Mem. Fac. Sci. Shimane Univ.,
22, pp. 43–49 Dec. 20, 1988

Linear Topological Spaces and Simplicial Complexes with the Locally Convex Topology

Dedicated to Professor Miyuki Yamada on his 60th birthday

Takuo Miwa

Department of Mathematics, Shimane University, Matsue, Japan
(Received September 7, 1988)

In the previous papers ([8], [10]), we introduced the locally convex topology to simplicial complexes and real vector spaces. In this paper, we deal with continuity of maps and product spaces in vector spaces and simplicial complexes with the locally convex topology. Further, we deal with subdivisions and neighborhood retracts of simplicial complexes with the locally convex topology.

§1. Introduction

In the previous paper [8], we introduced a locally convex topology to simplicial complexes, which is the strongest locally convex topology contained in the Whitehead topology. We called it the *locally convex topology*. The study of this topology was motivated by the fact that the Whitehead topology is not necessarily locally convex. The locally convex topology of full simplicial complex was applied to the closed embedding theorem; that is, each stratifiable space is embedded to an AR(stratifiable)-space as a closed subspace ([9]). The fact that the Whitehead topology is not necessarily locally convex was recognized by the following ([4; pp. 416, 4.3]):

PROPOSITION 1.1. *There is a real vector space with the finite topology such that it is neither a linear topological space nor a locally convex space.*

In Proposition 1.1 the *finite topology* in a real vector space is the weak topology determined by the Euclidean topology on each finite dimensional linear subspace (cf. [4; pp. 416, Definition 4.2]). On the other hand, by the fact of Proposition 1.1 the study of the other previous paper [10] was motivated; that is, in a real vector space E there exists a topology $\mathscr{T}$ such that $(E, \mathscr{T})$ is a linear topological space and $\mathscr{T}$ is the strongest locally convex topology contained in the finite topology. (We also call this topology $\mathscr{T}$ the *locally convex topology*. From now on, we use "the l.c. topology" as an abbreviation of the locally convex topology.)

In this paper, by considering the fact that a simplicial complex with the l.c. topology is a subspace of some real vector space with the l.c. topology, we show the following results: In section 3, we show that every linear map from a real vector space with the l.c. topology to a locally convex linear topological space is continuous. Further, we deal with continuity of linear maps in simplicial complexes. In section 4, we show that the cartesian product of two real vector spaces E and F with the l.c. topology is homeomorphic to the vector space $E \times F$ with the l.c. topology. Further we show that the cartesian product of two simplicial complexes with the l.c. topology is a simplicial complex with the l.c. topology. In section 5, we show that any subdivision of a simplicial complex with the l.c. topology is a simplicial complex with the l.c. topology which is homeomorphic to the original one. In section 6, we show that every subcomplex of a simplicial complex K with the l.c. topology is a neighborhood retract of K. This theorem was announced in [8; Theorem 4.1], but as we only gave an outline of the proof in [8], we give a complete proof of the theorem.

Throughout this paper, N and R denote the sets of all natural numbers and all real numbers, respectively. For M_1-spaces and stratifiable spaces, see [2] and [1], respectively. For ANR (or AR) and linear topological spaces, see [5] and [7], respectively. Every terminology is referred to [4] or [7], unless otherwise stated.

The author wishes to express his gratitude to Dr. K. Sakai for his valuable comments and discussions.

§2. Preliminaries

In the previous paper [10], we introduced the l.c. topology to a real vector space. This is useful because a real vector space with the l.c. topology is a locally convex linear topological space (cf. Proposition 1.1 and [10; Theorem 2.4]). Note that a countable dimensional vector space with the finite topology is a locally convex linear topological space (cf. [6; Theorem] or [3; Lemma 4.4]).

We now refer to the construction of the l.c. topology in a real vector space ([10; Construction 2.1]) so that it can be used in this paper.

CONSTRUCTION 2.1. Let E be a real vector space with a Hamel basis $\mathscr{B} = \{u_\alpha : \alpha \in \Lambda\}$ and let $\mathscr{E}_n$ be all n-dimensional linear subspaces of E generated by n elements of $\mathscr{B}$. For each $\alpha \in \Lambda$, pick up an $n_\alpha \in N$. Let $U_1 = \bigcup\{\{tu_\alpha : |t| < 1/n_\alpha\} : \alpha \in \Lambda\}$. By using induction, if U_{n-1} has been defined for $n \geq 2$, let $U_n = \bigcup\{\mathrm{conv}(F \cap U_{n-1}) : F \in \mathscr{E}_n\}$, where conv A is the convex hull of A. Let $U(n_\alpha : \alpha \in \Lambda) = \bigcup\{U_n : n \in N\}$ and $\mathscr{U}$ be all $U(n_\alpha : \alpha \in \Lambda)$.

By [10; Lemma 2.2], $\mathscr{U}$ satisfies the local base conditions. Therefore by [7; Theorem 5.1], $\mathscr{T} = \{W \subset E$: For each $x \in W$, there is $U \in \mathscr{U}$ with $x + U \subset W\}$ is a vector topology (i.e. $(E, \mathscr{T})$ is a linear topological space) and $\mathscr{U}$ is a local base for $\mathscr{T}$. We denote the space E equipped with this topology $\mathscr{T}$ by $|E|_C$. Furthermore we obtained the following result ([10; Theorem 2.4]).

THEOREM 2.2. *$(E, \mathscr{T})$ is a locally convex linear topological space, and $\mathscr{T}$ is the strongest locally convex topology contained in the finite topology.*

On the other hand, for a full simplicial complex K, we introduced the l.c. topology ([8]; also see [9]). We denote the space K equipped with the l.c. topology by $|K|_C$. (Note that $|K|_W$ is the space K with the Whitehead topology.) The l.c. topology of K is characterized by the strongest locally convex topology contained in $|K|_W$. We embed K in a suitable vector space E with the l.c. topology so that its vertices are at the unit points of E. By the definitions and constructions of the l.c. topology in vector spaces or full simplicial complexes ([8], [10]), it is easily verified that $|K|_C$ is a subspace of $|E|_C$. Therefore we use the same name "the l.c. topology" in vector spaces and full simplicial complexes. In general, if a simplicial complex H has a relative topology of the l.c. topology in a full simplicial complex containing H, we call that the space H has the *l.c. topology* and denote by $|H|_C$.

In the previous papers [8] and [10], we obtained the following results:

THEOREM 2.3. *Let E be a real vector space and K a full simplicial complex. Then*
(1) [8; Theorem 3.2] $|K|_C$ *is an M_1-space.*
(2) [8; Theorem 3.3] $|K|_C$ *is* AR(stratifiable).
(3) [10; Theorem 2.8] $|E|_C$ *is an M_1-space.*
(4) [10; Theorem 2.9] $|E|_C$ *is* AR(stratifiable).
Furthermore, we have

THEOREM 2.4. *Let H be a simplicial complex. Then $|H|_C$ is an M_1-space.*

PROOF. H can be embedded in a full simplicial complex K with the same vertics. To prove this theorem, we use the notation in the proof of [8; Theorem 3.2]. Let $\mathscr{U}_n^m(H)=\{U\cap H\colon U\in\mathscr{U}_n^m\}$ and $\mathscr{U}_0(H)=\{U\cap H\colon U\in\mathscr{U}_0\}$. Then it is obvious that $\{\mathscr{U}_n^m(H)\colon m, n\in H\}\cup\{\mathscr{U}_0(H)\}$ is a base of $|H|_C$. Therefore it is sufficient to prove that each $\mathscr{U}_n^m(H)$ and $\mathscr{U}_0(H)$ are closure preserving in $|H|_C$.

Let $\mathscr{U}\subset\mathscr{U}_n^m(H)$, $x\in H$ and $x\bar{\in} cl_H U$ for each $U\in\mathscr{U}$. Then there is a simplex S of H such that $x\in S$ and $S\in\mathscr{K}_k$ for some $k\in N$, and there is $U'\in\mathscr{U}_n^m$ with $U'\cap H=U$. Since $x\bar{\in} cl_H U$ and $U'\cap H=U$, it is easily verified that $x\bar{\in} cl_K U'$. Let $\mathscr{U}'=\{U'\in\mathscr{U}_n^m\colon U'\cap H=U$ for $U\in\mathscr{U}\}$. Since $\mathscr{U}_n^m$ is closure preserving by the proof of [8; Theorem 3.2], there is a neighborhood W' of x in $|K|_C$ such that $W'\cap U'=\phi$ for each $U'\in\mathscr{U}'$. Then $W=W'\cap H$ is a neighborhood of x in $|H|_C$ such that $W\cap U=\phi$ for each $U\in\mathscr{U}$. Thus $\mathscr{U}_n^m(H)$ is closure preserving. The closure preservingness of $\mathscr{U}_0(H)$ is much the same.

§3. Continuity of linear maps

Continuity of a map from a linear space with the finite topology (resp. a simplicial complex with the Whitehead topology) is verified by continuity of the restriction to each finite dimensional linear subspace (resp. each simplex). But in the l.c. topology, there do not exist such good and simple verifying methods, because a vector space (or a simplicial complex) with the l.c. topology need not be a k-space (cf. [8; Proposition 2.3]). The following theorems are useful to verify continuity of maps from vector spaces (or simplicial complexes) with the l.c. topology.

THEOREM 3.1. *Let E be a vector space and F a locally convex linear topological space. Then every linear map f: $|E|_C \to F$ is continuous.*

PROOF. By [7; pp. 37, 5.3], it is sufficient to prove the continuity of f at the origin 0 of E. Let V be any locally convex neighborhood of $0=f(0)$ in F. Using the notation of Construction 2.1, for each $\alpha \in A$, there exists an $n_\alpha \in N$ such that $f(\{tu_\alpha: |t|<1/n_\alpha\}) \subset V$ because f is linear. By the convexity of V, $f(U(n_\alpha: \alpha \in A)) \subset V$. This completes the proof.

THEOREM 3.2. *Let K and L be two simplicial complexes. Then every linear map f: $|K|_C \to |L|_C$ is continuous, where a linear map of a simplicial complex means that the map is linear on each simplex.*

PROOF. Let $\mathscr{B}=\{u_\alpha: \alpha \in A\}$ and $\mathscr{B}'$ be the sets of all vertices of K and L, respectively. Further, let E and F be two real vector spaces which have the bases $\mathscr{B}$ and $\mathscr{B}'$ as their Hamel bases, respectively. Then $|K|_C$ and $|L|_C$ are the subspaces of $|E|_C$ and $|F|_C$, respectively. Furthermore, we define a map g: $|E|_C \to |F|_C$ by

$$g(s \cdot u_\alpha + t \cdot u_\beta) = s \cdot f(u_\alpha) + t \cdot f(u_\beta) \quad (s,\ t \in R,\ u_\alpha,\ u_\beta \in \mathscr{B}).$$

Then it is easily verified that g is a linear map and $g|\,|K|_C = f$. By Theorem 3.1, g is continuous. Therefore f is continuous.

§4. Product spaces

In this section, we shall show that for two vector spaces E and F, $|E|_C \times |F|_C$ is homeomorphic to $|E \times F|_C$; further that for two simplicial complexes K and L, $|K|_C \times |L|_C$ is homeomorphic to $|K \times L|_C$. These results show that the l.c. topology behaves very well for the cartesian product.

THEOREM 4.1. *Let E and F be two vector spaces. Then the product space $|E|_C \times |F|_C$ is homeomorphic to $|E \times F|_C$.*

PROOF. Let Id: $|E|_C \times |F|_C \to |E \times F|_C$ be the identity. Then, by Theorem 3.1, it is clear that Id^{-1} is continuous. Next, let $\mathscr{B}=\{u_\alpha: \alpha \in A\}$ and $\mathscr{B}'=\{v_\beta: \beta \in M\}$ be Hamel bases of E and F, respectively. Then, $\mathscr{B}_1=\{u'_\alpha: \alpha \in A\} \cup \{v'_\beta: \beta \in M\}$ is a Hamel basis of $E \times F$, where $u'_\alpha=(u_\alpha, 0)$ and $v'_\beta=(0, v_\beta)$ for each $\alpha \in A$ and $\beta \in M$. Now, we shall prove that Id is continuous. Let $U(n_\alpha: \alpha \in A+M)$ be a canonical convex neighborhood of the origin in $|E \times F|_C$. Then $V=U(2n_\alpha: \alpha \in A)$ and $W=U(2n_\beta: \beta \in M)$ are neighborhoods of the origins of E and F, respectively. Pick up any $(x, y) \in V \times W$. Then, there are some sets $\{\alpha_1, \dots, \alpha_m\} \subset A$, $\{\beta_1, \dots, \beta_k\} \subset N$ and $\{a_1, \dots, a_m, b_1, \dots, b_k\} \subset R$ such that

$$x=\sum_{i=1}^{m} a_i(1/2n_{\alpha_i})u_{\alpha_i},\ y=\sum_{i=1}^{k} b_i(1/2n_{\beta_i})v_{\beta_i},$$

$$\sum_{i=1}^{m} |a_i|<1 \quad \text{and} \quad \sum_{i=1}^{k} |b_k|<1.$$

Therefore, since

$$(x, y)=\sum_{i=1}^{m} (a_i/2)\,(1/n_{\alpha_i})u'_{\alpha_i}+\sum_{i=1}^{k} (b_i/2)\,(1/n_{\beta_i})v'_{\beta_i},$$

and

$$\sum_{i=1}^{m} |a_i/2|+\sum_{i=1}^{k} |b_i/2|<1,$$

(x, y) belongs to $U(n_\alpha\colon \alpha\in A+M)$. Thus $Id(V\times W)\subset U(n_\alpha\colon \alpha\in A+M)$. This completes the proof.

THEOREM 4.2. *Let K and L be two simplicial complexes. Then the product space* $|K|_C\times|L|_C$ *is homeomorphic to* $|K\times L|_C$.

PROOF. Let E and F be two vector spaces such that K and L are naturally embedded to E and F, respectively. Then, $|K|_C$ and $|L|_C$ are subspaces of $|E|_C$ and $|F|_C$, respectively. Further, $|K\times L|_C$ is a subspace of $|E\times F|_C$. Therefore, by Theorem 4.1, $Id\colon |K|_C\times|L|_C\to|K\times L|_C$ is a homeomorphism. This completes the proof.

§5. Subdivisions

In this section, we consider subdivisions of simplicial complexes with the l.c. topology. For subdivisions, we have the following:

THEOREM 5.1. *If* $Sd\colon K\to K'$ *is a subdivision of a simplicial complex K, and if K and K' are given the l.c. topology, then Sd is a homeomorphism.*

PROOF. Since Sd^{-1} maps each closed simplex of K' linearly, by Theorem 3.2 Sd^{-1} is continuous. Next, we shall prove that Sd is continuous. Pick up a point $x\in K$ and a neighborhood U of $Sd(x)$ in K'. Then there is a simplex S with $x\in S$. Suppose that S is subdivised to $S_1\cup S_2\cup\cdots\cup S_n$ in K'. Then there is a convex neighborhood V_S of x in S such that $Sd(V_S)\subset U\cap(S_1\cup\cdots\cup S_n)$. For any simplex T with S as its face, by the same way there is a convex neighborhood V_T of x in T such that $Sd(V_T)\subset U$ and $V_T\cap S=V_S$. By the constructions of neighborhoods in simplicial complexes (cf. [8], proof of Theorem 3.2), the union V of these V_T's is a neighborhood of x in K such that $Sd(V)\subset U$. This completes the proof.

§6. Neighborhood retracts in simplicial complexes

The following theorem was announced in [8; Theorem 4.1]. In this section, we shall give a complete proof.

THEOREM 6.1. *Let K be a simplicial complex and H a subcomplex of K. Then $|H|_C$ is a neighborhood retract of $|K|_C$.*

PROOF. Consider the barycentric subdivisions K' and H' of K and H, respectively. Then, by Theorem 5.1, $|K'|_C$ and $|H'|_C$ are homeomorphic to $|K|_C$ and $|H|_C$, respectively. Let $\{u_\alpha: \alpha\in\Lambda\}$ denote the set of all vertices of K'. A point $x\in K'$ is determined by its barycentric coodinates $\{x_\alpha: \alpha\in\Lambda\}$, where

$$\sum_{\alpha\in\Lambda} x_\alpha = 1.$$

Let $M\subset\Lambda$ be defined by $M=\{\alpha\in\Lambda: u_\alpha\in H'\}$ and consider the real-valued function $f: K' \to I=[0, 1]$ defined by

$$f(x)=\sum_{\mu\in M} x_\mu .$$

Since f is clearly linear on each closed simplex of K', by Theorem 3.2 f is continuous on $|K'|_C$. Then the set $U=\{x\in K': f(x)>0\}$ is an open neighborhood of $|H'|_C$ in $|K'|_C$.

Next, define a map $r: U\to|H'|_C$ by taking as the image $r(x)$ of a point $x\in U$ the point whose barycentric coordinates are

$$[r(x)]_\alpha=\begin{cases} x_\alpha/f(x) & (\text{if } \alpha\in M) \\ 0 & (\text{if } \alpha\in\Lambda-M). \end{cases}$$

We shall prove that r is continuous. For a point $x\in U$, let W be any convex neighborhood of $r(x)$ in $|H'|_C$, and let S be a simplex of K' with $x\in S$. Since $r|U\cap S$ is clearly continuous, there is a convex neighborhood V_S of x in $U\cap S$ such that $r(V_S)\subset W$. For any simplex T with S as its face, by the definition of r and the convexity of W, there is a convex neighborhood V_T of x in $U\cap T$ such that $r(V_T)\subset W$ and $V_T\cap S=V_S$. By the constructions of neighborhoods in simplicial complexes with the l.c. topology (cf. [8], proof of Theorem 3.2), the union V of these V_T's is a neighborhood of x in U such that $r(V)\subset W$. Thus, r is a retraction. Therefore $|H'|_C$ is a neighborhood retract of $|K'|_C$. Since $|K'|_C$ and $|H'|_C$ are homeomorphic to $|K|_C$ and $|H|_C$, respectively, this completes the proof.

By this theorem, Theorems 2.3 (2) and 2.4, we have the following corollary, which was announced in [8; Corollary 4.2].

COROLLARY 6.2. *Every simplicial complex with the l.c. topology is* ANR(M_1).

References

[1] C. R. Borges, On stratifiable spaces, Pacific J. Math., **17** (1966), 1–16.

[2] J. G. Ceder, Some generalizations of metric spaces, Pacific J. Math., **11** (1961), 105–126.

[3] D. W. Curtis, Some theorems and examples on local equiconnectedness and its specializations, Fund. Math., **72** (1972), 101–113.

[4] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[5] S. T. Hu, Theory of retracts, Wayne State University Press, Detroite, 1965.

[6] S. Kakutani and V. Klee, The finite topology of a linear space, Arch. Math., **14** (1963), 55–58.

[7] J. L. Kelley and I. Namioka, Linear topological spaces, Van Nostrand Comp., 1963.

[8] T. Miwa, A locally convex topology of simplicial complexes, Mem. Fac. Sci., Shimane Univ., **20** (1986), 25–30.

[9] ——, Embeddings to AR-spaces, Bull. Pol. Acad. Sci. Math., **35** (1987), 565–572.

[10] ——, Topological properties of some locally convex linear topological spaces, Mem. Fac. Sci., Shimane Univ., **21** (1987), 17–20.