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Some Properties of Royden Boundary of an Infinite Network

Dedicated to Professor Miyuki Yamada on his 60th birthday

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Fine properties of Royden boundary of an infinite network are discussed in this paper. In particular, we study the problem whether every one-point set of the Royden boundary is a G_δ-set or not.

§1. Introduction

The concept of Royden boundary is one of the most important notions in the theory of Riemann surfaces. In order to obtain a fine theory of the ideal boundary of an infinite network, we studied in [1] and [6] discrete analogues of Royden boundary $\Gamma=\Gamma^{(p)}$ and harmonic boundary $\Gamma_h=\Gamma_h^{(p)}$ of an infinite network $N=\{X, Y, K, r\}$ of order $p>1$. Our aim is to study the discrete analogue to the fact in [3] that a point x of the Royden compactification of a Riemannian manifold is a Royden boundary point if and only if the set $\{x\}$ is not a G_δ-set.

We shall show in §2 that if x is a Royden boundary point and not a Royden harmonic boundary point, i.e., $x\in\Gamma-\Gamma_h$, then the set $\{x\}$ is not a G_δ-set. This result was proved in [1] in case $p=2$. In contrast with the continuous case, we can not assure that the set $\{x\}$ is not a G_δ-set for $x\in\Gamma_h$ (cf. [3; Chap. III, Theorem 2D]). Our proof in §3 shows a difference between the continuous case and the discrete case. Some supplementary remarks will be given in §4. We shall give an example which shows that $\Gamma=\Gamma_h$ is a singleton and a G_δ-set. It should be noted that the closure of $\Gamma-\Gamma_h$ in the Royden compactification is equal to Γ in the continuous case (cf. [3; Chap. III, Theorem 2E]). In §5, we shall study the Royden boundary of a network defined by a binary tree. For this network, there exists $x\in\Gamma_h$ such that the set $\{x\}$ is not a G_δ-set.

We shall freely use the notation in [2], [5] and [6].

§2. Royden boundary

Let $L(X)$ be the set of all real functions on X and $L_0(X)$ be the set of all $u\in L(X)$ with finite support. For $u\in L(X)$, its discrete derivative $du\in L(Y)$ and its discrete Dirichlet integral $D_p(u)$ of order p $(1<p<\infty)$ are defined by

$$du(y) = -r(y)^{-1}\sum_{x\in X} K(x, y)u(x),$$

$$D_p(u) = \sum_{y\in Y} r(y)|du(y)|^p.$$

Denote by $\mathbf{D}^{(p)}(N)$ the set of all $u\in L(X)$ with finite Dirichlet integral of order p. It is easily seen that $\mathbf{D}^{(p)}(N)$ is a reflexive Banach space with the norm $\|u\|_p = [D_p(u) + |u(b)|^p]^{1/p}$ $(b\in X)$. Let $\mathbf{D}_0^{(p)}(N)$ be the closure of $L_0(X)$ in $\mathbf{D}^{(p)}(N)$ with respect to this norm. We call an element of $\mathbf{D}_0^{(p)}(N)$ a Dirichlet potential of order p. Denote by $\mathbf{BD}^{(p)}(N)$ and $\mathbf{BD}_0^{(p)}(N)$ the subsets of $\mathbf{D}^{(p)}(N)$ and $\mathbf{D}_0^{(p)}(N)$ which consist of bounded functions respectively.

By a compactification of X which is a locally compact Hausdorff space with respect to the discrete topology, we mean a compact Hausdorff space X^* containing X as a dense open subset. There is a unique (up to a homeomorphism) compactification X^* of X such that every $f\in\mathbf{BD}^{(p)}(N)$ can be continuously extended to X^* and the class of extended functions separates points of X^*-X. This compactification is called the Royden p-compactification of N and Γ and $\Gamma=\Gamma^{(p)}=X^*-X$ is called the p-Royden boundary of N. The extension of $f\in\mathbf{BD}^{(p)}(N)$ to X^* is denoted by f again.

Next, we define the Royden p-harmonic boundary Γ_h of N by

$$\Gamma_h = \Gamma_h^{(p)} = \{x\in\Gamma;\ f(x)=0 \text{ for all } f\in\mathbf{BD}_0^{(p)}(N)\}.$$

Note that Γ_h is a compact subset of Γ.

We proved in [4] that N is of parabolic type of order p if and only if $1\in\mathbf{D}_0^{(p)}(N)$. Thus we have

THEOREM 2.1. *An infinite network N is of parabolic type of order p if and only if $\Gamma_h = \phi$.*

Since $\{x\}$ is a G_δ-set for every $x\in X$, we have

THEORREM 2.2 *If $x\in X^*$ and $\{x\}$ is not a G_δ-set, then $x\in\Gamma$.*

We shall discuss the converse of this fact. We shall prove

THEOREM 2.3. *For any $\alpha\in\Gamma-\Gamma_h$, the set $\{\alpha\}$ is not a G_δ-set in X^*.*

In contrast with the continuous case, we can not assure that $\{\alpha\}$ is not a G_δ-set for every $\alpha\in\Gamma$ (cf. [3; Chap. III, Theorem 2D]).

§3. Proof of Theorem 2.3

For a subset B of X, let us define $Y(B)$ and $X(B)$ by

(3.1) $$Y(B) = \{y\in Y;\ e(y)\cap B \neq \phi\},$$

(3.2) $$X(B) = \cup\{e(y);\ y\in Y(B)\}.$$

For the proof of Theorem 2.3, we need some lemmas.

We proved in [6]

LEMMA 3.1. *For a closed subset F of X^* such that $F \cap \Gamma_h = \phi$, there exists $f \in \mathbf{BD}_0^{(p)}(N)$ such that $f=1$ on F and $0 \leqslant f \leqslant 1$ on X^*.*

LEMMA 3.2. *Let $\{u_n\}$ be a sequence in $\mathbf{D}_0^{(p)}(N)$ which converges pointwise to $u \in L(X)$. If $\{D_p(u_n)\}$ is bounded, then $u \in \mathbf{D}_0^{(p)}(N)$.*

We shall prove

LEMMA 3.3. *Let $u \in \mathbf{BD}_0^{(p)}(N)$ and A be a finite subset of X. For any $\varepsilon > 0$, there exists $f \in L_0(X)$ such that $f=u$ on A, $\sup|f| \leqslant \sup|u|$ and $D_p(u-f) < \varepsilon$.*

PROOF. Since $u \in \mathbf{BD}_0^{(p)}(N)$, there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\sup|f_n| \leqslant \sup|u|$ and $\|u-f_n\|_p \to 0$ as $n \to \infty$. Define u_n by $u_n(x)=u(x)$ on A and $u_n(x) = f_n(x)$ on $X-A$. Then $u_n \in L_0(X)$. It suffices to show that $D_p(u-u_n) \to 0$ as $n \to \infty$. Since $f_n(x) \to u(x)$ as $n \to \infty$ for each $x \in X$, $d(u-u_n)(y) \to 0$ as $n \to \infty$ for each $y \in Y$. We have

$$D_p(u-u_n) \leqslant \textstyle\sum_{y \in Y(A)} r(y) |d(u-u_n)(y)|^p + D_p(u-f_n).$$

Since $Y(A)$ is a finite set, we see that

$$\textstyle\sum_{y \in Y(A)} r(y) |d(u-u_n)(y)|^p \to 0 \quad \text{as} \quad n \to \infty,$$

and hence $D_p(u-u_n) \to 0$ as $n \to \infty$.

LEMMA 3.4. *Let $u \in \mathbf{BD}_0^{(p)}(N)$ and $\{V_n\}$ be a sequence of infinite subsets of X. Then there exist a function $\varphi \in \mathbf{BD}_0^{(p)}(N)$ and two sequences $\{a_m\}$ and $\{b_m\}$ of nodes satisfying the following conditions:*

(3.1) $$a_m \in V_{2m-1},\ b_m \in V_{2m},\ m=1, 2, \cdots,$$

(3.2) $$a_m \neq a_n \quad \textit{and} \quad b_m \neq b_n \qquad \textit{for} \quad m \neq n,$$

(3.3) $$\varphi(a_m)=0 \quad \textit{and} \quad \varphi(b_m)=u(b_m) \qquad \textit{for all } m.$$

PROOF. First, choose any $x_1 \in V_1$ and set $A_1=\{x_1\}$. By Lemma 3.3, there is $f_1 \in L_0(X)$ such that $f_1(x_1)=u(x_1)$, $\sup|f_1| \leqslant \sup|u|$ and $D_p(u-f_1) < 1/2$. Set $B_1 = X(Sf_1) \cup A_1$, where $Sf = \{x \in X; f(x) \neq 0\}$ for $f \in L(X)$. Since B_1 is a finite set, we can choose $x_2 \in V_2 - B_1$. Set $A_2 = X(B_1) \cup \{x_2\}$. By Lemma 3.3 again, there is $f_2 \in L_0(X)$ such that $f_2=u$ on A_2, $\sup|f_2| \leqslant \sup|u|$ and $D_p(u-f_2) < (1/2)^2$. Repeating this process, we can find sequences $\{x_n\}$ in X and $\{f_n\}$ in $L_0(X)$ which satisfy the following conditions:

$$x_n \in V_n - B_{n-1}, \text{ where } B_0 = \phi \text{ and } B_n = X(Sf_n) \cup A_n,$$

$$f_n = u \quad \text{on} \quad A_n = X(B_{n-1}) \cup \{x_n\},$$

$$\sup|f_n| \leqslant \sup|u| \text{ and } D_p(u-f_n) < (1/2)^n,$$

$n=1, 2, \cdots$. Note that each B_n is a finite set, $X(B_{n-1}) \subset B_n$, $x_n \in B_n - B_{n-1}$ and $f_n(x_n) = u(x_n)$, $n=1, 2, \cdots$. Now we define

$$a_m = x_{2m-1} \quad \text{and} \quad b_m = x_{2m}, \ m = 1, 2, \cdots$$

and

$$\varphi_k(x) = u(x) - f_{2m-1}(x) \quad \text{if} \quad x \in B_{2m-1} - B_{2m-2} \quad (m=1, 2, \cdots, k),$$
$$\varphi_k(x) = f_{2m}(x) \quad \text{if} \quad x \in B_{2m} - B_{2m-1} \quad (m=1, 2, \cdots, k),$$
$$\varphi_k(x) = 0 \quad \text{if} \quad x \in X - B_{2k},$$

for $k=1, 2, \cdots$. Then $\varphi_k \in L_0(X)$, $\varphi_k(a_m)=0$ and $\varphi_k(b_m)=u(b_m)$ for $m=1, 2, \cdots, k$. Thus $\{\varphi_k\}$ converges to a function φ satisfying (3.3). Obviously, $\{a_m\}$ and $\{b_m\}$ satisfy (3.1) and (3.2).

In order to show that $\varphi \in \mathbf{BD}_0^{(p)}(N)$, we evaluate $D_p(\varphi_k)$. Let

$$Y_n = \{y \in Y;\ e(y) \subset B_n - B_{n-1}\},$$
$$Y'_n = \{y \in Y;\ e(y) \cap B_n \neq \phi,\ e(y) \not\subset B_n\},$$

$n=1, 2, \cdots$. Since $B_{n+1} \supset X(B_n)$, we see that

$$\cup_{n=1}^{2k} (Y_n \cup Y'_n) = Y(B_{2k}),$$

so that

$$D_p(\varphi_k) = \textstyle\sum_{n=1}^{2k} \{\sum_{y \in Y_n} r(y)|d\varphi_k(y)|^p + \sum_{y \in Y'_n} r(y)|d\varphi_k(y)|^p\}.$$

If $y \in Y_{2m-1}$, $m \leqslant k$, with $e(y) = \{z_1, z_2\}$, then $\varphi_k(z_j) = u(z_j) - f_{2m-1}(z_j)$, $j=1, 2$, so that

$$|d\varphi_k(y)|^p = |d(u-f_{2m-1})(y)|^p.$$

If $y \in Y_{2m}$, $m \leqslant k$, with $e(y) = \{z_1, z_2\}$, then $\varphi_k(z_j) = f_{2m}(z_j)$, $j=1, 2$, so that

$$|d\varphi_k(y)|^p = |df_{2m}(y)|^p \leqslant 2^{p-1}\{|du(y)|^p + |d(u-f_{2m})(y)|^p\}.$$

If $y \in Y'_n$, $n \leqslant 2k$, with $e(y) = \{z_1, z_2\}$, $z_1 \in B_n$ and $z_2 \notin B_n$, then $z_1 \notin Sf_n$ (for, otherwise $z_2 \in X(Sf_n) \subset B_n$) and $z_2 \in X(B_n) \subset A_{n+1}$. Hence $f_n(z_1)=0$ and $f_{n+1}(z_2)=u(z_2)$. It follows that $\varphi_k(z_1)=\varphi_k(z_2)=0$ if n is even, and $\varphi_k(z_j)=u(z_j)$, $j=1, 2$, if n is odd. Therefore

$$\begin{aligned} D_p(\varphi_k) &\leqslant \textstyle\sum_{m=1}^{k} [\sum_{y \in Y_{2m-1}} r(y)|d(u-f_{2m-1})(y)|^p + \sum_{y \in Y'_{2m-1}} r(y)|du(y)|^p \\ &\quad + 2^{p-1}\textstyle\sum_{y \in Y_{2m}} r(y)\{|du(y)|^p + |d(u-f_{2m})(y)|^p\}] \\ &\leqslant 2^{p-1}\{D_p(u) + \textstyle\sum_{n=1}^{2k} D_p(u-f_n)\} \end{aligned}$$

$$\leqslant 2^{p-1}\{D_p(u)+\sum_{n=1}^{2k}(1/2)^n\} \leqslant 2^{p-1}\{D_p(u)+1\}$$

for all k. Hence, by Lemma 3.2, we see that $\varphi\in\mathbf{BD}_0^{(p)}(N)$, and the lemma is proved.

PROOF OF THEOREM 2.3. Suppose that $\{\alpha\}$ is a G_δ-set. Then there exists a sequence $\{U_n\}$ of open neighborhoods of α in X^* such that $Cl(U_{n+1})\subset U_n$ and $\bigcap_{n=1}^{\infty}U_n=\{\alpha\}$. Since $\alpha\in\Gamma-\Gamma_h$, there exists $u\in\mathbf{BD}_0^{(p)}(N)$ such that $u(\alpha)=1$ and $0\leqslant u\leqslant 1$ on X by Lemma 3.1. Set

$$V_n=\{x\in U_n\cap X;\ u(x)>1/2\},\quad n=1, 2,\cdots.$$

Then each V_n is an infinite set. Hence, by Lemma 3.4, there exist a function $\varphi\in\mathbf{BD}_0^{(p)}(N)$ and two sequences $\{a_m\}$ and $\{b_m\}$ of nodes such that $a_m\in U_{2m-1}$, $b_m\in U_{2m}$ and $0=\varphi(a_m)<1/2<\varphi(b_m)$ for all m. This contradicts the continuity of φ at α, since $a_m\to\alpha$ and $b_m\to\alpha$. Therefore $\{\alpha\}$ is not a G_δ-set.

§4. Supplementary remarks

By Theorems 2.1 and 2.3, we have

THEOREM 4.1. *Assume that N is of parabolic type of order p. Then $\alpha\in\Gamma$ if and only if the one-point set $\{\alpha\}$ is not a G_δ-set.*

As a criterion for a singleton to be a G_δ-set, we have

THEOREM 4.2. *Let $\alpha\in\Gamma_h$. The singleton $\{\alpha\}$ is a G_δ-set if and only if there exists $v\in\mathbf{BD}^{(p)}(N)$ such that $0\leqslant v(x)<1$ on $X^*-\{\alpha\}$ and $v(\alpha)=1$.*

PROOF. The "if" part is clear, since $v\in\mathbf{BD}^{(p)}(N)$ is continuous on X^*. To prove the "only if" part, let $\{U_n\}$ be a sequence of open sets in X^* such that $\bigcap_{n=1}^{\infty}U_n=\{\alpha\}$ and put $F_n=X^*-U_n$. Then F_n is compact and $\alpha\notin F_n$. There exists $u_n\in\mathbf{BD}^{(p)}(N)$ such that $u_n(\alpha)=1$, $u_n(x)=0$ on F_n and $0\leqslant u_n(x)\leqslant 1$ on X. Let us take $v=\sum_{n=1}^{\infty}2^{-n}u_n(x)$. Then $v\in\mathbf{BD}^{(p)}(N)$, $v(\alpha)=1$ and $0\leqslant v(x)<1$ on X. Since $\bigcup_{n=1}^{\infty}F_n=X^*-\{\alpha\}$, we see that $v(x)<1$ on $X^*-\{\alpha\}$.

For an infinite path P, denote by $e(P)$ the intersection of the Royden boundary and the closure of $C_X(P)$.

THEOREM 4.3. *Let P be an infinite path. If $\sum_{y\in C_Y(P)}r(y)<\infty$, then $e(P)$ is a singleton and $e(P)\subset\Gamma_h$.*

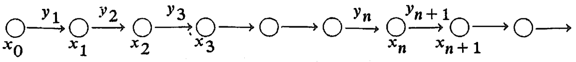
PROOF. Assume that $\sum_{y\in C_Y(P)}r(y)<\infty$. By Theorem 3.1 in [2], every $u(x)$ ($u\in\mathbf{BD}^{(p)}(N)$) has a limit as x tends to the ideal boundary along P, so that $e(P)$ is a singleton. By Theorem 3.2 in [2], every $v(x)$ ($v\in\mathbf{BD}_0^{(p)}(N)$) has a limit 0 as x tends to the ideal boundary along P, so that $e(P)\subset\Gamma_h$.

COROLLARY. *If $e(P)$ contains more than two points, then*

$$\sum_{y\in C_Y(P)} r(y)=\infty. \tag{4.1}$$

Now we remark that $\{\alpha\}$ may be a G_δ-set for $\alpha\in\Gamma_h$. This is shown by the following example:

EXAMPLE 4.1. Let us consider the infinite graph $G=\{X, Y, K\}$ shown as in the following figure, where $X=\{x_n; n=0, 1, 2,\cdots\}$ and $Y=\{y_n; n=1, 2,\cdots\}$:



Here $K(x_n, y_n)=1$ and $K(x_{n-1}, y_n)=-1$ for every positive integer n and $K(x,y)=0$ for any other pair. Let $r\in L(Y)$ be strictly positive and $\sum_{y\in Y} r(y)<\infty$. Then $N=\{G, r\}$ is an infinite network which is of hyperbolic type of order p. We see by Theorem 4.3 that Γ $\Gamma$is a singleton $\{\alpha\}$ and $\Gamma=\Gamma_h$. Let us consider the function u defined by $u(x_n)=\sum_{k=1}^{n} r(y_k)$. Then $du(y_n)=-1$ and $D_p(u)=\sum_{y\in Y} r(y)$. Put $U_n=\{x\in X;\ u(x)>u(x_{n-1})\}\cup\{\alpha\}$. Then U_n is an open set in X^* and $\bigcap_{n=1}^{\infty} U_n=\{\alpha\}$. Namely $\{\alpha\}$ is a G_δ-set.

EXAMPLE 4.2. Let $\{X, Y, K\}$ be the same graph as in Example 4.1 and let $r(y)=1$ on Y. Since N is of parabolic type of order p, $\Gamma_h=\phi$ by Theorem 2.1. We show that Γ contains uncountable points. In fact, consider a function f on X defined by

$$f(x_k)=2^{-m}t \quad \text{for} \quad k=2^{m+1}+t-2 \quad (t=0, 1,\cdots, 2^m),$$

$$f(x_k)=1-2^{-m}t \quad \text{for} \quad k=2^{m+1}+2^m+t-2 \quad (t=1, 2,\cdots, 2^m),$$

where $m=0, 1, 2,\cdots$. Then we have

$$D_p(f)=\sum_{k=0}^{\infty}|f(x_k)-f(x_{k+1})|^p$$
$$=2\sum_{m=0}^{\infty}\sum_{t=1}^{2^m}(2^{-m})^p=2\sum_{m=0}^{\infty}2^{-m(p-1)}<\infty.$$

Then $f\in \mathbf{BD}^{(p)}(N)$. Note that the closure of $f(X)$ in the real line is equal to the interval $[0, 1]$, and hence $f(X^*)=[0, 1]$ by the continuity of f on X^*. Since $f(X)$ is a countable set and $f(\Gamma)\supset f(X^*)-f(X)$, $f(\Gamma)$ is an uncountable set. Thus Γ contains uncountable points.

§5. Royden boundary of a tree

We shall study the Royden bouundary of a tree. A binary tree stemmed from $x_0^{(0)}$ is defined as the graph $\{X, Y, K\}$, where

$$X=\{x_n^{(k)}; n=0, 1,\cdots, 2^k-1, k=0, 1, 2,\cdots\},$$

$$Y=\{y_n^{(k+1)}; n= 0, 1, 2,\cdots, 2^{k+1}-1, k=0, 1, 2,\cdots\},$$

$$K(x_n^{(k)}, y_{2n}^{(k+1)}) = K(x_n^{(k)}, y_{2n+1}^{(k+1)}) = -1 \quad \text{for} \quad n=0, 1, \cdots, 2^k-1,$$
$$K(x_n^{(k+1)}, y_n^{(k+1)}) = 1 \quad \text{for} \quad n=0, 1, \cdots, 2^{k+1}-1,$$

$(k=0, 1, 2, \cdots)$ and $K(x, y)=0$ for any other pair. Take $r(y)=1$ on Y. Then $N=\{X, Y, K, r\}$ is an infinite network. It was shown in [4] that N is of hyperbolic type of any order $p>1$. For each k, we define a finite family of subnetworks $\{N_n^{(k)}; n=0, 1, 2, \cdots, 2^k-1\}$ $(N_n^{(k)} = <X_n^{(k)}, Y_n^{(k)}>)$ by

$$X_n^{(k)} = \cup_{t=k}^{\infty} \{x_m^{(t)}; m=2^{t-k}n, 2^{t-k}n+1, \cdots, 2^{t-k}n+2^{t-k}-1\},$$
$$Y_n^{(k)} = \{y \in Y; e(y) \subset X_n^{(k)}\}.$$

We may call the subgraph $\{X_n^{(k)}, Y_n^{(k)}, K\}$ the binary tree stemmed from $x_n^{(k)}$.

Denote by $P_{a,\infty}(N)$ the set of all paths $P=\{C_X(P), C_Y(P), p\}$ from node a to the ideal boundary of N such that $C_X(P)$ and $C_Y(P)$ are contained in the binary tree from a. We call an element of the union P_∞ of the set $\{P_{a,\infty}(N); a \in X\}$ an infinite path.

PROPOSITION 5.1. *For every infinite path P, $e(P)$ contains uncountable points.*

PROOF. Without any loss of generality, we may assume that $a=x_0^{(0)}$ and $P \in P_{a,\infty}(N)$ with $C_X(P)=\{x_0^{(k)}; k=0, 1, 2, \cdots\}$ and $C_Y(P)=\{y_0^{(k)}; k=1, 2, \cdots\}$. Then this path P is identified with the network considered in Example 4.2. Let f be the function defined in Example 4.2 with $x_0^{(k)}=x_k$ and extend f to a function u on X by

$$u(x)=f(x_0^{(k)}) \text{ on } X_n^{(k)} \text{ for } n=1, 2, \cdots, 2^k-1 \text{ and } k=1, 2, \cdots.$$

Then we see easily that $u \in \mathbf{BD}^{(p)}(N)$ and $u(C_X(P))=f(C_X(P))$. Since $u(e(P))=f(e(P))$ contains uncountable points by Example 4.2, $e(P)$ contains uncountable points.

Let us consider the following extremum problem:

$$\text{(5.1)} \qquad \text{Minimize } D_p(u; N_n^{(k)}) \text{ subject to } u \in \mathbf{D}_0^{(p)}(N_n^{(k)}) \text{ and } u(x_n^{(k)})=1,$$

where $D_p(u; N_n^{(k)}) = \sum_{y \in Y_n^{(k)}} |du(y)|^p$ and $\mathbf{D}_0^{(p)}(N_n^{(k)})$ is defined similarly to $\mathbf{D}_0^{(p)}(N)$ replacing N, $L_0(X)$ and $D_p(u)$ by $N_n^{(k)}$, $L_0(X_n^{(k)})$ and $D_p(u; N_n^{(k)})$.

Denote by $d_0(x_n^{(k)}, \infty; N_n^{(k)})$ the value of problem (5.1). By the similarity of $N=N_0^{(0)}$ and $N_n^{(k)}$, we see that

$$d_0(x_n^{(k)}, \infty; N_n^{(k)}) = d_0(x_0^{(0)}, \infty; N_0^{(0)}) \text{ with } N=N_0^{(0)}.$$

We proved in Example 5.2 in [4] that $d_0(x_0^{(0)}, \infty; N)>0$. Thus we have

LEMMA 5.1. *All the values of problems* (5.1) *for n and k* $(n=0, 1, \cdots, 2^k-1; k=0, 1, 2, \cdots)$ *are equal to a positive constant.*

Now we shall prove

PROPOSITION 5.2. *Let* $u \in \mathbf{BD}_0^{(p)}(N)$. *For every infinite path* P, $u(x)$ *has a limit* 0 *as* x *tends to the ideal boundary along* P.

PROOF. Without any loss of generality, we may assume that $C_X(P) = \{x_0^{(k)}; k = 0, 1, 2, \cdots\}$. Suppose that $\limsup_{n \to \infty} |u(x_0^{(n)})| = \rho > 0$. There exists a subsequence $\{b_m\}$ of $\{x_0^{(k)}\}$ such that $|u(b_m)| > \rho - 1/m$ for every m. Denote by a_m the node $x_1^{(k)}$ with $b_m = x_0^{(k)}$. Since the restriction of u to $X_1^{(k)}$ belongs to $\mathbf{BD}_0^{(p)}(N_1^{(k)})$, we have by Lemma 5.1 in case $u(x_1^{(k)}) \neq 0$

$$D_p(u/u(x_1^{(k)}); N_1^{(k)}) \geqq d_0(x_1^{(k)}, \infty; N_1^{(k)}) = d_0 > 0,$$

or equivalently

$$D_p(u; N_1^{(k)}) \geqq |u(x_1^{(k)})|^p d_0. \tag{5.2}$$

In case $u(x_1^{(k)}) = 0$, (5.2) is clear. It follows that

$$D(u) = \sum_{y \in C_Y(P)} |du(y)|^p + \sum_{k=1}^{\infty} |u(x_0^{(k)}) - u(x_1^{(k)})|^p + \sum_{k=1}^{\infty} D_p(u; N_1^{(k)})$$
$$\geqq \sum_{m=1}^{\infty} |u(b_m) - u(a_m)|^p + \sum_{m=1}^{\infty} |u(a_m)|^p d_0,$$

so that $u(b_m) - u(a_m) \to 0$ and $u(a_m) \to 0$ as $m \to \infty$. This is a contradiction. Therefore $\lim_{n \to \infty} u(x_n) = 0$.

COROLLARY. $e(P) \subset \Gamma_h$ *for every infinite path* P.

By this corollary, we see that $Cl(\cup\{e(P); P \in P_\infty\}) \subset \Gamma_h$. Since the inverse inclusion relation was proved in [6; Theorem 6.4], we have

PROPOSITION 5.3. $Cl(\cup\{e(P); P \in P_\infty\}) = \Gamma_h$.

REMARK 5.1. If $P \neq P'$, then $e(P) \cap e(P') = \phi$. In fact, if $P \neq P'$, then there exists a node $x_n^{(k)}$ and infinite subpaths P_1 and P'_1 of P and P' respectively such that $C_X(P_1) \subset X_n^{(k)}$ and $C_X(P'_1) \subset X_{n+1}^{(k)}$. Define $u \in L(X)$ by $u(x) = 1$ on $X_n^{(k)}$ and $u(x) = 0$ on $X - X_n^{(k)}$. Then $u \in \mathbf{BD}^{(p)}(N)$, $u(x) = 1$ on $e(P)$ and $u(x) = 0$ on $e(P')$. Thus $e(P) \cap e(P') = \phi$.

PROPOSITION 5.4. *Let* P *be an infinite path and* $\alpha \in e(P)$. *Then* $\{\alpha\}$ *is not a* G_δ*-set.*

PROOF. Let $\alpha \in e(P)$ and assume that $\{\alpha\}$ is a G_δ-set. By Theorem 4.2, there exists $h \in \mathbf{BD}^{(p)}(N)$ such that $h(x) < 1$ on $X^* - \{\alpha\}$ and $h(\alpha) = 1$. Note that $V_n = \{x \in C_X(P); h(x) > 1 - 1/n\}$ is an infinite set. Since the subnetwork $N_P = \langle C_X(P), C_Y(P) \rangle$ is of parabolic type of order p (cf. Example 4.2), $u = 1 \in \mathbf{BD}_0^{(p)}(N_P)$. By Lemma 3.4, we can find a function $\varphi \in \mathbf{BD}_0^{(p)}(N_P)$ and two sequences $\{a_m\}$ and $\{b_m\}$ in $C_X(P)$ such that $a_m \in V_{2m-1}$, $b_m \in V_{2m}$ and $\varphi(a_m) = 0 < u(b_m) = \varphi(b_m)$ for all m. We extend φ to a function v on X by the same way as in Proposition 5.1. Then $D_p(v) = D_p(\varphi; N_P)$, $v \in \mathbf{BD}^{(p)}(N)$ and $v(x) = \varphi(x)$ on $C_X(P)$. Since $\{a_m\}$ and $\{b_m\}$ converge to α and v is continuous at α, we arrive at a contradiction. Thus $\{\alpha\}$ is not a G_δ-set.

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