 \leq t_1 \leq \bar{t}_1) \\[2ex] \dfrac{t_1 - \bar{t}_1}{1 - \bar{t}_1} x_1 & (\text{if } \bar{t}_1 \leqq t_1 \leqq 1). \end{cases}$$

In case $i = 3$, let $x = (x_1, x_2, x_3) \in (E_1)^3$ and $t = (t_1, t_2, t_3) \in P_2$. First, we consider the case that there is $F \in \mathscr{E}_1$ such that $x_i \in F (i = 1, 2, 3)$. Then $h_3: (E_1)^3 \times P_2 \to E_1$ is defined by

$$h_3(x, t) = t_1 x_1 + t_2 x_2 + t_3 x_3.$$

Next, in case that there is $F \in \mathscr{E}_1$ such that $x_1 \in F$, $x_k \notin F (k = 2, 3)$, h_2 is defined by

$$h_3(x, t) = \begin{cases} h_2\left(\left(x_1, h_2\left(\delta_1 x, \delta_1\left(\dfrac{t}{1 - t_1}\right)\right)\right), (t_1, 1 - t_1)\right) & (\text{if } t_1 \neq 1) \\[2ex] x_1 & (\text{if } t_1 = 1). \end{cases}$$

Any other case (i.e. $x_2 \in F$, $x_1 \notin F$, $x_3 \notin F$; etc.) is similar.

We assume that, for $k \leq n-1$, $h_k: (E_1)^k \times P_{k-1} \to E_1$ were defined inductively. In case $i = n$, let $x = (x_1, \dots, x_n) \in (E_1)^n$ and $t = (t_1, \dots, t_n) \in P_{n-1}$. First, we consider the case that there is $F \in \mathscr{E}_1$ such that $x_i \in F (i = 1, \dots, n)$. Then $h_n: (E_1)^n \times P_{n-1} \to E_1$ is defined by

$$h_n(x, t) = t_1 x_1 + t_2 x_2 + \cdots + t_n x_n.$$

Next, in case that there is $F \in \mathscr{E}_1$ such that $x_1 \in F$, $x_i \notin F (i = 2, \dots, n)$, h_n is defined by

$$h_n(x, t) = \begin{cases} h_2\left(\left(x_1, h_{n-1}\left(\delta_1 x, \delta_1\left(\dfrac{t}{1 - t_1}\right)\right)\right), (t_1, 1 - t_1)\right) & (\text{if } t_1 \neq 1) \\ x_1 & (\text{if } t_1 = 1). \end{cases}$$

Any other case (i.e. $x_2 \in F$, $x_1 \notin F$, $x_i \notin F (i = 3, \dots, n)$; etc.) is similar.

It is easily verified by the constructions of h_n and the locally convex topology that these functions h_n satisfy the conditions (a), (b), (c) of Definition 4.1.

LEMMA 4.3. *E_2 is hyperconnected.*

PROOF. $h_1: E_2 \times P_0 \to E_2$ is defined by $h_1(x, \{1\}) = x$.

In case $i = 2$, let $x = (x_1, x_2) \in (E_2)^2$ and $t = (t_1, t_2) \in P_1$. First, we consider the case that there is $F \in \mathscr{E}_2$ such that $x_1, x_2 \in F$. Then $h_2: (E_2)^2 \times P_1 \to E_2$ is defined by

$$h_2(x, t) = t_1 x_1 + t_2 x_2.$$

Secondly, in case that there are $F_i \in \mathscr{E}_2 (i = 1, 2)$ such that $x_i \in F_i (i = 1, 2)$, $x_1 = x_{1\alpha} u_\alpha + x_{1\beta} u_\beta$ and $x_2 = x_{2\alpha} u_\alpha + x_{2\gamma} u_\gamma$. If $x_{1\beta} x_{2\gamma} > 0$, the segment $[x_1, x_2] (= \{s_1 x_1 + s_2 x_2 : s_1 + s_2 = 1, s_1, s_2 \geq 0\})$ and the plane $\langle u_\alpha, u_\beta + u_\gamma \rangle$ cross at $\bar{t}_1 x_1 + (1 - \bar{t}_1) x_2$, where $\bar{t}_1 = \dfrac{x_{2\gamma}}{x_{1\beta} + x_{2\gamma}}$. If $x_{1\beta} x_{2\gamma} < 0$, the segment $[x_1, x_2]$ and the plane $\langle u_\alpha, u_\beta - u_\gamma \rangle$ cross at $\bar{t}_1 x_1 + (1 - \bar{t}_1) x_2$, where $\bar{t}_1 = \dfrac{x_{2\gamma}}{x_{2\gamma} - x_{1\beta}}$. Then, for the point

$$z_0 = (\bar{t}_1 x_{1\alpha} + (1 - \bar{t}_1) x_{2\alpha}) u_\alpha,$$

h_2 is defined by

$$h_2(x, t) = \begin{cases} \dfrac{t_1}{\bar{t}_1} z_0 + \dfrac{\bar{t}_1 - t_1}{\bar{t}_1} x_2 & (\text{if } 0 \leq t_1 \leq \bar{t}_1) \\ \dfrac{1 - t_1}{1 - \bar{t}_1} z_0 + \dfrac{t_1 - \bar{t}_1}{1 - \bar{t}_1} x_1 & (\text{if } \bar{t}_1 \leq t_1 \leq 1). \end{cases}$$

Thirdly, in case that there are $F_i \in \mathscr{E}_2 (i = 1, 2)$ such that $x_i \in F_i (i = 1, 2)$ and $F_1 \cap F_2 = \{0\}$. Then h_2 is defined by the same method in the above. (For general cases, see the proof of Lemma 4.4.)

In case $i = 3$, let $x = (x_1, x_2, x_3) \in (E_2)^3$ and $t = (t_1, t_2, t_3) \in P_2$. First, we consider the case that there is $F \in \mathscr{E}_2$ such that $x_i \in F (i = 1, 2, 3)$. Then $h_3: (E_2)^3 \times P_2 \to E_2$ is defined by

$$h_3(x, t) = t_1 x_1 + t_2 x_2 + t_3 x_3.$$

Next, in case that there is $F \in \mathscr{E}_2$ such that $x_1 \in F$, $x_i \notin F (i = 2, 3)$, h_3 is defined by

$$h_3(x, t) = \begin{cases} h_2\left(\left(x_1, h_2\left(\delta_1 x, \delta_1\left(\dfrac{t}{1 - t_1}\right)\right)\right), (t_1, 1 - t_1)\right) & (\text{if } t_1 \neq 1) \\ x_1 & (\text{if } t_1 = 1). \end{cases}$$

Any other case (i.e. $x_2 \in F$, $x_1 \notin F$, $x_3 \notin F$; etc.) is similar.

We assume that, for $k \leq n - 1$, $h_k: (E_2)^k \times P_{k-1} \to E_2$ were defined, inductively. In case $i = n$, $h_n: (E_2)^n \times P_{n-1} \to E_2$ is defined as same as in Lemma 4.2.

Furthermore, it is easily verified by the constructions of h_n and the locally convex topology that these functions h_n satisfy the conditions (a), (b), (c) of Definition 4.1.

LEMMA 4.4. *For each $n \geq 3$, E_n is hyperconnected.*

PROOF. We assume that the index set Λ of the Hamel basis $\mathscr{B}$ is a well-ordered set with the order $\leq$, and we introduce the lexicographic order to $\Lambda \times \Lambda$.

In case $i = 1$, $h_1: E_n \times P_0 \to E_n$ is trivially defined.

In case $i = 2$, let $x = (x_1, x_2) \in (E_n)^2$ and $t = (t_1, t_2) \in P_1$. First, we consider the case that there is $F \in \mathscr{E}_n$ such that $x_1, x_2 \in F$. Then $h_2: (E_n)^2 \times P_1 \to E_n$ is defined by

$$h_2(x, t) = t_1 x_1 + t_2 x_2.$$

Secondly, in case that there are $F_i \in \mathscr{E}_n (i = 1, 2)$ such that $x_i \in F_i (i = 1, 2)$ and $F_1 = \langle u_{\alpha_1}, \ldots, u_{\alpha_k}, u_{\alpha_{k+1}}, \ldots, u_{\alpha_n} \rangle$, $F_2 = \langle u_{\alpha_1}, \ldots, u_{\alpha_k}, u_{\beta_{k+1}}, \ldots, u_{\beta_n} \rangle$, where $1 \leq k \leq n - 1$, $\alpha_i \neq \beta_j$ for $k + 1 \leq i, j \leq n$ and $\alpha_{k+1} \leq \alpha_{k+2} \leq \cdots \leq \alpha_n$, $\beta_{k+1} \leq \beta_{k+2} \leq \cdots \leq \beta_n$. Let $A = \{(\alpha_i, \beta_j): k + 1 \leq i, j \leq n\}$ be a subset of $\Lambda \times \Lambda$. Further let

$$x_1 = x_{1\alpha_1} u_{\alpha_1} + \cdots + x_{1\alpha_k} u_{\alpha_k} + x_{1\alpha_{k+1}} u_{\alpha_{k+1}} + \cdots + x_{1\alpha_n} u_{\alpha_n}$$

$$x_2 = x_{2\alpha_1} u_{\alpha_1} + \cdots + x_{2\alpha_k} u_{\alpha_k} + x_{2\beta_{k+1}} u_{\beta_{k+1}} + \cdots + x_{2\beta_n} u_{\beta_n}.$$

Then, since A is a well-ordered set, there exists

$$(\alpha_p, \beta_q) = \min\{(\alpha, \beta) \in A: x_{1\alpha} x_{2\beta} \neq 0\}.$$

If $x_{1\alpha_p} x_{2\beta_q} > 0$, the segment $[x_1, x_2]$ and the plane $\langle u_{\alpha_1}, \ldots, u_{\alpha_k}, u_{\alpha_p} + u_{\beta_q}, u_{\alpha_{p+1}}, \ldots, u_{\alpha_n}, u_{\beta_{q+1}}, \ldots, u_{\beta_n} \rangle$ cross at $\bar{t}_1 x_1 + (1 - \bar{t}_1) x_2$, where $\bar{t}_1 = \dfrac{x_{2\beta_q}}{x_{1\alpha_p} + x_{2\beta_q}}$. If $x_{1\alpha_p} x_{2\beta_q} < 0$, the segment $[x_1, x_2]$ and the plane $\langle u_{\alpha_1}, \ldots, u_{\alpha_k}, u_{\alpha_p} - u_{\beta_q}, u_{\alpha_{p+1}}, \ldots,$

$u_{\alpha_n}, u_{\beta_{q+1}},\dots, u_{\beta_n}\rangle$ cross at $\bar{t}_1 x_1 + (1-\bar{t}_1)x_2$, where $\bar{t}_1 = \dfrac{x_{2\beta_q}}{x_{2\beta_q} - x_{1\alpha_p}}$. Then, for the point

$$z_0 = (\bar{t}_1 x_{1\alpha_1} + (1-\bar{t}_1)x_{2\alpha_1})u_{\alpha_1} + \cdots + (\bar{t}_1 x_{1\alpha_k} + (1-\bar{t}_1)x_{2\alpha_k})u_{\alpha_k},$$

h_2 is defined by

$$h_2(x, t) = \begin{cases} \dfrac{t_1}{\bar{t}_1} z_0 + \dfrac{\bar{t}_1 - t_1}{\bar{t}_1} x_2 & (\text{if } 0 \le t_1 \le \bar{t}_1) \\ \dfrac{1-t_1}{1-\bar{t}_1} z_0 + \dfrac{t_1 - \bar{t}_1}{1-\bar{t}_1} x_1 & (\text{if } \bar{t}_1 \le t_1 \le 1). \end{cases}$$

Thirdly, in case that there are $F_i \in \mathscr{E}_n (i = 1, 2)$ such that $x_i \in F_i (i = 1, 2)$ and $F_1 \cap F_2 = \{0\}$. Then h_2 is defined by the same method in the above; the case $k = 0$.

In case $i = 3$ or $i = k (k \ge 4)$, h_i: $(E_n)^i \times P_{i-1} \to E_n$ is defined as same as in Lemma 4.3.

Furthermore, it is easily verified by the constructions of h_n and the locally convex topology that these functions h_n satisfy the conditions (a), (b), (c) of Definition 4.1.

By Lemmas 4.2–4.4 and [2; Theorem 4.1], we have

THEOREM 4.5. *For each $n \in N$, E_n is hyperconnected. Therefore, E_n is* AE(stratifiable).

§5. Adjunction spaces

It is obvious that:

PROPOSITION 5.1. *Let X, Y be σ-metric, A a closed subset of X and $f: A \to Y$ a map. Then the adjunction spase $X \cup_f Y$ is also σ-metric.*

Since the adjunction space of two stratifiable space is stratifiable [1, Theorem 6.2], the following is obtained by the well-known method which uses adjunction spaces (cf. [9]).

THEOREM 5.2. *For a σ-metric stratifiable space X, X is* AR(σ-metric stratifiable) (*resp.* ANR) *if and only if X is* AE(σ-metric stratifiable) (*resp.* ANE).

It is well known [9; pp178] that if X, A and Y are ANR(metric)'s and $f: A \to Y$ a map, then the adjunction space $X \cup_f Y$ is ANR(metric) provided that it is metrizable. This result was essentially proved in successive stages by Borsuk [3], Whitehead [22] and Hanner [8]. For attempt to generalize this theorem, Hyman [10] proved the case of Hyman's M-spaces. Cauty [4] announced the case of stratifiable spaces, but his proof was false. This was pointed out by San-nou [21]. Therefore the case of stratifiable spaces is still open. Even the case of σ-metric stratifiable spaces is still open.

PROBLEM 5.3. Let X and Y be two stratifiable spaces, A a closed subset of X and $f: A \to Y$ a map. If X, A and Y are ANR(stratifiable)'s, is the adjunction space $X \cup_f Y$ ANR(stratifiable)?

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