Mem. Fac. Sci. Shimane Univ.,
22, pp. 1–9 Dec. 20, 1988

Periodic solutions of the equation $\dot{x}(t)=-f(x(t))(g(x(t))+h(x(t-1)))$

Dedicated to professor Tosihusa Kimura on his 60th birthday

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(Received September 7, 1988)

The existence of nontrivial periodic solutions of the scalar equation $\dot{x}(t)=-f(x(t))(g(x(t))+h(x(t-1)))$ is mainly discussed by using a fixed point theorem for a closed convex set. As an application of the main results, we show that a conjecture by G. Seifert is right. Moreover we give a negative answer to a question by G. Seifert.

§1. Introduction

Recently, G. Seifert [8] has obtained some results concerning the boundedness and the asymptotic behavior of the solutions, and the existence of periodic solutions of the scalar generalized logistic equation

$$\dot{N}(t)=N(t)\ (a-bN(t)-N(t-1)),\quad t\geqq 0, \tag{1}$$

which arises in population dynamics. Here the superposed dot denotes the right-hand derivative, a and b are positive constants. We are concerned with solutions of (1) such that $N(t)=N_0(t)$, where $N_0(t)$ is a given initial function defined on $[-1, 0]$ which is positive and continuous. In [8], concerning the existence of periodic solutions, it is shown that (1) has nontrivial periodic solutions for a fixed b $(0<b<1)$ and a near $a_0(b)(=\sqrt{(1+b)/(1-b)}Cos^{-1}(-b))$ by using a Hopf bifurcation. In addition, G. Seifert presented the following conjecture and a question for $b<1$ in [8].

(C) For all $a>a_0(b)$, there exist nontrivial periodic solutions of (1).

(Q) Is it possible that there exists a sequence $t_k\to\infty$ as $k\to\infty$ such that $N(t_k)\to 0$ as $k\to\infty$?

In this paper, we shall show that Conjecture (C) is right and give a negative answer to Question (Q).

There are various methods and many results for the existence of periodic solutions of functional differential equations [cf. 1–5, 7]. In §2, we shall show the existence of nontrivial periodic solutions of a more general system than (1) by using a fixed point theorem for a closed convex set which can be found in [5]. In §3, we shall obtain a result concerning the existence of nontrivial periodic solutions of (1) as an application of the results obtained in §2. Finally we shall give a negative answer to Question (Q).

Let R denote the interval $-\infty < t < \infty$, and let C be the Banach space of continuous functions ϕ: $[-1,0] \to R$ with the uniform norm $|\phi| = \sup_{-1<\theta<0} |\phi(\theta)|$. For any $M>0$, let $S_M = \{\phi \in C: |\phi| = M\}$. For any continuous function $x(s)$ defined on $-1 \leqq s < T$ $(0 < T \leqq \infty)$, and any fixed t $(0 \leqq t < T)$, $x_t \in C$ is defined by $x_t(\theta) = x(t+\theta)$, $-1 \leqq \theta \leqq 0$.

§2. Existence of nontrivial periodic solutions

If we put $x(t) = N(t) - a/(b+1)$ for a solution $N(t)$ of (1), we obtain from (1) the equation equivalent to (1):

$$\dot{x}(t) = -\left(x(t) + \frac{a}{b+1}\right)(bx(t) + x(t-1)), \quad t \geqq 0. \tag{2}$$

The zero solution of (2) corresponds to the constant solution $N(t) = a/(b+1)$ of (1).

In this section, we shall discuss the existence of nontrivial periodic solutions of the equation

$$\dot{x}(t) = -f(x(t))\,(g(x(t)) + h(x(t-1))), \quad t \geqq 0, \tag{3}$$

where f, g, and h satisfy the following conditions for $A_0 > A_1 > 0$, $A_2 > 0$, and $B > 0$.

(H1) f, g, h: $R \to R$ are continuously differentiable, $f(-A_0) = 0$, $f(x) > 0$ for $x > -A_0$, $f(x) > B$ for $-A_1 < x < A_2$, $xg(x) > 0$ and $xh(x) > 0$ for $x \neq 0$, and $g(x^*) + h_* = 0$, $g(x) + h_* < 0$ for $0 < x < x^*$, where $x^* > 0$ is a constant and $h_* = \inf_{-A_0 < x < 0} h(x)$.

(H2) $|h(x)| \geqq q|x|$ for $-A_1 \leqq x \leqq A_2$, where $q > 0$ is a constant.

(H3) For $G, H > 0$, $g(x) - Gx = o(|x|)$ and $h(x) - Hx = o(|x|)$ as $x \to 0$.

The function $x(t) = x(t, \phi)$ is said to be the solution of (3) through $(0, \phi)$ if for T with $0 < T \leqq \infty$, $x(t)$ is defined and continuous on $[-1, T)$ and satisfies (3) on $[0, T)$, and $x_0 = \phi$. For any $k > 0$, the set $K(k)$ is defined by

$$K(k) = \left\{ \phi \in C: \begin{array}{l} |\phi| \leqq k,\ \phi(-1) = 0,\ \phi(\theta) \geqq 0 \quad \text{and} \quad |\phi(\theta_1) - \phi(\theta_2)| \\ \leqq L|\theta_1 - \theta_2| \quad \text{for} \quad -1 \leqq \theta,\ \theta_1,\ \theta_2 \leqq 0 \end{array} \right\}$$

where $L = (\sup_{0<x<x^*} f(x)) \max\{\sup_{0<x<x^*} g(x), -h^*\}$. Then $K(k)$ is a compact convex set in C, $0 \in K(k)$, and we have:

LEMMA 1. *If* $Bq \geqq 1$, *then there are positive constants* $t_0 = t_0(k)$, $k_0 = k_0(k)$, *and* k_1 *such that if* $\phi \in K(k) \backslash \{0\}$, *then*

(i) $x(t) = x(t, \phi) = 0$ *for some* $t \in [0, t_0]$,

(ii) $x(t) \geqq -k_0$ *as long as* $\sup_{-1<s<t} x(s) \leqq k$ *for* $t \geqq 0$, *and*

(iii) *there is a finite* $\tau(\phi) > 2$ *such that*

$$x_{\tau(\phi)}(\phi)\in K(k_1),$$

where the set $\{\tau(\phi):\ \phi\in K(k)\backslash\{0\}\}$ *is bounded.*

PROOF. Let $t_0=2$ if $k\leqq A_2$, and let $t_0=(k-A_2)/u(v+w)+3$ if $k>A_2$, where $u=\inf_{A_2<x<k} f(x)$, $v=\inf_{A_2<x<k} g(x)$, and $w=\inf_{A_2<x<k} h(x)$. We show that $x(t)=x(t,\phi)=0$ for some $t\in[0, t_0]$. First we consider the case $k>A_2$ and $\phi(0)>A_2$. In this case, $x(t)$ is non-increasing for $t\geqq 0$ as long as $x(t)>0$, and $\dot{x}(t)\leqq -u(v+w)$ for $t\geqq 1$ as long as $x(t)>A_2$. Hence if $x(1)>A_2$, we have

$$x(t)\leqq x(1)-(t-1)u(v+w),\quad t\geqq 1$$

as long as $x(t)>A_2$. Suppose that $x(t)>A_2$ on $[0, t_0-2]$. Then we obtain

$$x(t_0-2)\leqq x(1)-(t_0-3)u(v+w)\leqq k-(k-A_2)=A_2.$$

This contradiction shows that $x(t)=A_2$ for some $t\in[0, t_0-2]$. Next suppose that $x(t)>0$ on $[t_0-2, t_0]$. Then $x(t)$ is decreasing on $[t_0-2, t_0]$, and

$$\dot{x}(t)\leqq -Bqx(t-1),\quad t_0-1\leqq t\leqq t_0.$$

Hence we have

$$x(t_0)\leqq x(t_0-1)-Bq\int_{t_0-1}^{t_0} x(s-1)ds\leqq x(t_0-1)\,(1-Bq)\leqq 0,$$

and this contradiction shows that $x(t)=0$ for some $t\in[0, t_0]$.

In other cases, we can prove similarly that $x(t)=0$ for some $t\in[0, t_0]$.

(ii) It is clear that

$$\dot{x}(t)\geqq -f(x(t))\,(g(x(t))+h^*) \tag{4}$$

holds as long as $-A_0<x(s)\leqq k$, $-1\leqq s\leqq t$, for $t\geqq 0$, where $h^*=\sup_{0<x<k} h(x)$. Let $x_0(t)$ be the solution of the equation $\dot{x}=-f(x)\ (g(x)+h^*)$ through (0, 0). Then $x_0(t)$ is decreasing for $0\leqq t\leqq 1$ and $-A_0<x_0(1)<0$. Let $k_0=k_0(k)$ be a number such that $-x_0(1)\leqq k_0<A_0$. Now we show that the following holds.

$$x(t)\geqq -k_0\quad\text{as long as}\quad \sup_{-1<s<t} x(s)\leqq k. \tag{5}$$

Suppose that for some $t_1>0$, $x(t_1)<-k_0$ and $x(t_1)<x(t)\leqq k$ for $-1\leqq t<t_1$. Define t_2 by $t_2=\sup\{t\in[0,\ t_1):\ x(t)=0\}$. First we show that $t_1-t_2\leqq 1$. Suppose that $t_1-t_2>1$. Then we obtain

$$x(t_2+1)\geqq x_0(1)\geqq -k_0$$

from (4). Since we have $x(t)\leqq 0$ for $t_2\leqq t\leqq t_1$ and $\dot{x}(t)\geqq 0$ for $t_2+1\leqq t\leqq t_1$, we obtain

$$x(t_1) \geqq x(t_2+1) \geqq -k_0,$$

which contradicts the choice of t_1. Thus we have $t_1 - t_2 \leqq 1$ and

$$x(t_1) \geqq x_0(t_1 - t_2) \geqq -k_0,$$

which contradicts the choice of t_1 again. Hence (5) holds.

(iii) First we show that $x(t)<0$ for some $t>0$. Suppose that $x(t) \geqq 0$ for $t \geqq 0$. Then $x(t)$ is nonincreasing for $t \geqq 0$ and we obtain from (i) that $x(t) \equiv 0$ for $t \geqq t_0$, and consequently $x(t) \equiv 0$ for $t \geqq -1$. But this contradicts the fact that $\phi \neq 0$. Now let $\tau_0 = \inf\{t>0: x(t)<0\}$. We show that

$$x(t)<0, \quad \tau_0 < t \leqq \tau_0 + 1 \tag{6}$$

holds. Suppose that (6) does not hold. Then there are t_3 and t_4 such that $\tau_0 < t_3 < t_4 \leqq \tau_0 + 1$, $x(t)<0$ for $t_3 \leqq t < t_4$, and $x(t_4)=0$. Since we have $x(t-1) \geqq 0$ for $t_3 \leqq t \leqq t_4$, we obtain

$$\dot{x}(t) \leqq -f(x(t))g(x(t)), \quad t_3 \leqq t \leqq t_4.$$

Let $x_1(t)$ be the solution of the equation $\dot{x} = -f(x)g(x)$ through $(t_3, x(t_3))$. Then $x_1(t)$ is increasing on $[t_3, t_4]$ and $x_1(t_4)<0$. Thus we obtain

$$x(t_4) \leqq x_1(t_4) < 0,$$

which contradicts the choice of t_4, and hence, (6) holds.

Let α, β, γ, and δ be numbers such that $0<\alpha<\min\{k_0, A_1\}$, $\beta = \inf_{-k_0<x<-\alpha} f(x)$, $\gamma = \sup_{-k_0<x,y<-\alpha} (g(x)+h(y))$, and $\delta = (\alpha - k_0)/\beta\gamma$. First we show that $x(t) = -\alpha$ for some $t \in [\tau_0, \tau_0 + \delta + 2]$ even if $x(t_5) < -\alpha$ for some $t_5 \in (\tau_0, \tau_0 + 1)$. If $x(\tau_0 + 2) < -\alpha$, then we have

$$\dot{x}(t) \geqq -\beta\gamma, \quad t \geqq \tau_0 + 2$$

as long as $x(t) < -\alpha$. Thus it is easily seen that $x(t) = -\alpha$ for some $t \in [\tau_0, \tau_0 + \delta + 2]$.

Next let $t_6 \in [\tau_0 + 1, \tau_0 + \delta + 2]$ be a number such that $-\alpha \leqq x(t_6) < 0$. If $x(t)<0$ on $[t_6, t_6+2]$, then $x(t)$ is increasing on $[t_6, t_6+2]$ and we have

$$\dot{x}(t) \geqq -Bqx(t-1), \quad t_6 + 1 \leqq t \leqq t_6 + 2,$$

which implies

$$x(t_0) \leqq x(t_0 - 1) - Bq \int_{t_0-1}^{t_0} x(s-1)ds \leqq x(t_0 - 1)\,(1 - Bq) \leqq 0,$$

Since this is a contradiction, $x(t)=0$ for some $t \in (\tau_0, \tau_0 + \delta + 4]$.

If we define τ_1 by $\tau_1 = \inf\{t > \tau_0: x(t) = 0\}$, then by a similar argument as in the proof of (6), we can easily prove that $x(t)>0$ for $\tau_1<t\leqq\tau_1+1$.

Since we have $x(t)\geqq -k_0$ for $-1\leqq t\leqq\tau_1$, we obtain

$$\dot{x}(t)\leqq -f(x(t))\,(g(x(t))+h_0),\quad \tau_1\leqq t\leqq\tau_1+1,$$

where $h_0=\inf_{-k_0<x<0} h(x)$. Let $x_*=\inf\{x>0\colon g(x)+h_0=0\}$, and let $x_2(t)$ be the solution of the equation $\dot{x}=-f(x)\,(g(x)+h_0)$ through (0, 0). Then $x_2(t)$ is increasing on $[\tau_1, \tau_1+1]$ and $x_2(\tau_1+1)<x_*$. Thus for $\tau(\phi)=\tau_1+1$ and $k_1=x_*$, we have

$$0\leqq x(t)<k_1,\quad \tau_1\leqq t\leqq\tau(\phi).$$

Moreover, since we have $|\dot{x}(t)|\leqq f(x(t))\max\{g(x(t)),\ -h(x(t-1))\}\leqq(\sup_{0<x<k_1} f(x))\max\{\sup_{0<x<k_1} g(x),\ -h_0\}\leqq L$ for $\tau_1\leqq t\leqq\tau(\phi)$, it follows that $x_{\tau(\phi)}(\phi)\in K(k_1)$. Finally $2<\tau(\phi)<t_0+\delta+5$ implies that the set $\{\tau(\phi)\colon \phi\in K(k)\backslash\{0\}\}$ is bounded.

The linear part of (3) is

$$\dot{y}(t)=-F(Gy(t)+Hy(t-1)),\quad t\geqq 0, \tag{7}$$

where $F=f(0)$. The characteristic equation for (7) is

$$\frac{\lambda}{F}+G+He^{-\lambda}=0. \tag{8}$$

Concerning the existence of a characteristic root of (8) with positive real part, we have:

LEMMA 2. *If* $0<G<H$ *and* $F>cos^{-1}(-G/H)/\sqrt{H^2-G^2}$ $(\pi/2<cos^{-1}(-G/H)<\pi)$, *there is a characteristic root* $\lambda=\alpha+i\beta$ *of* (8) *with* $0<\alpha<log(H/G)$ *and* $\pi/2<\beta<\pi$.

PROOF. Suppose that $\lambda=\alpha+i\beta$ $(\beta>0)$ solves (8). Then we have

$$\begin{cases} \dfrac{\alpha}{F}+G+He^{-\alpha}cos\beta=0 \\[2ex] \dfrac{\beta}{F}-He^{-\alpha}sin\beta=0. \end{cases} \tag{9}$$

By eliminating F from these equations, we obtain

$$-G=He^{-\alpha}\left(\frac{\alpha sin\beta}{\beta}+cos\beta\right)\equiv f(\alpha,\beta). \tag{10}$$

Since we have $f(\alpha, \pi/2)=2\alpha He^{-\alpha}/\pi$, $f(\alpha, \pi)=-He^{-\alpha}$, and

$$f_\beta(\alpha,\beta)=He^{-\alpha}\left(\frac{\alpha(\beta cos\beta-sin\beta)}{\beta^2}-sin\beta\right)<0,\ \alpha>0,\ \frac{\pi}{2}\leqq\beta\leqq\pi,$$

there is a continuous function $\beta(\alpha)$ defined on $[0, log(H/G)]$ such that $\pi/2<\beta(\alpha)<\pi$ and

$f(\alpha, \beta(\alpha)) = -G$ for $0<\alpha<log(H/G)$, $\beta(0)=cos^{-1}(-G/H)$, and $\beta(log(H/G))=\pi$. From the second equation of (9), we obtain

$$\frac{1}{F}=\frac{He^{-\alpha}sin\beta}{\beta}\equiv g(\alpha, \beta),$$

and since $G(\alpha)=g(\alpha, \beta(\alpha))$ is continuous on $[0, log(H/G)]$ and we have

$$G(0)=\frac{\sqrt{H^2-G^2}}{cos^{-1}(-G/H)}>\frac{1}{F}>0=G(log(H/G)),$$

there is a characteristic root $\lambda=\alpha+i\beta$ of (8) such that $0<\alpha<log(H/G)$ and $\pi/2<\beta<\pi$.

Now we state a known result for (7). For any characteristic root λ of (8), there is a decomposition of C as $C=P_\lambda\oplus Q_\lambda$, where P_λ and Q_λ are invariant under the solution operator $T(t)$ of (7), $T(t)\phi=y_t(\phi)$, $\phi\in C$. Let the projection operators defined by the above decomposition of C be π_λ and $I-\pi_\lambda$, where I denotes the identity operator and the range of π_λ *is* P_λ.

For $k>k_1$, let $K=K(k)$. For $\phi\in K\backslash\{0\}$, define the mapping A by

$$A\phi=x_{\tau(\phi)}(\phi).$$

Since we have $x(t)<0$ for $\tau_1-1\leqq t<\tau_1$ from (6) and the definition of τ_1, we obtain $\dot{x}(\tau_1)>0$. Thus by the continuity of $x(t, \phi)$ in t and ϕ, $\tau(\phi)$ is continuous on $K\backslash\{0\}$, and hence, τ: $K\backslash\{0\}\to[2, \infty)$ is completely continuous by Lemma 1 (iii). On the other hand, A is continuous and $A\phi\in K(k_1)\subset K$ on $K\backslash\{0\}$. Thus A takes $K\backslash\{0\}$ into K and is completely continuous. Moreover, we have the following lemma.

LEMMA 3. (i) *Let λ be the characteristic root of* (8) *given in Lemma* 2. *Then there is a* $\delta>0$ *such that*

$$\inf\{|\pi_\lambda\phi|: \phi\in K\cap S_\delta\}>0. \tag{11}$$

(ii) *There is an $M>0$ such that $A\phi=\mu\phi$, $\phi\in K\cap S_M$ implies $\mu<1$.*

PROOF. (i) For the characteristic root $\lambda=\alpha+i\beta$ of (8) given in Lemma 2, let $\xi(\theta)=e^{\lambda\theta}$, $-1\leqq\theta\leqq 0$, and $\eta(s)=e^{-\lambda s}$, $0\leqq s\leqq 1$. The adjoint equation of (7) is

$$\dot{z}(t)=F(Gz(t)+Hz(t+1))$$

and the bilinear form is given by

$$(\eta, \xi)=\eta(0)\xi(0)-FH\int_{-1}^{0}\eta(\theta+1)\xi(\theta)d\theta.$$

Let $Y=\mathrm{col}(\eta, \bar{\eta})$ and define $\Xi=(\xi_1, \xi_2)$ by

$$\xi_1(\theta)=\frac{1}{\Delta}((\bar{\eta},\ \bar{\xi})\xi-(\bar{\eta},\ \xi)\bar{\xi}),$$
$$\qquad\qquad\qquad -1\leqq\theta\leqq 0$$
$$\xi_2(\theta)=\frac{1}{\Delta}((\eta,\ \xi)\bar{\xi}-(\eta,\ \bar{\xi})\xi),$$

where $\bar{\eta}$ denotes the complex conjugate of η and $\Delta=(\eta, \xi)(\bar{\eta}, \bar{\xi})-(\eta, \bar{\xi})(\bar{\eta}, \xi)\neq 0$. Then it is easy to see that $(\eta, \xi_1)=(\bar{\eta}, \xi_2)=1$ and $(\eta, \xi_2)=(\bar{\eta}, \xi_1)=0$. Therefore, for any $\phi\in C$, $\pi_\lambda\phi=(\eta, \phi)\xi_1+(\bar{\eta}, \phi)\xi_2$ (cf. [5, p. 177, Lemma 3.4]).

Let δ be a number with $0<\delta<k$. If (11) does not hold, then $\pi_\lambda\phi=0$ for some $\phi\in K\cap S_\delta$, since $|\pi_\lambda\phi|$ is a continuous function in ϕ on the compact set $K\cap S_\delta$. Thus we have $(\eta, \phi)=(\bar{\eta}, \phi)=0$. If we denote by $I(\phi)$ the imaginary part of (η, ϕ), then

$$I(\phi)=FH\int_{-1}^{0} e^{-\alpha(\theta+1)}sin\beta(\theta+1)\phi(\theta)d\theta.$$

Since $\pi/2<\beta<\pi$, we have $\sin\beta(\theta+1)\geqq 0$ for $-1\leqq\theta\leqq 0$, and hence, $\phi\in K\cap S_\delta$ implies $I(\phi)>0$. But this contradicts the fact that $(\eta, \phi)=0$.

(ii) For M with $k_1<M<k$, where k_1 is given in Lemma 1, $A\phi=\mu\phi$, $\phi\in K\cap S_M$ implies $\mu<1$ by Lemma 1 (iii).

We are now ready to prove the existence of a nontrivial periodic solution of (3) by using the following theorem, which can be found in [5].

THEOREM 1. *Suppose that the following conditions are satisfied:*

(i) *There is a characteristic root λ of* (8) *with $Re\ \lambda>0$.*

(ii) *There is a closed convex set $K\subset C$, $0\in K$, and $\delta>0$, such that*

$$\inf\{|\pi_\lambda\phi|:\ \phi\in K\cap S_\delta\}>0.$$

(iii) *There is a completely continuous function $\tau: K\backslash\{0\}\to[\varepsilon, \infty)$, $\varepsilon\geqq 0$ such that the mapping defined by*

$$A\phi=x_{\tau(\phi)}(\phi),\quad \phi\in K\backslash\{0\}$$

takes $K\backslash\{0\}$ into K and is completely continuous.

(iv) *There is an $M>0$ such that $A\phi=\mu\phi$, $\phi\in K\cap S_M$ implies $\mu<1$.*

Then there is a nontrivial periodic solution of (3) *with initial function in $K\backslash\{0\}$.*

Among the assumptions of Theorem 1, (i) holds by Lemma 2, (ii) and (iv) hold by Lemma 3, and (iii) holds for $\varepsilon=2$ by Lemma 1 and the continuity of $\tau(\phi)$, under the conditions in Lemmas 1 and 2. Hence we have the following theorem.

THEOREM 2. *Under the conditions of Lemmas* 1 *and* 2, *there is a nontrivial periodic solution $x(t)$ of* (3) *with $-k_0<x(t)<k_1$, its period is greater than* 2 *and less than $t_0+\delta+5$, $x(t)$ has at most one zero point in any interval $[s, s+1]$, and $x(t)$ crosses the t-axis at its zero point.*

§3. The equation $\dot{N}(t)=N(t)\ (a-bN(t)-N(t-1))$

Equation (2), which is equivalent to Equation (1), is the equation with $f(x)=x+a/(b+1)$, $g(x)=bx$, and $h(x)=x$ in Equation (3). Therefore (H1)–(H3) hold for $A_0=a/(b+1)$, any B with $0<B<a/(b+1)$, $A_1=a/(b+1)-B$, any $A_2>0$, $x_*=a/b(b+1)$, $q=1$, $G=b$, and $H=1$, and $F=a/(b+1)$ in (7). Moreover, x_* in the proof of Lemma 1 satisfies $x_*=k_0/b<a/b(b+1)$. Since $a/(b+1)>cos^{-1}(-b)/\sqrt{1-b^2}$ $(\pi/2<cos^{-1}(-b)<\pi)$ for $0<b<1$, we have the following corollary from Lemmas 1–3 and Theorem 2.

COROLLARY. *Suppose that* $0<b<1$. *If* $a>\sqrt{(1+b)/(1-b)}cos^{-1}(-b)$ $(\pi/2<cos^{-1}(-b)<\pi)$, *then there is a nontrivial periodic solution* $x(t)$ *of* (2) *with* $-a/(b+1)<x(t)<a/b(b+1)$, $x(t)$ *has at most one zero point in any interval* $[s, s+1]$, *and* $x(t)$ *crosses the t-axis at its zero point.*

REMARK. *By a similar argument as in the proof of Lemma* 1, *it is easily seen that the period of the periodic solution* $x(t)$ *of* (2) *in the above corollary is greater than* 2 *and less than* $\alpha+7$, *where* $\alpha=(k_0-\gamma)/\beta\gamma(b+1)$, $\beta=a/(b+1)-k_0$, $0<\gamma<min\{k_0, a/(b+1)-1\}$, *and* k_0 *is a suitable constant with* $0<k_0<a/(b+1)$.

Finally we give a negative answer to Question (Q).

THEOREM 3. *Each solution* $N(t)$ *of* (1) *such that* $N(t)>0$ *for* $-1\leqq t\leqq 0$ *is bounded away from zero.*

PROOF. Suppose that for some $t_1\geqq 0$, $N(t)\geqq a/(b+1)$ for $t\geqq t_1$. Then there is a $\delta_1>0$ such that $N(t)\geqq\delta_1$ for $-1\leqq t\leqq t_1$, which implies $N(t)\geqq\min\{a/(b+1), \delta_1\}$ for $t\geqq -1$, and hence, $N(t)$ is bounded away from zero. Now consider the case that the set $S=\{t>0: N(t)<a/(b+1)\}$ contains an arbitrary large t. By Theorem 3 in [8], we have

$$0<N(t)\leqq\frac{a}{b},\quad t\geqq t_2 \tag{12}$$

for some $t_2>0$. We show that there is a $\delta_2>0$ such that

$$N(t)\geqq\delta_2,\quad \alpha\leqq t\leqq\beta \tag{13}$$

holds for any interval $(\alpha, \beta)\subset S$ with $t_2+2\leqq\alpha<\beta\leqq\infty$ and $N(\alpha)=a/(b+1)$.

First consider the case $\beta>\alpha+1$. From (12), we obtain

$$\dot{N}(t)=N(t)\ (a-bN(t)-N(t-1))\geqq-\frac{a^2}{b^2},\quad \alpha-1\leqq t\leqq\alpha.$$

Thus if we define ϕ by

$$\phi(\theta)=min\left\{\frac{a}{b}, \frac{a}{b+1}-\frac{a^2\theta}{b^2}\right\}, \quad -1\leqq\theta\leqq 0,$$

then we have $N(t-1)\leqq\phi(t-\alpha-1))$, $\alpha\leqq t\leqq\alpha+1$, and hence,

$$\dot{N}(t)\geqq N(t)\,(a-bN(t)-\phi(t-\alpha-1)), \quad \alpha\leqq t\leqq\alpha+1. \tag{14}$$

Let $N_1(t)$ be the solution of $\dot{N}=N(a-bN-\phi(t-\alpha-1))$ on $[\alpha, \alpha+1]$ through $(\alpha, a/(b+1))$. Then we obtain from (14) that $N(t)\geqq N_1(t)$ for $\alpha\leqq t\leqq\alpha+1$. Since $N_1(t)>0$ for $\alpha\leqq t\leqq\alpha+1$, there is a $\delta_2>0$ such that

$$N(t)\geqq\delta_2, \quad \alpha\leqq t\leqq\alpha+1.$$

Moreover we have

$$\dot{N}(t)=N(t)\,(a-bN(t)-N(t-1))>0, \quad \alpha+1<t<\beta,$$

and consequently (13) holds. We can similarly prove that (13) holds in the case $\beta\leqq\alpha+1$.

On the other hand, for any $T>0$, there is a $\delta_3>0$ such that

$$N(t)\geqq\delta_3, \quad -1\leqq t\leqq T,$$

which together with (13) imply that $N(t)$ is bounded away from zero.

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