Mem. Fac. Sci. Shimane Univ.,
22, pp. 51–62 Dec. 20, 1988

A Construction of Anti-Lie Triple Systems from a Class of Triple Systems.

Dedicated to Professor Kiyosi Yamaguti on his 60th birthday

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(Received September 7, 1988)

Abstract. In this paper, it is shown that the construction of Lie algebras "$Id_w \otimes$ Der $J \oplus T \otimes$ Anti-Der $J \oplus W \otimes J$" due to U. Hirzebruch can be modified to a construction of Lie superalgebras.

Introduction

In [12], J. Tits constructed a Lie algebra from the tensor product $Y \otimes A$ of a Lie algebra Y and a Jordan algebra A. This construction is investigted by U. Hirzebruch in [3] and generalized for the tensor product $W \otimes J$, where W is a two dimensional Jordan triple system and J is any Jordan triple system. In this paper, it is shown that the construction can be modified to get a construction of a Lie superalgebra.

The aim of our article is to introduce a class of triple systems defined by certain identities and to construct an anti-Lie triple system from it. We shall obtain a construction of Lie superalgebras by the standard imbedding of anti-Lie triple systems.

In §1, we introduce a class of triple systems, called anti-Jordan triple systems and give some examples. We study correspondence of polarized anti-Jordan triple systems with anti-Jordan pairs.

In §2, we define a bilinear form on an anti-Jordan triple system and consider it. Also we study the bilinear form for an anti-Lie triple system and the Lie superalgebra which is related to an anti-Jordan triple system.

In §3, we study a construction of Lie superalgebras by a slightly different method than in §2.

We shall be concerned with algebras and triple systems which are finite dimensional over a field of characteristic different from 2 or 3, unless otherwise specified.

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* This research was partially supported by a Grant-in-Aid for Scientific Research (No 59540041) from the Ministry of Education, Japan.

1.

A triple system is a vector space V over a field K together with a K-linear map $V\times V\times V\to V$. We consider a triple system V as satisfying;

(1) $\{xyz\} = -\{zyx\}$

(2) $\{xy\{zuv\}\} = \{\{xyz\}uv\} + \{z\{yxu\}v\} + \{zu\{xyv\}\}.$

for all $x, y, z, u, v\in V$

and call it an anti-Jordan triple system. This notion first appeared in [2]. Also this triple system can be regarded as a Freudenthal-Kantor triple system with $\varepsilon=1$, $\delta=-1$ and $K(a, b)=0$ in [14]. We use the following notation in an anti-Jordan triple system;

$$S(a, b) := L(a, b) + L(b, a),$$

$$A(a, b) := L(a, b) - L(b, a),$$

where $L(a, b)c = \{abc\}$.

Then by straightforward calcuulations, we get

(3) $[S(a, b), L(c, d)] = L(S(a, b)c, d) + L(c, S(a, b)d)$

(4) $[A(a, b), L(c, d)] = L(A(a, b)c, d) - L(c, A(a, b)d),$

that is to say;

$S(a, b)$ is a derivation and $A(a, b)$ is an anti-derivation of the anti-Jordan triple system. (cf. [6, 14])

EXAMPLE 1. Let V be a vector space with an anti-symmetric bilinear form $< -, - >$. Then $\{xyz\} = <x, y>z + <y, z>x$ defines on V an anti-Jordan triple system.

EXAMPLE 2. Under the assumption of Example 1. $\{xyz\} = <x, y>z + <y, z>x - <z, x>y$ defines on V an anti-Jordan triple system.

EXAMPLE 3a. Let q be the standard nondegenerate anti-symmetric bilinear form on K^{2n} and $s: K^{2n}\to K^{2n}$ an orthogonal reflection, i.e. s is orthogonal and satisfies $s^2 = Id$. Then $\{xyz\} = q(x, y)z + q(y, z)x - q(z, sx)y$ defines on K^{2n} an anti-Jordan triple system. Especially, if $s = id$, this reduces to Example 2.

EXAMPLE 3b. One defines on $(K\oplus K)^{2n}$ a triple product $\{xyz\} = q(x, \bar{y})z + q(\bar{y}, z)x - q(z, x)\bar{y}$, where q is the standard non-degenerate anti-symmetric bilinear form on $(K\oplus K)^{2n}$ and $\overline{(x_1^+, x_1^-, \cdots, x_{2n}^+, x_{2n}^-)} = (x_1^-, x_1^+, \cdots, x_{2n}^-, x_{2n}^+)$. From this triple product, we obtain an anti-Jordan triple system.

EXAMPLE 4. For an anti-Jordan triple system V, we put $T = V\oplus \bar{V}$, where $\bar{V}$ is a of V. Then T becomes an anti-Jordan triple system with respect to the trilinear product defined by

(5) $$\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right\}=\begin{pmatrix} L(x_1, y_2)z_1 \\ L(x_2, y_1)z_2 \end{pmatrix}$$

We recall an anti-Lie triple system defined by J. R. Faulkner and J. C. Ferrar that satisfies

(6) $[xyz]=[yxz]$

(7) $[xyz]+[yzx]+[zxy]=0$

(8) $[xy[uvz]]-[uv[xyz]]=[[xyu]vz]+[u[xyv]z]$.

$x, y, z, u, v\in T$.

The starting point of this paper is the following fundamental theorem.

THEOREM 1.1. *If* $(V, \{-, -, -,\})$ *is an anti-Jordan triple system, then* $(V, [-, -, -])$ *with*

(9) $$[xyz]=\{xyz\}+\{yxz\}$$

is an anti-Lie triple system.

We denote the anti-Lie triple system induced from an anti-Jordan triple system V by V^+.

For $\delta=\pm 1$, we consider the following identities;

$$\{xyz\}=\delta\{zyx\}$$
$$\{xy\{uvw\}\}=\{\{xyu\}vw\}-\delta\{u\{yxv\}w\}+\{uv\{xyw\}\}$$

In the case $\delta=1$, these reduce to those of a Jordan triple system, while in the case $\delta=-1$ they reduce to those of an anti-Jordan triple system.

For $\delta=\pm 1$, we put $[xyz]=\{xyz\}-\delta\{yxz\}$. *If* $(V, \{\ \})$ is an anti-Jordan triple system $(\delta=-1)$, then $(V^+, [-, -, -,])$ becomes an anti-Lie triple system. If $(V, \{\ \})$ is a Jordan triple system $(\delta=+1)$, then $(V^-, [-, -, -])$ becomes a Lie triple system.

Hence we have the following correspondences.

Jordan triple system$\cdots\cdots$Lie triple system$\cdots\cdots$ $\delta=1$

(resp. anti-) $\cdots\cdots$(resp. anti-) $\cdots\cdots\delta=-1$

Next we shall discuss a relation of an anti-Jordan pair to a polarized anti-Jordan triple system.

A polarized anti-Jordan tripple system is ananti-Jordan system V together with a direct sum decomposition $V=V_+\oplus V_-$ into submodules V such that

$$\{V_\sigma V_\sigma V_\sigma\}=0,\ \{V_\sigma V_\sigma V_{-\sigma}\}=\{V_\sigma V_{-\sigma} V_{-\sigma}\}=0$$
and $\{V_\sigma V_{-\sigma} V_\sigma\}\subset V_\sigma$
holds for $\sigma=\pm$

In this case, we have

$$\{xyz\}=\{x_+y_-z_+\}\oplus\{x_-y_+z_-\}$$

for $x=x_+\oplus x_-$, $y=y_+\oplus y_-$, $z=z_+\oplus z_-$.

As in [11, Th A.3], we can show that;

PROPOSITION 1.2. *Let $V=V_+\oplus V_-$ and $W=W_+\oplus W_-$ be two simple polarized anti-Jordan triple system and $\mathfrak{V}=(V_+, V_-)$ resp. $\mathfrak{W}=(W_+, W_-)$ the corresponding anti-Jordan pairs. Then the followings are equivalent;*

i) *V and W are isomorphic*

ii) *$\mathfrak{V}$ is isomorphic to $\mathfrak{W}$ or to $\mathfrak{W}^{op}$,*

where $\mathfrak{W}^{op}=(W_-, W_+)$.

We now consider the correspondence between polarized anti-Jordan and anti-Lie triple systems. We begin with the definition. An anti-Lie triple system T is called polarized if it has a direct sum decomposition $T=T_+\oplus T_-$ satisfying

$$[T_\sigma T_\sigma T_\sigma]=[T_\sigma T_\sigma T_{-\sigma}]=0$$

and $[T_\sigma T_{-\sigma} T_\sigma]\subset T_\sigma$ for $\sigma=\pm$.

In this case, we have

$$[xyz]=[x_+y_-z_+]+[x_-y_+z_+]\oplus[x_-y_+z_-]+[x_+y_-z_-].$$

Hence we have the following proposition.

PROPOSITION 1.3.(a) *Let $T=T_+\oplus T_-$ be a polarized anti-Lie triple system. Define a new triple product on T by $\{x_+\oplus x_-, y_+\oplus y_-, z_+\oplus z_-\}=[x_+y_-z_+]\oplus[x_-y_+z_-]$. Then $(T, \{\ \})$ is a polarized anti-Jordan triple system.*
(b) *Conversely, if V is a polarized anti-Jordan triple system, then V^+ is a polarized anti-Lie triple system.*

Therefore from the above proposition and Proposition 5.1 in [2], we obtain the following.

THEOREM 1.4. *There exists a one to one correspondence between each pair of the following classes:*

(i) *equivalence classes of simple anti-Jordan pairs.*
(ii) *isomorphism classes of simple polarized anti-Jordan triple systems and*

(iii) *isomorphism classes of simple polarized anti-Lie triple systems.*

These correpondences are given by

$$(V_+, V_-)\leftrightarrow V = V_+ \oplus V_- \leftrightarrow V^+.$$

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Let V be an anti-Jordan triple system. Then we can define an anti-Lie triple system on $V\oplus \bar{V}$ with respect to the triple product defined by

$$\text{(10)} \qquad \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \begin{pmatrix} L(x_1, y_2)z_1 + L(y_1, x_2)z_1 \\ L(x_2, y_1)z_2 + L(y_2, x_1)z_2 \end{pmatrix}.$$

This anti-Lie tripple system is obtained from Example 4 and Theorem 1 by straightforward calculation. We call it the anti-Lie triple system associated with an anti-Jordan triple system. Using the notation $S(a, b)$, $A(a, b)$ of Section 1, we have

$$\begin{pmatrix} 2L(a, b) & 0 \\ 0 & 2L(b, a) \end{pmatrix} = \begin{pmatrix} S(a, b) + A(a, b) & 0 \\ 0 & S(a, b) - A(a, b) \end{pmatrix}.$$

Hence the inner derivations of the anti-Lie triple system are determined by derivations and anti-derivations of the anti-Jordan triple system. This construction is parallel to the construction of Lie algebras from Jordan triple systems in [8, 10]. On the other hand, these constructions of Lie algebras was extended to get all simple Lie algebras in [6, 7, 13].

As in [6], we may define a bilinear form $\gamma(x, y)$ of an anti-Jordan triple system by

$$\text{(11)} \qquad \gamma(x, y) := 1/2\ Tr\{L(x, y) - L(y, x)\}.$$

PROPOSITION 2.1. *For the bilinear form γ of an anti-Jordan triple system V, we have*

(i) $\gamma(z, \{yxw\}) + \gamma(\{xyz\}, w) = 0$

(ii) $\gamma(x, \{yzw\}) + (\{xwz\}, y) = 0$

for all $x, y, z, w \in V$.

If γ is nondegenerate, this implies

(i)′ $L(x, y)^* = -L(y, x)$

(ii)′ $R(z, w)^* = -R(w, z)$

*where * denotes the adjoint relative to γ.*

PROOF. (i): We have the following identity in anti-Jordan triple system:

$$[L(x, y), L(z, w)] = L(L(x, y)z, w) + L(z, L(y, x)w).$$

Therefore we get $TrL(L(x, y)z, w) = -TrL(z, L(y, x)w)$, so we obtain (i) of Lemma.

(ii): From $\{yxw\} = -\{wxy\}$ and (i), it is clear. ■

It should be noted that (i) and (ii) of the above proposition coincide with identities (10) in [2].

Next we shall define a bilinear form $\alpha(x, y)$ on an anti-Lie triple system T by

(12) $$\alpha(x, y) := 1/2\ Tr\{R(x, y) - R(y, x)\}, \qquad \text{where} \quad R(x, y)z = [zxy].$$

The correspondence between the bilinear form of an anti-Jordan triple system V and the bilinear form of the anti-Lie triple system T associted with it is given by

PROPOSITION 2.2.

(13) $$\alpha\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \gamma(x_1, y_2) + \gamma(x_2, y_1)$$

where $\gamma(\ ,\ )$ is the bilinear form of the anti-Jordan triple system, and $\alpha(\ ,\ )$ is the bilinear form of the anti-Lie triple system.

PROOF. From the definition of the anti-Lie triple system associated with an anti-Jordan triple system we have

$$\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] - \left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} L(z_1, x_2)y_1 + L(x_1, z_2)y_1 - L(z_1, y_2)x_1 - L(y_1, z_2)x_1 \\ L(z_2, x_1)y_2 + L(x_2, z_1)y_2 - L(z_2, y_1)x_2 - L(y_2, z_1)x_2 \end{pmatrix}$$

$$= \begin{pmatrix} L(x_1, y_2) - L(y_1, x_2) & K(x_1, y_1) \\ K(x_2, y_2) & L(x_2, y_1) - L(y_2, x_1) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

where $K(x, y)z = \{xzy\} - \{yzx\}$.

Hence we get

$$\alpha\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \gamma(x_1, y_2) + \gamma(x_2, y_1). \ \blacksquare$$

There exists an almost complex structure $J=\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ on anti-Lie triple system assodiated with an anti-Jordantriple system. As in [6] (cf. Remark in §4), this structure has the following property;

$$\alpha\left(J\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, J\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = -\alpha\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right).$$

Roughly speaking, this concept corresponds to an anti-Hermite structure on supersymmetric space.

Assume that T is an anti-Lie triple system and D is a Lie subalgebra of derivations of T containing the inner derivation $L(T, T)$. Consider $L(T, D)=L_0 \oplus L_1$ with $L_0=D$ and $L_1=T$ and with the product given by $[a_1, a_2]=L(a_1, a_2)$, $-[a_1, D_1]=[D_1, a_1]=D_1a_1$, $[D_1, D_2]=D_1D_2-D_2D_1$ for $a_i \in T$, $D_i \in D$. Then from the definition of an anti-Lie triple system, it follows that $L(T, D)$ is a Lie superalgebra. Hence $L(T, T)\oplus T$ is an ideal of the Lie superalgebra $L(T, D)$. We denote $L(T, T)\oplus T$ by L and call it the standard imbedding Lie superalgebra of T (cf. [2]).

We consider the correspondence between the bilinear form of anti-Lie triple system T and the bilinear form of Lie superalgebra $L=L(T, T)\oplus T$ which is defined by supertrace β in [5], that is, $\beta(x, y) := \text{strace } ad\, x\, ad\, y$, for $x, y \in L$.

PROPOSITION 2.3. *Let α be the bilinear form of an anti-Lie triple system T and let β be the bilinear form of the standard imbedding Lie superalgebra $L=L(T, T)\oplus T$. Then*

(14) $$\alpha(x, y)=\beta(y, x) \quad \textit{for} \quad x, y \in T.$$

Especially, $\beta(x, D)=0$ for $x \in T$, $D \in L(T, T)$.

PROOF. For $x, y \in T$, from the definition of standard imbedding Lie superalgebra of T, we obtain

$$\begin{aligned}&[zxy]-[zyx]\\ &\quad =[[z, x], y]-[[z, y], x]\\ &\quad =-ad\, y\, ad\, x\, z+ad\, x\, ad\, y\, z.\end{aligned}$$

On the other hand, if we put $\text{End}_0 T=\{a \in \text{End } L \mid aT \subset T\}$, then it contains $ad\, y\, ad\, x$, and $ad\, x\, ad\, y$. Hence from the properties of supertrace in [5], we obtain

$$\begin{aligned}2\alpha(x, y) &= Tr\{R(x, y)-R(y, x)\}\\ &=\text{strace } ad\, y\, ad\, x-\text{strace } ad\, x\, ad\, y\\ &=-2\ \text{strace } ad\, x\, ad\, y\\ &=2\beta(y, x).\end{aligned}$$

For $x, y \in T$, $D \in L(T, T)$, we have

$$ad\ x\ ad\ D\ y = [x, Dy] \in L(T, T).$$

If we put $\mathrm{End}_1 T = \{b \in \mathrm{End}\, L \mid bT \subset L(T, T)\}$, then $ad\, x\, ad\, D$ is contained in $\mathrm{End}_1 T$. Hence, we get

strace $ad\ x\ ad\ D = 0$. ■

From Proposition 2.2 and Proposition 2.3, we obtain the following theorem.

THEOREM 2.4. *Let V be an anti-Jordan triple system, T be the anti-Lie triple system associated with V, and L be the standard imbedding Lie superalgebra of T. Let γ, α, β be the respective forms.*
Then the following statements are equivalent:

(i) *the bilinear form γ is nondegenerate,*
(ii) *the bilinear form α is nondegenerate,*
(iii) *the bilinear form β is nondegenerate.*

3

We shall now study a construction of an anti-Lie triple system and of a Lie superalgebra by means of a slightly different way to the construction in §2. The conception in this chapter is a variation of the construction of Lie algebras due to U. Hirzebruch in [3]. Following J. Tits [12], in construction of a Lie superalgebra, it is natural for us to make tensor products $Y \otimes J$ of a Lie superalgebra Y with a bilinear form $< , >$ and Jordan algebras J. Hence, applying this idea to [3], we first consider an anti-Jordan triple system of a two dimensional vector space W over a field K defined by

(*) $$\{abc\} := <a, b>c + <b, c>a - <c, a>b \qquad \text{for} \quad a, b, c \in W.$$

where $< , >$ is an anti-symmetric nondegenerate bilinear form on W. (see Example 2). If we denote the map $c \to \{abc\}$ by $l(a, b)$, the expression $l(a, b) - <a, b> id_w$ is symmetric in a and b. Since W has dimension two over K, the linear span of the endomorphims $l(a, b) - <a, b> id_w$ has dimension one over K. Hence it follows that

$$[l(a, b), l(c, d)] = 0 \qquad \text{for all} \quad a, b, c, d \in W.$$

Together with the definition of an anti-Jordan triple system, by straightforward calculations,
we have

(15) $$\{ab\{cde\}\} = \{\{abc\}de\}$$

(16) $$\{ab\{cde\}\} = -\{c\{bad\}e\}$$

(17) $$\{ab\{cde\}\}=\{cd\{abe\}\}.$$

Now we can prove the following:

PROPOSITION 3.1. *Let W be a two dimensional anti-Jordan triple system over K defined by* (∗) *and J be any Jordan triple system over K. Then the triple product on $W\otimes J$ defined by*

$$\{a\otimes x,\ b\otimes y,\ c\otimes z\}=\{abc\}\otimes\{xyz\}$$

for $a, b, c\in W$ and $x, y, z\in J$,
is an anti-Jordan triple product on $W\otimes J$.

PROOF. It is clear that

$$\{a\otimes x,\ b\otimes y,\ c\otimes z\}=-\{c\otimes z,\ b\otimes y,\ a\otimes x\}.$$

By the definition, $\{a\otimes x,\ b\otimes y,\ \{c\otimes z,\ d\otimes u,\ e\otimes v\}\}$

$$\begin{aligned}&=\{ab\{cde\}\}\otimes\{xy\{zuv\}\}\\&=\{ab\{cde\}\}\otimes(\{\{xyz\}uv\}-\{z\{yxu\}v\}+\{zu\{xyv\}\})\\&=\{ab\{cde\}\}\otimes\{\{xyz\}uv\}-\{ab\{cde\}\}\otimes\{z\{yxu\}v\}+\\&\quad\{ab\{cde\}\}\otimes\{zu\{xyv\}\}\end{aligned}$$

Hence, by using (15), (16) and (17) we have

$$\begin{aligned}&\{a\otimes x,\ b\otimes y,\ \{c\otimes z,\ d\otimes u,\ e\otimes v\}\}\\&=\{\{abc\}de\}\otimes\{\{xyz\}uv\}+\{c\{bad\}e\}\otimes\{\{z\{yxu\}v\}\\&\quad+\{cd\{abe\}\}\otimes\{zu\{xyv\}\}\\&=\{\{a\otimes x,\ b\otimes y,\ c\otimes z\},\ d\otimes u,\ e\otimes v\}+\{c\otimes z\{b\otimes y,\ a\otimes x,\ d\otimes u\}\\&\quad e\otimes v\}+\{c\otimes z,\ d\otimes u,\ \{a\otimes x,\ b\otimes y,\ e\otimes v\}\}.\end{aligned}$$

This completes the proof. ■

REMARK 3.2. Let W be a two dimensional Jordan triple system (resp. anti-) which is defined by

$$\{abc\}=<a,b>c+<b,c>a-<c,a>b$$

where $<a,b>=<b,a>$ (resp. $<a,b>=-<b,a>$). And let J be any Jordan triple system or anti-Jordan triple system. Then we can consider the four possibilities of tensor product $W\otimes J$ as follows;

	J: J. T. S.	J: *anti-*
W: J. T. S.	(A) $W\otimes J$: J. T. S.	(B) $W\otimes J$:*anti-*
W: *anti-*	(C) $W\otimes J$:*anti-*	(D) $W\otimes J$: J. T. S.

Giving an anti-Jordan triple system on $W\otimes J$ (in the case of (B) and (C)), one gets an anti-Lie triple system $(W\otimes J)^+$ on the vector space $W\otimes J$ by symmetrizing the first two variables in the anti-Jordan triple system. From the anti-Lie triple system one obtains a Lie superalgebra by taking the standard imbedding (see Section 2).

For the anti-Jordan triple system W, since

$$\{abc\}-<a, b>c=<b, c>a-<c, a>b \quad \text{and} \quad l(a, b)-<a, b>id_w$$

is symmetric in a, b, there are a symmetric bilinear form τ and an endomorphism S on W such that

$$l(a, b)=<a, b>id_w+\tau(a, b)S.$$

Let J be any Jordan triple system over K. Then it can be easily seen that

$$L(x, y)-L(y, x)\in Der\ J.$$

$$L(x, y)+L(y, x)\in Anti\text{-}Der\ J, \qquad \text{for} \quad x, y\in J.$$

For the left-multiplication in the anti-Lie triple system on $W\otimes J$, we get

$$\begin{aligned} &L(a\otimes x, b\otimes y)+L(b\otimes y, a\otimes x)\\ &\qquad =<a, b>id_w\otimes(L(x, y)-L(y, x))\\ &\qquad +\tau(a, b)S\otimes(L(x, y)+L(y, x)), \end{aligned}$$

which is contained in

$$id_w\otimes Der\ J+S\otimes Anti\text{-}Der\ J.$$

From the assumption that the form $<\ ,\ >$ on W is nondegenerate, there exists a basis $\{e_1, e_2\}$ of W such that

$$<e_1, e_1>=0, <e_2, e_2>=0 \quad \text{and} \quad <e_1, e_2>\neq 0.$$

Then we have

$$\tau(e_1, e_2)Se_1=<e_2, e_1>e_1 \text{ and } \tau(e_1, e_2)Se_2=-<e_2, e_1>e_2.$$

Hence it has been shown that the linear endomorphisms id_w and S are linearly independent. Therefore, by choosing an element $w\in W$ such that w and Sw form a basis of W, we can see that an element of

$$(id_w\otimes Der\ J)\cap(S\otimes Anti\text{-}Der\ J) \text{ is zero on } w\otimes J+Sw\otimes J.$$

Hence the sum

$$id_w \otimes Der\ J + S \otimes Anti\text{-}Der\ J \text{ is direct sum.}$$

Next we shall show that

$$id_w \otimes Der\ J \in Der\ (W \otimes J)^+$$
$$S \otimes Anti\text{-}Der\ J \in Der\ (W \otimes J)^+.$$

For $D \in Der\ J$, we have

$$\begin{aligned} &< a, b > id_w \otimes D \\ &= 1/2\{< a, b > id_w \otimes D + \tau(a, b) S \otimes D\} \\ &- 1/2\{< b, a > id_w \otimes D + \tau(b, a) S \otimes D\} \\ &= 1/2 l(a, b) \otimes D - 1/2 l(b, a) \otimes D \\ &= 1/2(l(a, b) - l(b, a)) \otimes D. \end{aligned}$$

By straightforward calculation, we can show that $(l(a, b) - l(b, a)) \otimes D$ is a derivation of the anti-Jordan triple system $W \otimes J$. Hence $id_w \otimes D$ is a derivation of the anti-Lie triple system on $W \otimes J$. Similarly, using $\tau(a, b)S = l(a, b) - < a, b > id_w$, we get $S \otimes Anti\text{-}Der\ J \in Der\ (W \otimes J)^+$. So we have the following,

THEOREM 3.3. *Under the assumption of Proposition* 3.1, *there exists a natural Lie superalgebra structure on*

(18) $$id_w \otimes Der\ J \oplus S \otimes Anti\text{-}Der\ J \oplus W \otimes J$$

such that this Lie superalgebra contains the standard imbedding of the anti-Lie triple system on $W \otimes J$.

REMARK 3.4. Theorem in [3], Remark 3.2 and Theorem 3.3 in this section suggest that simultaneous treatment for the four cases on $W \otimes J$ is natural.

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