Mem. Fac. Sci. Shimane Univ.,
22, pp. 67–73 Dec. 20, 1988

Minkowski Stochastic Quantization with Gaussian White Noises

Jiro SAKAMOTO

Department of Physics, Shimane University, Matsue, 690 Japan
(Received September 7, 1988)

We have proposed stochastic quantization procedure in Minkowski space-time in previous papers. That formulation, however, was done with non-Gaussian white noises. In this article we show that the equivalent formulation is possible with Gaussian white noises. This will be useful for the application to numerical analysis though manifest covariance of the theory under Lorentz transformation is lost. The covariance is restored at the Green function level.

§1. Introduction

After the proposal of stochastic quantization method as a new quantization procedure by Parisi and Wu [1], many people have discussed the application of this method to many types of fields [2]. In most of them including the Parisi and Wu's original paper, however, quantization is done in Euclidean space-time. This is because Minkowski action is not positive-definite in general and then a solution of the Langevin equation does not approach an equilibrium state. Minkowski stochastic quantization has been formulated by analytic continuation from Euclidean to Minkowski space-time, i.e. $t \to it$. [3, 4, 5] In this formulation action becomes a complex number so that the solution of the Langevin equation oscillates. Then a dumping factor must be introduced to action in order to make the solution converge. However, such introduction of the dumping factor to action may violate some symmetries such as gauge or chiral invariances because this dumping factor is a very imaginary mass term. [4]

In previous papers the present author has proposed a new Minkowski quantization procedure. [6, 7] Instead of addition of such a dumping factor to action, we introduced a kernel to the Langevin equation in order to make a solution converge. We discussed application of this method to a scalar, a non-Abelian gauge and a fermion fields. We found that we can derive chiral anomaly, which coincides with that from the other conventional quantization methods, though this formulation is chirally symmetric even if it is regularized. [7]

In this formulation, however, the white noises are not Gaussian but have the correlation as

$$<\eta(x, t)\eta_*(y, t')> = i\delta^4(x-y)\delta(t-t'), \tag{1.1}$$

for a scalar field for example. Here we suppose that the noises η and η_* are both real numbers but their distribution function is a complex one. The above equation does not have any trouble for formal calculation. But it won't be applicable straight- forwardly to numerical computation.

We show in this paper that an equivalent formulation is possible with Gaussian white noises. It is done by recombining the white noises. The white noises for the scalar field are only rearranged while the noises for the vector field must be rearranged and doubled at the same time. Such a doubling of freedom of the noises were originally proposed by Nakazato et. al. [5] Manifest covariance of the theory for the vector field under the Lorentz transformation is lost though it is restored at the Green function level.

The plan of this paper is as follows. In the next section we shortly review our original proposal for Minkowski stochastic quantization with non-Gaussian white noises. We will discuss the scalar and the vector field cases. In §3 we show the equivalent formulation with the Gaussian white noises. The last section is devoted to conclusion. We use the metric convention $g_{\mu\nu}=(-, +, +, +)$.

§2. Minkowski stochastic quantization

We consider at first the scalar field whose Minkowski action is given by

$$S=\int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-V(\phi)\right]. \tag{2.1}$$

The Langevin equation for this field is supposed as

$$\frac{\partial}{\partial t}\phi(x, t)=\int d^4y\ K(x-y)\left\{-\frac{\delta S}{\delta\phi}\bigg|_{\phi\to\phi(y,t)}+\eta(y, t)\right\}+\eta_*(x, t). \tag{2.2}$$

The correlations of the white noises η and η_* are

$$<\eta(x, t)\eta_*(y, t')> = i\delta^4(x-y)\delta(t-t'),$$
$$<\eta\ \eta> = <\eta_*\ \eta_*> = 0, \tag{2.3}$$

Then it is obvious as mentioned before that these noises are not Gaussian.

The kernel $K(x-y)$ is defined by

$$[\square-m^2]_x K(x-y)=\delta^4(x-y), \tag{2.4}$$

Then the Langevin equation (2.2) is rewritten as

$$\frac{\partial}{\partial t}\phi(x, t)=-\phi(x, t)+\int d^4y\ K(x-y)\left\{\frac{\partial S}{\delta\phi}\bigg|_{\phi\to\phi(y,t)}+\eta(y, t)\right\}+\eta_*(x, y). \tag{2.2'}$$

A solution of the above equation is given iteratively from

$$\phi(x, t)=\int_0^t d\tau\, e^{\tau-t}\left\{\int d^4y\ K(x-y)\eta(y, \tau)+\eta_*(x, \tau)\right\}$$
$$+\int_0^t d\tau\, e^{\tau-t}\int d^4y\ K(x-y)\frac{\partial V}{\delta\phi}\bigg|_{\phi\to\phi(y,t)}+e^{-t}\phi(x, 0). \tag{2.5}$$

Convergence of the solution is obvious at least in perturbative sense.

The Schwinger-Dyson equation [8] is given as

$$\int d^4x d^4y < \left\{\frac{\delta O(\phi)}{\delta\phi(x)}K(x-y)\frac{\delta S}{\delta\phi(y)}-i\frac{\delta^2 O(\phi)}{\delta\phi(x)\delta\phi(y)}K(x-y)\right\} >_{eq}=0, \tag{2.6}$$

where $< \ >_{eq}$ means equilibrium limit of $t\to\infty$ and $O(\phi)$ denotes a product of $\phi(x)$.[5] From this equation we can derive the Green function. If we put $O(\phi)=\phi(x)\phi(y)$ as an example, we have

$$<\phi(x)\phi(y)>_{eq}=\Delta_F(x-y)+(\text{interaction terms}). \tag{2.7}$$

Next we consider the non-Abelian gauge field A^a_μ whose action is

$$S=-\frac{1}{4}\int d^4x\ F^a{}_{\mu\nu}F_a{}^{\mu\nu}. \tag{2.8}$$

The Langevin equation is given by

$$\frac{\partial}{\partial t}A^a_\mu(x, t)=$$
$$-\int d^4y\ K^{ab}(x, y; t)*\left\{\frac{\partial S}{\delta A^\mu_b(y, t)}-\eta^b_\mu(y, t)\right\}+i\int d^4y\frac{\delta K^{ab}(x, y; t)}{\delta A^\mu_b(y, t)}$$
$$+\eta^a_{*\mu}(x, t). \tag{2.9}$$

The correlations of white noises η_* and $\eta_{*\mu}$ are defined as

$$<\eta^a_\mu(x, t)\eta^b_{*\nu}(y, t)> =i\ \delta^{ab}g_{\mu\nu}\delta^4(x-y)\delta(t-t'),$$
$$<\eta^a_\mu\eta^b_\nu> = <\eta^a_{*\nu}\eta^b_{*\nu}> =0. \tag{2.10}$$

Then they are not Gaussian again. The kernel $K(x, y; t)$ is defined by

$$\nabla^{ab}_\mu(x, t)\nabla^\mu_{bc}(x, t)K^{cd}(x, y; t)=\delta^{ad}\delta^4(x-y),$$

which depends on gauge field A^a_μ for the sake of the gauge invariance of the Langevin equation. The additional term in (2.9) is necessary for the equivalence to the other quantization methods. [6] The Schwinger-Dyson equation is derived as

$$<\int d^4x d^4y \left\{ \frac{\delta S}{\delta A^a_\mu(x)} - i\frac{\delta}{\delta A^a_\mu(x)} \right\} K^{ab}(x, y) \frac{\delta O}{\delta A^\mu_b(y)} \Big\} >_{eq} = 0. \tag{2.12}$$

§3. Gaussian white noises

In the previous section we review our procedure for Minkowski stochastic quantization. It is formulated with non-Gaussian white noises. This point, of course, does not cause any trouble for formal calculation. This procedure, however, is not applicable to numerical analysis straightforwardly.

One should note that in derivation of the Green function correlation between the terms consisting of white noises in the Langevin equation is essential but the detail structures of these terms are unimportant.

Now we construct the equivalent formulation with Gaussian white noises. We discuss the scalar and the gauge fields separately.

i) Scalar field

We rearrange the white noises η and η_* into complex numbers as

$$\begin{aligned} \eta &= \eta_1 + i\eta_2, \\ \eta_* &= i\eta_1 + \eta_2, \end{aligned} \tag{3.1}$$

where η_1 and η_2 are both real numbers. Correlation between them is given as

$$<\eta_i(x, t)\eta_j(y, t')> = 1/2\ \delta_{ij}\delta^4(x-y)\delta(t-t'), \tag{3.2}$$

Then it is obvious that they are Gaussian. Correlation between η and η_* is given identically as (2.3). Though white noises η and η_* are complex numbers in this formulation, it is easily confirmed that the same Schinger-Dyson equation as (2.6) holds. Thus recombining the white noises we obtain the equivalent formulation for the scalar field with Gaussian white noises.

ii) Gauge field

As is shown below, we must extend freedom of the white noises. We put

$$\begin{aligned} \eta_\mu &= \sum_{i=1}^{N} C_i \eta^i_\mu, \\ \eta_{*\mu} &= \sum_{i=1}^{N} C_{*i} \eta^i_\mu, \end{aligned} \tag{3.3}$$

where η^i_μ is a real number while C_i and C_{*i} are complex nuumbers. Note that C_{*i} is not complex conjugate to C_i.

Now we prove that N must be more than three. Correlations we have to set up are

$$<\eta_\mu(x,\ t)\eta_{*\nu}(y,\ t')> = ig_{\mu\nu}\delta^4(x-y)\delta(t-t'),$$

$$<\eta_\mu\ \eta_\nu> = <\eta_{*\mu}\ \eta_{*\nu}> = 0. \tag{3.4}$$

Let us suppose that correlation of η^i_μ is given as

$$<\eta^i_\mu(x,\ t)\eta^i_\nu(y,\ t')> = \delta^{ij}T^i_{\mu\nu}\delta^4(x-y)\delta(t-t'), \tag{3.5}$$

where $T^i_{\mu\nu}$ is a positive definite matrix. Then η^i_μ is a Gaussian white noise.

Substituting (3.3) into (3.4) we have

$$\sum_{i=1}^{N} C_i C_{*i} T^i_{\mu\nu} = ig_{\mu\nu}, \tag{3.6}$$

$$\sum_{i=1}^{N} C_i C_i T^i_{\mu\nu} = 0, \tag{3.7}$$

$$\sum_{i=1}^{N} C_{*i} C_{*i} T^i_{\mu\nu} = 0. \tag{3.8}$$

It is obvious that $N=1$ is forbidden because from (3.7) or (3.8) we have $T^i_{\mu\nu}=0$, which contradicts (3.6).

If $N=2$, we have $T^1 \backsim T^2$ from (3.7) or (3.8). Then we find $T^1_{\mu\nu} \backsim T^2_{\mu\nu} \backsim g_{\mu\nu}$ which is inconsistent with the assumption that $T^i_{\mu\nu}$ is positive definite.

For $N=3$ we must discuss two possible cases: a) eqs. (3.6)–(3.8) are all linearly independent of each other, and b) they are not independent.

a) We can eliminate one of T^i, say T^3, from (3.7) and (3.8). Then we have $T^1_{\mu\nu} \backsim T^2_{\mu\nu}$. This show that $T^3_{\mu\nu}$ is proportional to the other $T^i_{\mu\nu}$'s. From (3.6) T^i is equal to $g_{\mu\nu}$ up to a numerical factor. T^i is not positive definite again.

b) When (3.7) and (3.8) are not independent of each other, we have

$$\frac{(C_2)^2}{(C_1)^2} = \frac{(C_{*2})^2}{(C_{*1})^2}, \qquad \frac{(C_3)^2}{(C_1)^2} = \frac{(C_{*3})^2}{(C_{*1})^2}, \tag{3.9}$$

which give

$$C_2 = \pm\frac{C_1 C_{*2}}{C_{*1}}, \qquad C_3 = \pm\frac{C_1 C_{*3}}{C_{*1}}. \tag{3.10}$$

From (3.6) and the above equations we have

$$(C_{*1})^2 T^1_{\mu\nu} \pm (C_{*2})^2 T^2_{\mu\nu} \pm (C_{*3})^2 T^3_{\mu\nu} = i\frac{C_{*1}}{C_1} g_{\mu\nu}. \tag{3.11}$$

With (3.8) we can eliminate two T^i's in L.H.S. on the above equation. Then we find that T^i left is proportional to $g_{\mu\nu}$, which contradicts the assumption on T^i again.

When $N=4$, we may put

$$\begin{aligned} C_1 &= 1, & C_{*1} &= i, \\ C_2 &= 1, & C_{*2} &= -i, \\ C_3 &= -i, & C_{*3} &= 1, \\ C_4 &= i, & C_{*4} &= 1. \end{aligned} \tag{3.12}$$

We obtain from (3.6)–(3.8)

$$\begin{aligned} T^1_{\mu\nu} - T^2_{\mu\nu} - T^3_{\mu\nu} + T^4_{\mu\nu} &= g_{\mu\nu} \\ T^1_{\mu\nu} + T^2_{\mu\nu} - T^3_{\mu\nu} - T^4_{\mu\nu} &= 0. \end{aligned} \tag{3.13}$$

Then we set

$$\begin{aligned} T^1_{\mu\nu} = T^4_{\mu\nu} &= (1/2,\ 1,\ 1,\ 1), \\ T^2_{\mu\nu} = T^3_{\mu\nu} &= (1,\ 1/2,\ 1/2,\ 1/2), \end{aligned} \tag{3.14}$$

for example so that η_μ and $\eta_{*\mu}$ satisfy (3.4). We conclude, therefore that N must be more than 3. $N=4$ implies that freedom of the white noises is doubled.

The positive-definite matrices T^i is not a constant Lorentz tensor. Therefore, this formulation with the Gaussian white noises is frame-dependent. The covariance under Lorentz transformation is restored at the Green function level.

§4. Conclusion

In this article we have proposed the Minkowski stochastic quantization method with Gaussian white noises. This will make it possible to deal with the Langevin equation numerically, e.g. we can obtain scattering cross section from the Langevin equation by numerical calculation.

The essence of this formulation is that we can make the non-Gaussian white noises Gaussian by recombining their components and expanding their freedom. The present author conjecture that every type of non-Gaussian white noises can be made Gaussian by such procedures.

The white noise in the Langevin equation is naively understood to correspond to quantum fluctuation of the field. Then what is the physical meaning to expand its freedom? To answer this question implies that we can understand stochastic quantization method more deeply.

References

[1] G. Parisi and Wu Yong-Shi, Sci. Sin. **24** (1981), 483.

[2] See as a review P. H. Damgaard and H. Hüffel, Phys. Rep. **152** (1987), 228.

[3] H. Hüffel and H. Rumpf, Z. Phys. **C29** (1985), 319.
H. Rumpf, Phys. Rev. **D33** (1986), 942.
E. Gozzi, Phys, Lett. **150B** (1985), 119.

[4] Hüffel and P. V. Landshoff, Nucl. Phys. **B260** (1985), 545.

[5] H. Nakazato and Y. Yamanaka, Phys. Rev. **D34** (1986), 492.
H. Nakazato, Prog. Theor. Phys. **77** (1987), 20, 802L.

[6] J. Sakamoto, Prog. Theor. Phys. **80** (1988), 190.

[7] J. Sakamoto and A. Sugisawa "Minkowski Stochastic Quantization of Fermion Field and Chiral Anomaly", Shimane Univ. preprint, 1988. To appear in Prog. Theor. Phys. Vol. **81** No. 1 (1989).

[8] Z. Bern, M. B. Halpern, L. Sadun and C. Taubes, Phys. Lett. **165B** (1985), 151, Nucl. Phys. **B284** (1987), 1, 35, 92.