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A Structure Theory of Freudenthal-Kantor Triple Systems III

Dedicated to Professor Nathan Jacobson on his 80th birthday

By

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In this paper, we give a construction of balanced Freudenthal-Kantor triple systems and investigate a structure of the Jordan triple systems associated with reduced balanced Freudenthal-Kantor triple systems.

Introduction

The triple systems studied here are a specialization of the class of Freudenthal-Kantor triple systems given in [21, 22, 13], which is called balanced by ourselves. This triple system is a variation of Freudenthal triple systems [7, 18], symplectic ternary algebras [6] and symplectic triple systems [23]. This paper is a continuation of the previous articles [13, 14]. The main purpose of this article is to give followings:

(i) A construction of Jordan triple systems from a vector space equipped with relations of a cross product and a bilinear form.

(ii) A construction of balanced Freudenthal-Kantor triple systems from a class of vector matrices as follows:

$$\begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix}, \quad \alpha, \beta \in \Phi, \quad a, b \in V$$

where Φ is a base field, V is the Jordan triple system defined by (i).

(iii) If a simple balanced Freudenthal-Kantor triple system $\mathfrak{M}$ is reduced, then $\mathfrak{M} \cong \mathfrak{M}(V)$, where $\mathfrak{M}(V)$ is the balanced Freudenthal-Kantor triple system defined by (ii).

We shall be concerned with algebras and triple systems which are finite dimensional over a field Φ of characteristic different from 2 or 3, unless otherwise specified. We shall mainly employ the notation and terminology in [13, 14].

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In this section, we shall give a construction of Jordan triple systems and consider the norm similarity.

THEOREM 1.1. *Let V be a vector space over an arbitrary field Φ equipped with a bilinear form $B(a, b)$ and a cross product $a \times b$ satisfying the following conditions:*

(1) $a \times b = b \times a$

(2) $B(a, b) = B(b, a)$

(3) $B(a, b \times d) = B(a \times b, d)$

(4) $((a \times b) \times e) \times d + ((b \times d) \times e) \times a + ((d \times a) \times e) \times b$
$= B(a \times b, d)e + B(a, e)b \times d + B(b, e)d \times a + B(d, e)a \times b$

for all $a, b, d, e \in V$.

Then V becomes a Jordan triple system with respect to the triple product

$$\{xyz\} = B(x, y)z + B(z, y)x - (x \times z) \times y.$$

PROOF. By the definition of the triple product, it is clear that

$$\{xyz\} = \{zyx\}.$$

We compute as follows;

$$\begin{aligned}
&\{uv\{xyz\}\} - \{\{uvx\}yz\} + \{x\{vuy\}z\} - \{xy\{uvz\}\} \\
&= B(u, v)(B(x, y)z + B(z, y)x - (x \times z) \times y) \\
&\quad + B(B(x, y)z + B(z, y)x - (x \times z) \times y, v)u \\
&\quad - (u \times (B(x, y)z + B(z, y)x - (x \times z) \times y)) \times v \\
&\quad - B(B(u, v)x + B(x, v)u - (u \times x) \times v, y)z \\
&\quad - B(z, y)(B(u, v)x + B(x, v)u - (u \times x) \times v) \\
&\quad + ((B(u, v)x + B(x, v)u - (u \times x) \times v) \times z) \times y \\
&\quad + B(x, B(v, u)y + B(y, u)v - (v \times y) \times u)z \\
&\quad + B(z, B(v, u)y + B(y, u)v - (v \times y) \times u)x \\
&\quad - (x \times z) \times (B(v, u)y + B(y, u)v - (v \times y) \times u) \\
&\quad - B(x, y)(B(u, v)z + B(z, v)u - (u \times z) \times v) \\
&\quad - B(B(u, v)z + B(z, v)u - (u \times z) \times v, y)x \\
&\quad + (x \times (B(u, v)z + B(z, v)u - (u \times z) \times v)) \times y \\
&= - B((x \times z) \times y, v)u + (u \times ((x \times z) \times y)) \times v \\
&\quad + B(x, v)(u \times z) \times y - (x \times z) \times B(y, u)v \\
&\quad - (((u \times x) \times v) \times z) \times y + (x \times z) \times ((v \times y) \times u) \\
&\quad + B(z, v)(x \times u) \times y - (x \times ((u \times z) \times v)) \times y
\end{aligned}$$

$$= (B(x, v)(u \times z) - ((u \times x) \times v) \times z$$
$$+ B(z, v)(x \times u) - x \times ((u \times z) \times v)) \times y$$
$$(- ((v \times (x \times z)) \times u) + B(x \times z, u)v + B(v, u)x \times z) \times y$$

(by the relation (4) of the assumption, that is, $B((x \times z) \times y, v)u - (u \times ((x \times z) \times y)) \times v + (x \times z) \times B(y, u)v - (x \times z) \times ((v \times y) \times u) = ((v \times (x \times z)) \times u) \times y - B(x \times z, u)v \times y - B(v, u)y \times (x \times x))$

$$= 0.$$

(by the relation (4))
This completes the proof. ■

If N is a cubic form on a vector space V and $c \in V$ a basepoint where $N(c) = 1$, then we can form the trace form

$$T(x, y) = -\partial_x \partial_y \log N|_c = (\partial_x N|_c)(\partial_y N|_c) - \partial_x \partial_y N|_c$$

of N at c. We say N is nondegenerate at c if its trace form is nondegenerate. For nondegenerate forms we have a unique quadratic mapping $x \to x^{\#}$ in V defined by $T(x^{\#}, y) = \partial_y N|_x$. We say a nondegenerate cubic form N and basepoint c are admissible if the adjoint identity $x^{\#\#} = N(x)x$ holds under all scalar extensions (see [17]). We denote this vector space V by $\mathfrak{J}(N, c)$. For ch $\Phi \neq 2, 3$, to apply the case of our construction, we put $2x^{\#} = x \times x$, $T(x, y) = B(x, y)$ and $N(x) = 1/3\, T(x^{\#}, x)$. We can easily show that if $N(x)x = x^{\#\#}$, then $4/3\, B(x \times x, x)x = (x \times x) \times (x \times x)$. Also these identities yield the relation $x \times (x^{\#} \times y) = N(x)y + T(x, y)x^{\#}$ (by the argument of density of V). Hence this result implies that

$$((x \times x) \times y) \times x = 1/3\, B(x \times x, x)y + B(x, y)x \times x,$$

which reduce the relation (4) of the assumption in Theorem 1. Thus we obtain the following corollary.

COROLLARY [17]. *If the cubic form N and basepoint c are admissible then $\mathfrak{J}(N, c)$ is a Jordan triple system with respect to the triple product*

$$\{xyz\} = T(x, y)z + T(z, y)x - (x \times z) \times y.$$

THEOREM 1.2. *Let V be a vector space over a field Φ of characteristic $\neq 2$ or 3 equipped with a bilinear form $B(a, b)$ and a cross product $a \times b$ satisfying the relations (1) ~ (4) of Theorem 1.1 and the following conditions;*

(5) $$(a \times b) \times (e \times d) + (b \times e) \times (d \times a) + (e \times a) \times (b \times d)$$
$$= B(a \times b, e)d + B(a \times e, d)b + B(a \times d, b)e + B(b \times e, d)a,$$

(6) *there exists an element $c \in V$ such that*

$$x \times c = B(x, c)c - x \text{ for all } x \in V.$$

Then it holds

$$x^3 - T(x)x^2 + S(x)x - N(x)c = 0$$

and $x \times x = 2x^2 - 2T(x)x + 2S(x)c$ *for all* $x \in V$,
where $x^3 = 1/2\{xxx\}$, $x^2 = 1/2\{xcx\}$, $T(x) = B(x, c)$, $S(x) = 1/2B(x \times x, c)$ *and* $N(x) = 1/6B(x \times x, x)$.

PROOF. From $x^3 = 1/2\{xxx\}$ and $x^2 = 1/2\{xcx\}$, we have

$$\begin{aligned} x^3 - B(x, c)x^2 &= 1/2\{xxx\} - 1/2B(x, c)\{xcx\} \\ &= B(x, x)x - 1/2(x \times x) \times x - 1/2B(x, c)(2B(x, c)x - (x \times x) \times c) \\ &= (B(x, x) - B(x, c)^2)x - 1/2(x - B(x, c)c) \times (x \times x). \end{aligned} \tag{1–1}$$

On the other hand, we have

$$\begin{aligned} B(x \times y, c) &= B(x, y \times c) \\ &= B(y, c)B(x, c) - B(x, y) \text{ (by the relation (6) of the assumption).} \end{aligned}$$

If we put $y = x$, then this implies that

$$B(x \times x, c) = B(x, c)^2 - B(x, x).$$

Combining this with the identity (1–1), we get

$$x^3 - B(x, c)x^2 = -B(x \times x, c)x + (x \times c) \times (x \times x). \tag{1–2}$$

By the relation (5) of the assumption, we have

$$(x \times c) \times (x \times x) = 1/3B(x \times x, x)c + B(x \times c, x)x. \tag{1–3}$$

From (1–2) and (1–3), it follows that

$$x^3 - B(x, c)x^2 + 1/2B(x \times x, c)x - 1/6B(x \times x, x)c = 0.$$

Hence this yields that

$$x^3 - T(x)x^2 + S(x)x - N(x)c = 0,$$

where $T(x) = B(x, c)$, $S(x) = 1/2B(x \times x, c)$ and $N(x) = 1/6B(x \times x, x)$. Also, it follows from $x \times c = B(x, c)c - x$ that

$$(x \times x) \times c = B(x \times x, c)c - x \times x.$$

From this identity and the identity $x^2 - T(x)x = -1/2(x \times x) \times c$, we obtain

$$x^2 - T(x)x = -1/2(B(x \times x, c)c - x \times x),$$

which implies $x \times x = 2x^2 - 2T(x)x + 2S(x)c$. This completes the proof. ■

THEOREM 1.3. *Let* V(resp. V') *be a vector space over an infinite field* Φ *of characteristic* $\neq 2$ *or* 3 *equipped with a nondegenerate bilinear form* $B(a, b)$ (resp. $B(a, b)'$) *and a cross product* $a \times b$(resp. $a' \times b'$) *satisfying the relations* (1), (2), *(3) and (5) of Theorem 1.2 (resp. (1)′, (2)′, (3)′ and (5)′). If a mapping g is invertible (= linear and bijective) from V onto V', then the followings are equivalent:*

(i) $B(ga \times ga, ga)' = \lambda B(a \times a, a)$ $\lambda \in \Phi^*$, for all $a \in V$

(ii) g is an isotopy of the Jordan triple system with respect to the triple product

$$\{xyz\} = B(x, y)z + B(z, y)x - (x \times z) \times y.$$

Furthermore, in the case of (ii), we have

$$g(x \times y) = \lambda \hat{g}x \times \hat{g}y, \quad \hat{g}(a \times b) = \lambda^{-1} ga \times gb$$

and $B(\hat{g}a, gb)' = B(a, b)$, where $B(g^* a', b) = B(a', gb)'$ and $\hat{g} = g^{*-1}$.

PRROF. (i) ⇒ (ii) If g is a bijective linear mapping, one may define a bijective linear mapping g^* of V' onto V by

$$B(g^* a', b) = B(a', gb)'.$$

Hence we have

$$B(ga \times ga, ga)' = B(g^*(ga \times ga), a). \tag{1–4}$$

From the assumption that $B(ga \times ga, ga)' = \lambda B(a \times a, a)$ and $B(,)$ is nondegenerate, we get

$$g^*(ga \times ga) = \lambda a \times a$$

and so $\hat{g}(a \times a) = \lambda^{-1} ga \times ga$, where $\hat{g} = g^{*-1}$. (1–5)

Using (1–5), we obtain

$$(ga \times ga) \times (ga \times ga) = \lambda^2 \hat{g}(a \times a) \times \hat{g}(a \times a). \tag{1–6}$$

By relation (5)′ of the assumption, we have

$$4B(ga \times ga, ga)' \, ga = 3(ga \times ga) \times (ga \times ga).$$

The left-hand side of equation (1–6) is equal to

$$\begin{aligned} &4/3 \; B(ga \times ga, ga)' \, ga \\ &= 4/3 \; \lambda B(a \times a, a) ga. \end{aligned}$$

Consequently, we get

$$4/3 \; B(a \times a, a) ga = \lambda \hat{g}(a \times a) \times \hat{g}(a \times a).$$

Replacing a by $a \times a$, we have

$$\begin{aligned} &4/3 \; B((a \times a) \times (a \times a), a \times a) g(a \times a) \\ &= \lambda \hat{g}((a \times a) \times (a \times a)) \times \hat{g}((a \times a) \times (a \times a)). \end{aligned}$$

Using the relation $(a \times a) \times (a \times a) = 4/3 \; B(a \times a, a) a$, we get

$$(4/3)^2\ (B(a \times a, a))^2\ g(a \times a) = \lambda(4/3\ B(a \times a, a))^2\ \hat{g}a \times \hat{g}a.$$

By using a density argument, that is, $B(a \times a, a) \neq 0$ for all $a \neq 0$ in V, we obtain

$$g(a \times a) = \lambda \hat{g}a \times \hat{g}a.$$

In $B(g^* a', b) = B(a', gb)'$, putting $a = g^* a'$, we have

$$B(a, b) = B(\hat{g}a, gb)'.$$

From the definition of the triple product

$$\{xyz\} = B(x, y)z + B(z, y)x - (x \times z) \times y,$$

we can see that

$$\begin{aligned} g\{xyz\} &= B(x, y)gz + B(z, y)gx - g((x \times z) \times y) \\ &= B(gx, \hat{g}y)' gz + B(gz, \hat{g}y)' gx - \lambda(\hat{g}(x \times z) \times \hat{g}y) \\ &= B(gx, \hat{g}y)' gz + B(gz, \hat{g}y)' gx - (gx \times gz) \times \hat{g}y \\ &= \{gx\hat{g}ygz\}'. \end{aligned}$$

Similarly we have

$$\hat{g}\{xyz\} = \{\hat{g}xgy\hat{g}z\}'.$$

(ii)$\Rightarrow$(i). Let g be an isotopy satisfying $\hat{g}(x \times y) = \lambda^{-1}(gx \times gy)$ and $g(x \times y) = \lambda \hat{g}x \times \hat{g}y$. From $g\{xyz\} = \{gx\hat{g}ygz\}'$ and the definition of the triple product, we have

$$\begin{aligned} &B(x, y)gz + B(z, y)gx - (gx \times gz) \times \hat{g}y \\ &\quad = B(gx, \hat{g}y)' gz + B(gz, \hat{g}y)' gx - (gx \times gz) \times \hat{g}y. \end{aligned} \tag{1–7}$$

Putting $x = z$ in the identity (1–7), we get

$$B(x, y) = B(gx, \hat{g}y)'. \tag{1–8}$$

Replacing y by $x \times x$ in the equation (1–8), we have

$$\begin{aligned} &B(x \times x, x) = \lambda^{-1} B(gx \times gx, gx)' \\ &(\text{by } \hat{g}(x \times x) = \lambda^{-1}(gx \times gx)). \end{aligned}$$

This completes the proof. ■

Theorem 1.3 can be regarded as a generalization of the following proposition for a Jordan triple system.

PROPOSITION 1.4. [12] *Let V and V' be reduced simple exceptional Jordan algebras. Then the following conditions are equivalent:*

(1) *V and V' are isotopic,*

(2) *V and V' are norm similar.*

If A is a linear mapping of a vector space V equipped with a bilinear form $B(x, y)$ and a cross product $x \times y$ into itself satisfying

$$B(Ax, x \times x) = \rho B(x \times x, x) \text{ for all } x \in V \tag{1-9}$$

where $\rho \in \Phi^*$ is fixed and satisfying (1–9) for all field extensions of Φ, then A is said to be a Lie similarity of V. Then we have the following;

THEOREM 1.5. *Let V be as in Theorem* 1.3. *If A is a linear mapping of V into itself, then the followings are equivalent;*
(i) *A is a Lie similarity of V.*
(ii) *There exists a linear mapping A of V into itself satisfying*

$$A\{xyz\} = \{Axyz\} + \{xA^*yz\} + \{xyAz\}$$

where A^* is the linear mapping of V into itself defined by $B(A^*x, y) = - B(x, Ay)$.

REMARK. *The above theorem implies that the notion of structure algebra of the Jordan triple system V coincides that of Lie similarity. (for the definition of structure algebra, see* [13]). *In particular, if the cross product is zero, then an arbitrary linear mapping A of V is a Lie similarity, hence if V has a nondegenerate bilinear form, the mapping A is a structure algebra of V.*

2

In this section, we shall study a construction of the prototype of a balanced Freudenthal-Kantor triple system with $\varepsilon = 1$.

For $\varepsilon = \pm 1$, a triple system $U(\varepsilon)$ with the triple product $\langle -, -, - \rangle$ is called a Freudenthal-Kantor triple system if

$$\text{(U1)} \quad [L(a, b), L(c, d)] = L(\langle abc \rangle, d) + \varepsilon L(c, \langle bad \rangle) \tag{2-1}$$

$$\text{(U2)} \quad K(K(a, b)c, d) - L(d, c)K(a, b) + \varepsilon K(a, b)L(c, d) = 0, \tag{2-2}$$

where $L(a, b)c = \langle abc \rangle$ and $K(a, b)c = \langle acb \rangle - \langle bca \rangle$.

DEFINITION. A Freudenthal-Kantor triple system is balanced if there exists an anti-symmetric bilinear form $\langle , \rangle$ such that $K(x, y) = \langle x, y \rangle \operatorname{Id}$, $\langle x, y \rangle \in \Phi^*$.

REMARK. *From results in* [14], *we note the following:*
(i) *The case of $\varepsilon = -1$ does not occur in a balanced Freudenthal-Kantor triple system.*
(ii) *A balanced Freudenthal-Kantor triple system is simple if and only if the anti-symmetric bilinear form $\langle , \rangle$ is nondegenerate.*
(iii) *The derivation of semisimple Freudenthal-Kantor triple systems over a field of characteristic* 0 *is a finite sum of inner derivations of $L(a, b) + \varepsilon L(b, a)$ (denoted by $S(a, b)$).*

Let V be a vector space over an arbitrary field Φ equipped with a bilinear form $B(a, b)$ and a cross product $a \times b$ satisfying the following conditions:

(1) $a \times b = b \times a$

(2) $B(a, b) = B(b, a)$

(3) $B(a, b \times d) = B(a \times b, d)$

(4) $((a \times b) \times e) \times d + ((b \times d) \times e) \times a + ((d \times a) \times e) \times b$
$= B(a \times b, d)e + B(a, e)b \times d + B(b, e)d \times a + B(d, e)a \times b$

(5) $(a \times b) \times (e \times d) + (b \times e) \times (d \times a) + (e \times a) \times (b \times d)$
$= B(a \times b, e)d + B(a \times e, d)b + B(a \times d, b)e + B(b \times e, d)a$

for all $a. b. d. e \in V$.

In particular, for ch $\Phi \neq 2, 3$, $3(a \times a) \times (a \times a) = 4B(a, a \times a)a$ holds under two conditions that "$a \times b = 0 \Rightarrow a = 0$ or $b = 0$" (division property) and $((a \times a) \times b) \times a = 1/3B(a \times a, a)b + B(a, b)a \times a$.

EXAMPLE. $\mathfrak{J}(N, c)$ satisfies the conditions (1) ~ (5).

We can consider the set of vector matrices with coefficients in the vector space V as follows:

$$\mathfrak{M}(V) = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \middle| \alpha, \beta \in \Phi, a, b \in V \right\}.$$

In $\mathfrak{M}(V)$, we shall introduce an operation $\circ$, that is,

$$\begin{pmatrix} \alpha_1 & a_1 \\ b_1 & \beta_1 \end{pmatrix} \circ \begin{pmatrix} \alpha_2 & a_2 \\ b_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + B(a_1, b_2) & \alpha_1 a_2 + \beta_2 a_1 + b_1 \times b_2 \\ \alpha_2 b_1 + \beta_1 b_2 + a_1 \times a_2 & \beta_2\beta_1 + B(a_2, b_1) \end{pmatrix}.$$

Next we shall use the following mapping to consider a triple product

$$P: \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \longrightarrow \begin{pmatrix} -\alpha & a \\ -b & \beta \end{pmatrix}$$

and

$$- : \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \longrightarrow \begin{pmatrix} \beta & a \\ b & \alpha \end{pmatrix}.$$

Thereby we can define a triple product on $\mathfrak{M}(V)$ as follows:

$$\langle x_1 x_2 x_3 \rangle = x_1 \circ (\overline{Px}_2 \circ x_3) + x_3 \circ (\overline{Px}_2 \circ x_1) - Px_2 \circ (\overline{x}_1 \circ x_3) \tag{2–3}$$

where $x_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix} \in \mathfrak{M}(V)$.

We have the following result on this vector matrix $\mathfrak{M}(V)$.

THEOREM 2.1. *Let* $\mathfrak{M}(V)$ *be the set of vector matrices of the above. Then* $(\mathfrak{M}(V), \langle -, -, - \rangle)$ *is a balanced Freudenthal-Kantor triple system with respect to the above triple product* (2–3).

PROOF. From the assumptions (1), (2), (3), (4) and (5) of vector space V, we can obtain this theorem by straightforward but very long calculations and we omit it. ■

We call $\mathfrak{M}(V)$ the balanced Freudenthal-Kantor triple system induced from the Jordan triple system V satisfying the conditions (1) ~ (5).

REMARK. For ch $\Phi \neq 2$, we note that

$$\langle x_1, x_2 \rangle = \beta_1\alpha_2 - \alpha_1\beta_2 + B(a_1, b_2) - B(b_1, a_2) \tag{2–4}$$

and

$$\gamma(x_1, x_2) = 4\langle x_1, x_2 \rangle \tag{2–5}$$

where $\gamma(x_1, x_2) = 1/2[\mathrm{tr}2(R(x_1, x_2) - R(x_2, x_1)) + L(x_1, x_2) - L(x_2, x_1)]$.

DEFINITION [7, 18]. A Freudenthal triple system is a vector space $\mathfrak{M}$ with trilinear product $(x, y, z) \rightarrow [xyz]$ and anti-symmetric bilinear form $(x, y) \rightarrow \langle x, y \rangle_F$ such that

(A1) $[xyz]$ is symmetric in all arguments;

(A2) $q_F(x, y, z, w) = \langle x, [yzw] \rangle_F$ is a nonzero symmetric 4-linear form;

(A3) $[[xxx]xy] = \langle y, x \rangle_F [xxx] + \langle y, [xxx] \rangle_F x$

for $x, y, z, w \in \mathfrak{M}$.

PROPOSITION 2.2. *If* $(\mathfrak{M}, \langle -, -, - \rangle)$ *is a balanced Freudenthal-Kantor triple system equipped with* $K(x, y) = \langle x, y \rangle \, Id$ *over a field ch* $\Phi \neq 2$, *then* $(\mathfrak{M}. [-, -, -])$ *is a Freudenthal triple system satisfying* $\langle x, y \rangle_F = 1/2\langle x, y \rangle$ *with respect to the triple product*

$$[xyz] := 1/2(\langle xyz \rangle + \langle xyz \rangle).$$

PROOF. (i) By the balanced condition, we have

$$\langle xyz \rangle - \langle yxz \rangle = -\langle xzy \rangle + \langle yzx \rangle.$$

Hence we have

$$\begin{aligned}[xyz] &= 1/2(\langle xyz \rangle + \langle xzy \rangle) \\ &= 1/2(\langle yzx \rangle + \langle yxz \rangle) \\ &= [yzx].\end{aligned}$$

From the definition of triple product, we have

$$[xyz] = [xzy].$$

(ii) Since $L(x, y) - L(y, x) = \langle y, x\rangle \mathrm{Id}$, we have

$$[L(x, y) - L(y, x), L(z, w)] = 0.$$

Similarly, $[L(y, x), L(z, w) - L(w, z)] = 0$ holds. Hence we get $[L(x, y), L(z, w)] = [L(y, x), L(w, z)]$. From (U1) with $\varepsilon = 1$, it follows that

$$L(\langle xyz\rangle, w) + L(z, \langle yxw\rangle) - L(\langle yxw\rangle, z\rangle - L(w, \langle xyz\rangle) = 0.$$

Therefore we obatain

$$\langle\langle xyz\rangle, w\rangle + \langle z, \langle yxw\rangle\rangle = 0. \tag{2–6}$$

Similarly, $\langle\langle xyz\rangle, w\rangle + \langle x, \langle wzy\rangle\rangle = 0$ holds. (2–7)

On theother hand, we have

$$\langle z, [ywx]\rangle_F = 1/4(\langle z, \langle ywx\rangle\rangle + \langle z, \langle yxw\rangle\rangle) \tag{2–8}$$

where $\langle a, b\rangle_F = 1/2\langle a, b\rangle$ (i.e., $\langle,\rangle_F$: the anti-symmetric bilinear form induced from an anti-symmetric form $\langle,\rangle$ of balanced Freudenthal-Kantor triple system). Combining this with (2–7), we get

$$\langle z, [ywx]\rangle_F = \langle y, [zwx]\rangle_F.$$

(iii) From (U1) with $\varepsilon = 1$, we have

$$\langle x\langle xxx\rangle y\rangle = -\langle\langle xxx\rangle xy\rangle. \tag{2–9}$$

Putting $z = x$ in $K(y, z)x = \langle zyx\rangle - \langle yzx\rangle$, we have

$$2\langle yxx\rangle = \langle xyx\rangle + \langle xxy\rangle. \tag{2–10}$$

Linearizing this relation, we get

$$\langle yxz\rangle + \langle yzx\rangle = 1/2(\langle xyz\rangle + \langle xzy\rangle + \langle zyx\rangle + \langle zxy\rangle). \tag{2–11}$$

Replacing $z = \langle xxx\rangle$, we have

$$\begin{aligned}\langle yx\langle xxx\rangle\rangle + \langle y\langle xxx\rangle x\rangle \\ = 1/2(\langle xy\langle xxx\rangle\rangle + \langle x\langle xxx\rangle y\rangle \\ + \langle\langle xxx\rangle yx\rangle + \langle\langle xxx\rangle xy\rangle).\end{aligned}$$

Combining this with (2–9), we have

$$\begin{aligned}\langle yx\langle xxx\rangle\rangle + \langle y\langle xxx\rangle x\rangle \\ = 1/2(\langle xy\langle xxx\rangle\rangle + \langle\langle xxx\rangle yx\rangle).\end{aligned} \tag{2–12}$$

From $K(x, y)\langle xxx\rangle = -L(x, y)\langle xxx\rangle + L(y, x)\langle xxx\rangle$, *we have*

$$\langle xy\,\langle xxx\rangle\rangle - \langle yx\,\langle xxx\rangle\rangle = -\langle x, y\rangle\langle xxx\rangle. \tag{2–13}$$

From $L(\langle xxx\rangle, y)x - L(y, \langle xxx\rangle)x = -K(\langle xxx\rangle, y)x$, we have

$$\langle\langle xxx\rangle\, yx\rangle - \langle y\,\langle xxx\rangle\, x\rangle = -\langle\langle xxx\rangle, y\rangle x. \tag{2–14}$$

Therefore by (2–13) and (2–14), we have

$$\begin{aligned}&\langle xy\,\langle xxx\rangle\rangle + \langle\langle xxx\rangle\, yx\rangle - \langle yx\,\langle xxx\rangle\rangle - \langle y\,\langle xxx\rangle\, x\rangle\\ &= -\langle x, y\rangle\langle xxx\rangle - \langle\langle xxx\rangle, y\rangle x.\end{aligned} \tag{2–15}$$

From (2–12) and (2–15), we obtain

$$\begin{aligned}&\langle yx\,\langle xxx\rangle\rangle + \langle y\,\langle xxx\rangle\, x\rangle\\ &= -\langle x, y\rangle\langle xxx\rangle - \langle\langle xxx\rangle, y\rangle x.\end{aligned}$$

Consequently, by means of $[xyz] = 1/2(\langle xyz\rangle + \langle xyz\rangle)$ and $\langle x, y\rangle_F = 1/2\langle x, y\rangle$, we have

$$[yx\,[xxx]] = \langle y, x\rangle_F\,[xxx] + \langle y, [xxx]\rangle_F\, x.$$

This completes the proof. ■

PROPOSITION 2.3. *If* $(\mathfrak{M}, [-,-,-], \langle,\rangle_F)$ *is a Freudenthal triple system over a field of ch* $\Phi \neq 2$, *then* $(\mathfrak{M}, \langle -,-,-\rangle)$ *is a balanced Freudenthal-Kantor triple system with respect to the triple product*

$$\langle xyz\rangle := [xyz] + \langle y, z\rangle_F\, x + \langle x, z\rangle_F\, y + \langle y, x\rangle_F\, z.$$

In this case, it holds $K(x, y) = 2\langle x, y\rangle_F\, Id$.

PROOF. From the definition of triple system. we have

$$\langle xyz\rangle - \langle yxz\rangle = 2\langle y, x\rangle_F\, z$$

and

$$\langle xzy\rangle - \langle yzx\rangle = 2\langle x, y\rangle_F\, z.$$

Hence we get

$K(x, y) = -L(x, y) + L(y, x) = 2\langle x, y\rangle_F$ Id (balanced property). Consequently, this yiels that

$$K(K(x, y)a, b) - L(b, a)K(x, y) + K(x, y)L(a, b) = 0.$$

We shall next show that the following equality holds,

$$\langle xy\,\langle abz\rangle\rangle = \langle\langle xyz\rangle\, bz\rangle + \langle a\,\langle yxb\rangle\, z\rangle + \langle ab\,\langle xyz\rangle\rangle.$$

This is verified by using (A2) and the following relation, which can be obtained from linearizations of (A3):

$$\begin{aligned}&[[xaz]\,by] - [[byz]\,ax] - [[bya]\,zx] - [[byx]\,za] \\ &\quad = -\langle b, [xaz]\rangle_F\, y - \langle y, [xaz]\rangle_F\, u + \langle y, a\rangle_F\, [xbz] + \langle y, z\rangle_F\, [xba] \\ &\qquad + \langle y, x\rangle_F\, [baz] + \langle b, z\rangle_F\, [yax] + \langle b, a\rangle_F\, [xyz] + \langle b, x\rangle_F\, [yza].\end{aligned}$$

This completes the proof. ■

The Freudenthal triple system $(\mathfrak{M}, [-, -, -])$ defined above is called the Freudenthal triple system associated with a balanced Freudenthal-Kantor triple system.

Let $V = \mathfrak{J}(N, c)$, and let the base field be characteristic zero. Then combining the above propositions with Satz 8, 4 in [18], we have dimensional formulas as follows;

THEOREM. 2.4. *Under the assumption of above, let $T(\mathfrak{M}(V))$ be the Lie triple system associated with $\mathfrak{M}(V)$ and $L(\mathfrak{M}(V))$ be the standard imbedding Lie algebra. If dim $\mathfrak{M}(V) = n$, then we have*

dim Der $\mathfrak{M}(V) = 3n(n+1)/(n+16)$.
dim Anti-Der $\mathfrak{M}(V) = 1$,
dim $T(\mathfrak{M}(V)) = 2n$ *and*
dim $L(\mathfrak{M}(V)) = (5n^2 + 38n + 48)/(n+16)$.

PROOF. Since the correspondence between the inner derivation $S(x, y)$ of a simple balanced Freudenthal-Kantor triple system and the derivation $D(x, y)$ of the Freudenthal triple system associated with it is given by

$$S(x, y)z = 2D(x, y)z := 2[xyz] - 2\langle z, y\rangle_F\, x - 2\langle z, x\rangle_F\, y,$$

the theorem is verified. ■

On the other hand, we have

dim $\mathfrak{A}$	1	2	4	8
dim V	6	9	15	27
dim $\mathfrak{M}(V)$	14	20	32	56

$$\text{where } V = \left\{ \begin{pmatrix} \xi_1 & c & \bar{b} \\ \bar{c} & \xi_2 & a \\ b & \bar{a} & \xi_3 \end{pmatrix} \middle| \begin{array}{l} \xi_i \in \Phi, a, b, c \in \mathfrak{A}.\ \text{(a composition} \\ \text{algebra over a field } \Phi) \\ - : \text{involution of the algebra } \mathfrak{A} \end{array} \right\}$$

(For composition algebras, see [12, 19]).

Therefore, for simple balanced Freudenthal-Kantor triple system over an algebraically closed field of characteristic 0, from the fact that $\mathfrak{M}(V)$ is simple if and only if $L(\mathfrak{M}(V))$ is simple [14], we can obtain simple Lie algebras;

dim $\mathfrak{A}$	1	2	4	8
Lie alg $L(\mathfrak{M}(V))$	F_4	E_6	E_7	E_8

For E_6, we note the followings: From dimension 78 of simple Lie algebras, it follows that there exist the type B_6, C_6 and E_6. In our case, since the dimension of the simple Lie triple system is 40's, we can obtain the type of E_6.

REMARK. If dim $\mathfrak{A} = 0$, then we have

$$\mathfrak{M}(V) := \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \middle| a = (\xi_1, \xi_2, \xi_3),\ b = (\eta_1, \eta_2, \eta_3),\ \xi_i,\ \eta_i \in \Phi \right\},$$

and $B(a, b) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$, $a \times a = 2(\xi_2\xi_3,\ \xi_1\xi_3,\ \xi_1\xi_2)$. Hence by straightforward calculations it is shown that the Lie algebra $L(\mathfrak{M}(V))$ is a simple Lie algebra of type D_4.

$$\text{Let}\quad \mathfrak{M}(\Phi) := \left\{ \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in \Phi \right\}. \quad B(\alpha, \beta) = \alpha\beta$$

and cross product be identically zero. Then it is clear that this matrix set $\mathfrak{M}(\Phi)$ satisfies conditions (1) $\sim$ (5). Therefore if Φ is an algebraically closed field of characteristic 0, then the standard imbedding Lie algebra $L(\mathfrak{M}(\Phi))$ is a simple Lie algebra of type G_2.

3

In this section, we shall consider a coordinatization theorem of simple reduced balanced Freudenthal-Kantor triple systems.

From now on we restrict our attention to simple balanced Freudenthal-Kantor triple systems $\mathfrak{M}$ over a field of characteristic $\neq 2$ or 3.

DEFINITION. *$u \in \mathfrak{M}$ is rank one if*

$$L(u, u) = 0. \tag{3–1}$$

REMARK. If an element a is rank one in the vector space V equipped with the conditions (1) $\sim$ (5) in Section 2(that is, $a \times a = 0$ and $a \neq 0$), then the element $\begin{pmatrix} \alpha & a \\ 0 & 0 \end{pmatrix}$ is rank one in $\mathfrak{M}(V)$, where α is an arbitrary element in Φ.

LEMMA 3.1. *Let $(\mathfrak{M}, \langle -, -, - \rangle)$ be a balanced Freudenthal-Kantor triple* *system and* $(\mathfrak{M},[-,-,-])$ *be the Freudenthal triple system associated with it. Then an element u is rank one in* $(\mathfrak{M},\langle -,-,-\rangle)$ *if and only if u is strictly regular in* $(\mathfrak{M},[-,-,-])$. *(for the definition of strictly regular element, for example* [7])

PROOF. "only if": Let $\langle uux\rangle=0$ for all $x\in\mathfrak{M}$. Since $\langle u,x\rangle u=\langle uux\rangle-\langle xuu\rangle$, we get

$$\langle xuu\rangle=-\langle u,x\rangle u. \tag{3–2}$$

On the other hand, by the balanced property, we have

$$\langle u,x\rangle u=-\langle uxu\rangle+\langle xuu\rangle. \tag{3–3}$$

Form (3–2) and (3–3), it follows that

$$\langle uxu\rangle=-2\langle u,x\rangle u. \tag{3–4}$$

By the definition of the triple product

$$[xyz]=1/2(\langle xyz\rangle+\langle xzy\rangle),$$

we obtain

$$[uxu]=-\langle u,x\rangle u,$$

which implies that u is strictly regular in $(\mathfrak{M}.[-,-,-])$.
"if": Let u be strictly regular. From the equation (5) in [7, p317], we have $[uuy]=2\langle y,u\rangle_F u$, where $\langle,\rangle_F$ is the anti-symmetric bilinear form of Freudenthal triple sytem. From Proposition 2.3. we get

$$\langle uuy\rangle=[uuy]+2\langle u,y\rangle_F u.$$

Therefore we obtain $\langle uuy\rangle=0$ for all $y\in\mathfrak{M}$. This completes the proof. ■

DEFINITION. A balanced Freudenthal-Kantor triple system $\mathfrak{M}$ is said to be reduced if $\mathfrak{M}$ contains a rank one element u.

DEFINITION. A pair of rank one element (u,v) is said to be supplementary if

$$K(u,v)=2Id. \tag{3–5}$$

PROPOSITION 3.2. *Let* $\mathfrak{M}$ *be a simple balanced Freudenthal-Kantor triple system. Then* $\mathfrak{M}$ *is reduced if and only if* $\mathfrak{M}$ *contains a pair of supplementary rank one elements.*

PROOF. Combining the above lemma 3.1 with Theorem 3.3 in [7], we can easily show the proposition. ■

COROLLARY. *Let* $\mathfrak{M}$ *be a simple balanced Freudenthal-Kantor triple system and* $q(x):=\langle\langle xxx\rangle,x\rangle$ *be a nonzero 4-linear form of* $\mathfrak{M}$. *Then* $\mathfrak{M}$ *is reduced if and only if* $\mathfrak{M}$ *contains an element x with* $q(x)=-24\beta^2$, $\beta\in\Phi^*$.

REMARK. Let $\mathfrak{M}$ be a balanced Freudenthal-Kantor triple system. Then for the 4-linear form $q(x, y, z, w) = \langle\langle xyz\rangle, w\rangle$ in $x, y, z, w \in \mathfrak{M}$, we have the following identies by straightforward calculations;

$$q(x, y, z, w) = q(w, z, y, x) = q(y, x, w, z) = q(z, w, x, y).$$

In particular,

$$q(x, x, x, y) = q(x, x, y, x) = q(x, y, x, x) = q(y, x, x, x).$$

Furtheremore, we have

$$q(S(x, y) z, z, z, z) = 0 \quad \text{for all } x, y, z \in \mathfrak{M},$$

where $S(x, y) = L(x, y) + L(y, x)$.

PROPOSITION 3.3. *Let $\mathfrak{M}$ be a simple balanced Freudenthal-Kantor triple system. If the 4-linear form $q(x)$ is identically zero, then it holds $\langle xyz\rangle = 1/2(\langle y, x\rangle z + \langle y, z\rangle x + \langle x, z\rangle y)$, for all $x, y, z \in \mathfrak{M}$.*

PROOF. By the fact that $\langle , \rangle$ is nondegenerate if and only if $\mathfrak{M}$ is simple, and from linearizing of $\langle\langle xxx\rangle, x\rangle = 0$ and the above remark it follows that

$$\langle xxx\rangle = 0 \text{ for all } x \in \mathfrak{M}.$$

Linearizing the identity $\langle xxx\rangle = 0$, we have

$$\langle xxy\rangle + \langle xyx\rangle + \langle yxx\rangle = 0. \tag{3–6}$$

From the assumption to be balanced, we have

$$\langle xxy\rangle = 2\langle yxx\rangle - \langle xyx\rangle. \tag{3–7}$$

Combining (3–6) with (3–7), we get from ch $\Phi \neq 3$

$$\langle yxx\rangle = 0.$$

Hence we have $\langle xyx\rangle = -\langle x, y\rangle x$. Linearizing this identity, we have

$$\langle xyz\rangle + \langle zyx\rangle = -\langle x, y\rangle z - \langle z, y\rangle x. \tag{3–8}$$

On the other hand, we have

$$\langle xyz\rangle - \langle zyx\rangle = \langle x, z\rangle y. \tag{3–9}$$

From (3–8) and (3–9), we obtain

$$\langle xyz\rangle = 1/2(\langle y, x\rangle z + \langle y, z\rangle x + \langle x, z\rangle y). \qquad \blacksquare$$

LEMMA 3.4. *Let (u, v) be a pair of supplementary rank one elements of simple balanced Freudenthal-Kantor triple system $\mathfrak{M}$. Then it holds*

$$1/4(R(u, v) + R(v, u))^2 x = x + 3/2\langle u, x\rangle v - 3/2\langle v, x\rangle \mathrm{u}$$

for all $x \in \mathfrak{M}$.

PROOF. From (U1) with $\varepsilon = 1$, we obtain the following relation by straightforward calculations:

$$R(c, d) R(a, b) x = R(a, \langle bcd \rangle) x - L(b, c) R(a, d) x - M(b, d) M(a, c) x, \tag{3–10}$$

where $R(a, b)x = \langle xab \rangle$ and $M(a, c)x = \langle acx \rangle$. By making use the relation (3–10), we have

$$\begin{aligned} R(u, v) R(u, v) x &= R(u, \langle vuv \rangle) x - L(v, u) R(u, v) x \\ &\quad - M(v, v) M(u, u) x \\ &= R(u, -2\langle v, u \rangle v) x - L(v, u) R(u, v) x \\ &\quad - 4\langle v, \langle u, x \rangle u \rangle v \\ &\quad (\text{by } (3\text{–}4)) \\ &= 4R(u, v) x - L(v, u) R(u, v) x + 8\langle u, x \rangle v. \\ &\quad (\text{by } \langle u, v \rangle = 2) \end{aligned}$$

Similarly, we have

$$R(v, u) R(v, u) x = -4R(v, u) x - L(u, v) R(v, u) x - 8\langle v, x \rangle u.$$

Hence we get

$$\begin{aligned} (R(u, v) + R(v, u))^2 x &= (4R(u, v) - L(v, u) R(u, v) + R(u, v) R(v, u) \\ &\quad + R(v, u) R(u, v) - 4R(v, u) - L(u, v) R(v, u)) x \\ &\quad + 8\langle u, x \rangle v - 8\langle v, x \rangle u. \end{aligned} \tag{3–11}$$

We compute

$$\begin{aligned} &(4R(u, v) - L(v, u) R(u, v) + R(u, v) R(v, u) \\ &\quad + R(v, u) R(u, v) - 4R(v, u) - L(u, v) R(v, u)) x \\ &= (4R(u, v) x - 2R(u, v) x + \langle R(u, v) x, u \rangle v \\ &\quad + 2R(v, u) x + \langle R(v, u) x, v \rangle u - 4R(v, u) x \end{aligned}$$

(by means of the relations;

$$\begin{aligned} &-L(v, u) y + R(v, u) y = -2y + \langle y, u \rangle v \quad \text{for all } y \in \mathfrak{M} \\ &-L(u, v) z + R(u, v) z = 2z + \langle z, v \rangle u \quad \text{for all } z \in \mathfrak{M}) \\ &= 2R(u, v) x - 2R(v, u) x + \langle R(u, v) x, u \rangle v + \langle R(v, u) x, v \rangle u \\ &= 4x + \langle R(u, v) x - 2x, u \rangle v + \langle R(v, u) x + 2x, v \rangle u \end{aligned}$$

(by means of the relation;

$$\begin{aligned} &R(u, v) x - R(v, u) x = 2x + \langle x, v \rangle u + \langle u, x \rangle v) \\ &= 4x + 2\langle x, u \rangle v - 2\langle x, v \rangle u \end{aligned} \tag{3–12}$$

(by means of the relations;

$$\langle\langle xuv\rangle, u\rangle = -\langle x, \langle uvu\rangle\rangle = 4\langle x, u\rangle$$
$$\langle\langle xvu\rangle, v\rangle = -\langle x, \langle vuv\rangle\rangle = -4\langle x, v\rangle).$$

Combining (3–12) with (3–11), we obtain

$$(R(u, v) + R(v, u))^2 = 4x + 6\langle u, x\rangle v - 6\langle v, x\rangle u.$$

This completes the proof. ■

We denote $1/2(R(u, v) + R(v, u))$ by $J(u, v)$. Thus on $(\Phi u \oplus \Phi v)^{\perp}$, we have $J(u, v)^2 = \mathrm{Id}$, so $(\Phi u \oplus \Phi v)^{\perp} = \mathfrak{M}_1 \oplus \mathfrak{M}_{-1}$, where $\mathfrak{M}_{\varepsilon}$ is the eigenspace for the eigenvalue ε of $J(u, v)$ for $\varepsilon = \pm 1$. Moreover, since $\langle -, -\rangle$ is nondegenerate, and its restriction to $(\Phi u \oplus \Phi v)^{\perp}$ is nondegenerate, we have

$$\mathfrak{M} = \Phi u \oplus \Phi v \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_{-1}.$$

Since $J(u, v)u = -2u$(resp. $J(u, v)v = 2v$), these imply u(resp. v) is the eigenspace for $J(u, v)$ with eigenvalue -2(resp. 2). Consequently we have the following decomposirion of $\mathfrak{M}$;

$$\mathfrak{M} = \mathfrak{M}_{-2} \oplus \mathfrak{M}_{-1} \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2,$$

where $\mathfrak{M}_i$ is the eigenspace for the eigenvalue i of $J(u, v)(i = \pm 1, \pm 2)$. We call this decomposition the Peirce decomposition of a simple reduced balanced Freudenthal-Kantor triple system. We remark that all Peirce spaces $\mathfrak{M}_i$ are totally isotopic (that is, $\langle \mathfrak{M}_i, \mathfrak{M}_{-j}\rangle \neq 0$ if $i = j$ and $\langle \mathfrak{M}_i, \mathfrak{M}_{-j}\rangle = 0$ otherwise). Using results of the coordinatization of simple reduced Freudenthal triple system, we can prove following results in a manner analogous to that in [7].

Let $\mathfrak{M} = \mathfrak{M}_{-2} \oplus \mathfrak{M}_{-1} \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2$ be the Peirce decomposition relative to a pair of supplementary rank one elements u and v. We define $t: \mathfrak{M}_1 \to \mathfrak{M}_{-1}$ as follows; if for all $y \in \mathfrak{M}_1, \langle u, \langle yyy\rangle\rangle = 0$, let $a_1, \cdots, a_n$ be a basis for $\mathfrak{M}_1$, $a_{-1}, \cdots. a_{-n}$ a dual basis for $\mathfrak{M}_{-1}$ relative to $\langle a_i, a_{-i}\rangle = 2$ and define t by $ta_i = 2a_{-i}$; if there is $y \in \mathfrak{M}_1$, with $1/2\langle u, \langle yyy\rangle\rangle = \lambda \neq 0$, define t by $ta = -1/4(\langle ayu\rangle + \langle auy\rangle) + 3/8\lambda^{-1}\langle u, \langle ayy\rangle\rangle\langle uyy\rangle$.

Combining Propositions 2.2 and 2.3 with results of §4 in [7], we have the following lemma.

LEMMA 3.5. *For t as above,*

(i) $\langle a, tb\rangle = -\langle ta, b\rangle$

(ii) $\langle v, tatata\rangle = \lambda/12\langle u, aaa\rangle$

(iii) *t is nonsingular*

(iv) $t\langle vtatb\rangle = -\lambda/12\langle uab\rangle$

for all $a, b \in \mathfrak{M}_1$.

We can next define a bilinear form $B(,)$ and a cross product on $\mathfrak{M}_1$ as follows: $B(a, b) = \lambda^{-1}/6\langle a, tb\rangle$ and $a \times b = -\lambda^{-1}/2(\langle vtatb\rangle + \langle vtbta\rangle)$ if $\lambda \neq 0$.

$B(a, b) = 1/6\langle a, tb\rangle$ and $a \times b = 0$, if $\lambda = 0$.

PROPOSITION 3.6. *Under the above definition, we have the following identities on* $\mathfrak{M}_1$:

(1) $a \times b = b \times a$

(2) $B(a, b) = B(b, a)$

(3) $B(a \times b, d) = B(a, b \times d)$

(4) $((a \times a) \times b) \times a = 1/3B(a, a \times a)b + B(b, a)a \times a$

(5) $(a \times a) \times (a \times a) = 4/3B(a \times a, a)a$

PROOF. By the definition of the above bilinear form and cross product, the relations obtained from Lemma 3.5 yield the proof. ■

THEOREM 3.7. *Let* $\mathfrak{M}$ *be a reduced simple balanced Freudenthal-Kantor triple system over* Φ. *Then it holds* $\mathfrak{M} \cong \mathfrak{M}(V)$, *where* V *is a vector space equipped with the bilinear form* $B(a, b)$ *and the cross product* $\times$ *satisfying the relations* (1) $\sim$ (5) *of Proposition* 3.6.

PROOF. We can show that if $\lambda \neq 0$, then the map $f: \mathfrak{M}(V) \to \mathfrak{M}$ defined as follows is an isomorphism of balanced Freudenthal-Kantor triple systems;

$$\begin{pmatrix} \alpha_1 & a_1 \\ b_1 & \beta_1 \end{pmatrix} \longrightarrow 36\lambda\alpha_1 v + 1/72\lambda^{-1}\beta_1 u + a_1 + 1/6\lambda^{-1}tb$$

if $\lambda = 0$, similarly.

$$\begin{pmatrix} \alpha_1 & a_1 \\ b_1 & \beta_1 \end{pmatrix} \longrightarrow 36\alpha_1 v + 1/72\beta_1 u + a_1 + 1/6tb_1.$$

As the proof of this isomorphism is very long and of strightforward calculations, we omit it. ■

Finally, from results of this paper, Theorem 6.8 and Theorem 7.4 in [7], we can obtain the following.

THEOREM 3.8. *Let* V(resp. V') *be a Jordan triple system induced from an admissible cubic form* N(resp. N') *with basepoint* c(resp. c'). *Then the followings are equivalent*

(i) V *and* V' *are isotopic.*

(ii) $\mathfrak{M}(V)$ *and* $\mathfrak{M}(V')$ *are isomorpic.*

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