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Projectivity of Homogeneous Left Loops on Lie Groups I

(Algebraic Framework)

Michihiko KIKKAWA

Department of Mathematics, Shimane University, Matsue, Japan
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This is an algebraic generalization of the results in the previous paper [6]. The concept of projective relation in the class of abstract homogeneous left loops (on the same underlying set) is introduced, and all homogeneous left loops which are in projective relation with an arbitrarily given abstract group (G, μ^0) are characterized by a slass of mappings from G into the automorphism group $\mathrm{Aut}(G, \mu^0)$. This will be developed for geodesic homogeneous left loops on Lie groups elsewhere.

§1. Introduction

Let (G, μ) be a left loop with the associated ternary operation η (cf. [6]):

$$\eta(x, y, z) = L_x\, \mu(L_x^{-1} y, L_x^{-1} z),$$

where L_x denotes the left translation by x in G. We denote the unit of the left loop by e. The left loop (G, μ) is said to be *homogeneous* provided that it has the left inverse property; $L_x^{-1} = L_{x^{-1}}$ for $x^{-1} = L_x^{-1} e$, and that any left inner mapping $L_{x,y} = L_{\mu(x,y)}^{-1} L_x L_y$ is an automorphism of (G, μ), which is equivalent to the homogeneity equation:

$$\eta(u, v, \eta(x, y, z)) = \eta(\eta(u, v, x),\ \eta(u, v, y),\ \eta(u, v, z)) \tag{1.1}$$

for any u, v, x, y, z in G. This ternary operation η is called the *homogeneous system* of the homogeneous left loop (G, μ) (cf. [4], [5]). Let $(G, \tilde{\mu})$ and (G, μ) be homogeneous left loops on the same underlying set G. Then, $(G, \tilde{\mu})$ will be said to be *in projective relation* with (G, μ) if both of them are power associative, with the same unit e, the same inverse x^{-1} and powers x^p for any x, and have the following interrelations:

$$\begin{aligned} \tilde{\eta}(u, v, \eta(x, y, z)) &= \eta(\tilde{\eta}(u, v, x),\ \tilde{\eta}(u, v, y),\ \tilde{\eta}(u, v, z)), \\ \eta(u, v, \tilde{\eta}(x, y, z)) &= \tilde{\eta}(\eta(u, v, x),\ \eta(u, v, y),\ \eta(u, v, z)), \end{aligned} \tag{1.2}$$

where $\tilde{\eta}$ and η are homogeneous systems of $(G, \tilde{\mu})$ and (G, μ), respectively.

REMARK. The projective relation is not an equivalence relation. In fact, it is not transitive although it is reflective and symmetric.

We know that any group (G, μ^0) is a homogeneous (left) loop with the homogeneous system

$$\eta^0(x, y, z) = yx^{-1}z. \tag{1.3}$$

In this paper, the homogeneous left loops which are in projective relation with any group (G, μ^0) will be investigated, and a necessary and sufficient condition for the multiplication $\tilde{\mu}$ on G will be found under which $(G, \tilde{\mu})$ be in projective relation with the group (G, μ^0).

In the succeeding paper, this algebraic result will be applied to the studying of geodesic homogeneous left loops on Lie groups. The case where the given group is the additive group on $\mathbf{R}^n$ has been investigated in [6] (see Example 2 in this paper).

§2. Main theorem

Let (G, μ^0) be an abstract group with the associated homogeneous system η^0 given by (1.3). The group multiplication will be denoted by juxtaposition. Now, assume that there exists a homogeneous left loop $(G, \tilde{\mu})$ which is in projective relation with the group (G, μ^0). Then, by definition, the unit e of the group is the two-sided unit of the power associative left loop $(G, \tilde{\mu})$, the inverse x^{-1} and any powers x^p of any element x are coincident with each other in the group and in the left loop, respectively. Furthermore, the homogeneous system $\tilde{\eta}$ of $(G, \tilde{\mu})$ satisfies the following relations:

$$uv^{-1}\tilde{\eta}(x, y, z) = \tilde{\eta}(uv^{-1}x,\ uv^{-1}y,\ uv^{-1}z) \tag{2.1}$$

$$\tilde{\eta}(u, v, xy^{-1}z) = \tilde{\eta}(u, v, x)\ \tilde{\eta}(u, v, y)^{-1}\ \tilde{\eta}(u, v, z). \tag{2.2}$$

If we denote by L_x and $\tilde{L}_x$ the left translations by x in (G, μ^0) and $(G, \tilde{\mu})$, respectively, we have;

$$L_x^{-1}\,\tilde{L}_x = \tilde{L}_x\,L_x^{-1}, \qquad x \in G. \tag{2.3}$$

Now, consider the map $d\colon G \to \mathrm{Aut}(G, \mu^0)$; $x \mapsto d_x$, from G into the automorphism group $\mathrm{Aut}(G, \mu^0)$ of the group (G, μ^0) defined by

$$d_x y = L_x^{-1}\,\tilde{L}_x y. \tag{2.4}$$

In fact, the map d_x is a group automorphism because

$$\begin{aligned} d_x(yz) &= L_x^{-1}\,\tilde{\eta}(e, x, yz) \\ &= L_x^{-1}\,\tilde{\eta}(e, x, y)\ \tilde{\eta}(e, x, e)^{-1}\ \tilde{\eta}(e, x, z) \qquad \text{by (2.2)} \\ &= (L_x^{-1}\,\tilde{L}_x y)(L_x^{-1}\,\tilde{L}_x z) \\ &= (d_x y)(d_x z). \end{aligned}$$

By using (2.1) we can also show that d_x is an automorphism of the left loop $(G, \tilde{\mu})$. We will show that the map d above satisfies the followings:

(i) $d_e = \mathrm{Id}$ (The identity map)
(ii) $d_x x = x$
(iii) $d_x^{-1} = d_{x^{-1}}$
(iv) $d_x d_y = d_{d_x y} d_x$.

In fact, (i) and (ii) are clear. (iii) follows immediately from the left inverse property; $\tilde{\mu}(x^{-1}, \tilde{\mu}(x, y)) = y$. Since $\tilde{\mu}(x^{-1}, y) = \tilde{\eta}(x, e, y)$, (iv) is shown as follows:

$$\begin{aligned} d_x d_y z &= d_x \tilde{L}_y L_y^{-1} z \\ &= d_x \tilde{\mu}(y, \mu^0(y^{-1}, z)) \\ &= \tilde{\mu}(d_x y, (d_x y)^{-1}(d_x z)) \\ &= \tilde{L}_{d_x y} L_{d_x y}^{-1} d_x z \\ &= d_{d_x y} d_x z. \end{aligned}$$

Our main thorem contains also the converse of the fact above, that is;

THEOREM. *Let (G, μ^0) be an abstract group. If there exists a homogeneous left loop $(G, \tilde{\mu})$ on G which is in projective relation with (G, μ^0), then, for every x in G, the map d_x: $G \to G$ given by the relation*

$$\tilde{\mu}(x, y) = x\, d_x y \tag{2.5}$$

is an automorphism of the group (G, μ^0) satisfying (i), (ii), (iii) *and* (iv) *above. Conversely, let d be a map from G into the automorphism group of the group (G, μ^0) satisfying the conditions* (i)-(iv). *Then, the multiplication $\tilde{\mu}$ on G given by* (2.5) *forms a homogeneous left loop which is in projective relation with the group (G, μ^0).*

PROOF. Since the relation (2.5) is equivalent to (2.4), the first half of the theorem has been shown above. The converse part is shown as follows: Let d be a map from G into the automorphism group of the group (G, μ^0) satisfying the conditions (i)-(iv). Then, it is clear that the multiplication $\tilde{\mu}(x, y) = x\, d_x y$ has the two-sided unit e and bijective left translations $\tilde{L}_x = L_x d_x$ for x in G, that is, $(G, \tilde{\mu})$ forms a left loop. Since $d_x x = x$ holds, the left loop $(G, \tilde{\mu})$ is power associative with the same inverse x^{-1} and powers x^p (p is any integer) as those of the group (G, μ^0). The left inverse property $\tilde{L}_x^{-1} = \tilde{L}_{x^{-1}}$ is obtained from (iii) with $d_{x^{-1}} x = x$. Next, we show that the left loop (G, μ) is homogeneous. We can see that any left inner map $\tilde{L}_{x,y}$ of $(G, \tilde{\mu})$ is given by

$$\tilde{L}_{x,y} = d_{\tilde{\mu}(x,y)}^{-1} d_y. \tag{2.6}$$

Then, it is sufficient to show that any group automorphism d_u, $u \in G$, is an automorphism of the left loop $(G, \tilde{\mu})$. In fact, by using (iv), we get

$$\begin{aligned}\tilde{\mu}(d_u x, d_u y) &= (d_u x)(d_{d_u x} d_u y)\\ &= (d_u x)(d_u d_x y)\\ &= d_u(x\, d_x y)\\ &= d_u \tilde{\mu}(x, y).\end{aligned}$$

Here, we find that the explicit form of the homogeneous system $\tilde{\eta}$ of the homogeneous left loop $(G, \tilde{\mu})$ is given by

$$\tilde{\eta}(x, y, z) = y\, d_{x^{-1}y}(x^{-1}z). \tag{2.7}$$

Finally, we show that the left loop $(G, \tilde{\mu})$ is in projective relation with the group (G, μ^0). In fact, substituting $uv^{-1}x$, $uv^{-1}y$, $uv^{-1}z$ for x, y, z in (2.7), respectively, we get the interrelation (2.1). The other relation (2.2) is obtained by using (2.7) as follows:

$$\begin{aligned}\tilde{\eta}(u, v, xy^{-1}z) &= v\, d_{u^{-1}v}(u^{-1}xy^{-1}z)\\ &= v(d_{u^{-1}v}u)^{-1}(d_{u^{-1}v}x)(d_{u^{-1}v}y)^{-1}(d_{u^{-1}v}z)\\ &= v\, \tilde{\eta}(u, v, u)^{-1}\, \tilde{\eta}(u, v, x)\, \tilde{\eta}(u, v, y)^{-1}\, \tilde{\eta}(u, v, z)\\ &= \eta^0(\tilde{\eta}(u, v, x), \tilde{\eta}(u, v, y), \tilde{\eta}(u, v, z))\end{aligned}$$

q.e.d.

§3. Examples

Let (G, μ^0) be an abstract group. Before we consider some examples of homogeneous left loops in projective relation with (G, μ^0), we notice the following facts which are obtained easily:

COROLLARY. *Let* $d: G \to \mathrm{Aut}(G, \mu^0)$ *be a map satisfying* (i)–(iv) *in Theorem. Denote by* d^{-1} *and* d^m *(m is a positive integer) the maps from G into* $\mathrm{Aut}(G, \mu^0)$ *which send each x in G to the inverse automorphism* d_x^{-1} *and the m-time power* d_x^m *of* d_x, *respectively, and set* $d^{-m} = (d^m)^{-1}$. *Then, for each integer p, the map* $d^p: G \to \mathrm{Aut}(G, \mu^0)$ *satisfies* (i)–(iv) *again, and any homogeneous left loops* (G, μ^p) *and* (G, μ^q) *associated with the maps* d^p *and* d^q *by Theorem, respectively, are in projective relation with each other.*

EXAMPLE 1. Let δ_x, $x \in G$, denote the inner automorphism of the group (G, μ^0) by x. Then, it is evident that the map $\delta: G \to \mathrm{Aut}(G, \mu^0)$ satisfies the conditions (i)–(iv) in Theorem. Hence, we get a homogeneous left loop (G, μ^1) on G given by

$$\mu^1(x, y) = x^2 y x^{-1}, \tag{3.1}$$

which is in projective relation with the given group (G, μ^0). By Corollary above, we see that the power δ^p (p is any integer) satisfies (i)–(iv) and we get a series of mutually projective homogeneous left loops (G, μ^p), $p \in \mathbb{Z}$, on G given by

(3.2) $$\mu^p(x, y) = x^{p+1} y\, y^{-p}.$$

In [2] and [3], we consider a condition

(3.3) $$\mu(x, y)^{-1} = \mu(x^{-1}, y^{-1})$$

for any loop (G, μ), and call it a symmetric loop if this condition is satisfied. Hereafter, we will call a left loop (G, μ) to be *symmetric* if it satisfies this condition (3.3).

PROPOSITION 1. *Let (G, μ^0) be an abstract group with the inner automorphisms $\delta_x y = xyx^{-1}$. The homogeneous left loop (G, μ^p) given above is symmetric if and only if x^{2p+1} belongs to the center of the group (G, μ^0) for each x.*

REMARK. In [1], G. Glauberman developed an extensive theory of finite B-loops of odd order, a certain class of symmetric finite loops of odd order realized in some groups. If the order of the group (G, μ^0) is of $2m+1$, then the left loop (G, μ^m) above forms a B-loop of order $2m+1$.

EXAMPLE 2. Let V be a real Lie algebra of dimension n. We consider the additive group $\mu^0(X, Y) = X + Y$ on V. For each X in V, set

$$d_X = \exp \operatorname{ad} X,$$

where $\operatorname{ad}: V \to \operatorname{End}(V)$ denotes the adjoint representation of the Lie algebra V. Then, the followings are easily checked: (i) $d_0 = \mathrm{Id}$, (ii) $d_X X = X$, (iii) $d_X d_{-X} Y = Y$ and (iv) $d_X d_Y Z = \exp \operatorname{ad} d_X Y(d_X Z)$. Hence, by Theorem, we see that the map $d: V \to \mathrm{GL}(V)$ gives a homogeneous left loop (V, μ^1);

$$\mu^1(X, Y) = X + A(X)Y, \quad A(X) = \exp \operatorname{ad} X \in \mathrm{GL}(V),$$

which is in projective relation with additive group $(V, +)$. We have considered this left loop in [6]. By Corollary, we see that there exists a series of homogeneous left loops (V, μ^p), $p \in \mathbf{Z}$, on V which are in projective relation with each other. In this example, the multiplicatin μ^p, for every integer p, is induced from the other Lie algebra on V with the Lie bracket $[\,,]^p$ given by

$$[X, Y]^p = p[X, Y], \quad X, Y \in V,$$

which is isomorphic to the given Lie algebra. Since $(\exp \operatorname{ad} X)^p = \exp \operatorname{ad} pX$ holds, we get

$$\begin{aligned} \mu^p(X, Y) &= X + \exp \operatorname{ad} pX(Y) \\ &= \frac{1}{p}\mu^1(pX, pY), \end{aligned}$$

that is, the left loops (V, μ^p) and (V, μ^1) are isomorphic with each other for $p \neq 0$. Thus, we have;

PROPOSITION 2. *Any two of the series of homogeneous left loops* (V, μ^p), $p \in \mathbf{Z} - \{0\}$, *obtained from a real Lie algebra* V *as above are isomorphic with each other.*

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