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Projectivity of Left Loops on R^n

Dedicated to Professor Miyuki Yamada on his 60th birthday

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Left loops and their projective transformations are considered on analytic manifolds. It is shown that there exists a one-to-one correspondence between the isomorphism classes of the images of the abelian Lie group R^n under projective transformations of left loops and the isomorphism classes of real Lie algebras of dimension n (Theorem 1). For any left loop in projective relation with R^n, the correspondence between normal left subloops and ideals of the tangent Lie triple algebra is established (Theorem 2).

§1. Introduction

A set G with a multiplication $\mu: G\times G\to G$, denoted by $xy=\mu(x, y)$ for $x, y\in G$, will be called a *left loop* if it satisfies;

i) the multiplication μ has a (two-sided) unit e,

ii)$_L$ each left translation $L_x: G\to G$; $L_x y=xy$, is a bijection of G onto itself.

A *loop* is defined to be a left loop satisfying the additional condition;

ii)$_R$ each right translation is a bijection.

With each left loop (G, μ), a ternary operation $\eta: G\times G\times G\to G$ is associated by setting

$$\eta(x, y, z)=L_x\mu(L(x, y), L(x, z)) \tag{1.1}$$

for $x, y, z\in G$, where

$$L(x, y)=L_x^{-1}y.$$

This ternary operation satisfies the following equalities;

$$\eta(x, x, z)=z, \tag{1.2}$$

$$\eta(x, y, x)=y, \tag{1.3}$$

$$\eta(e, x, \eta(e, y, z))=\eta(x, \eta(e, x, y), \eta(e, x, z)) \tag{1.4}$$

for $x, y, z\in G$, where e is the unit. The multiplication μ is expressed by η as follows;

(1.5) $$\mu(x, y) = \eta(e, x, y).$$

Assume that (G, μ) is a left loop with the left inverse property (*left I. P. left loop*), that is, $L_x^{-1} = L_{x^{-1}}$ holds for every $x \in G$, where $x^{-1} = L_x^{-1}e$. Then the tenary operation η above satisfies the additional equality

(1.6) $$\eta(e, x, \eta(x, e, y)) = \eta(x, e, \eta(e, x, y)) = y.$$

Conversely, let $\eta: G \times G \times G \to G$ be a ternary operation on a set G satisfying the equalities (1.2), (1.3), (1.4) and (1.6) for some fixed element $e \in G$. Then, the multiplication μ on G given by (1.5) makes G a left I.P. left loop whose associated ternary operation is η itself. That is to say, every left I. P. left loop (G, μ) is uniquely determined by the ternary system (G, η) satisfying (1.2), (1.3), (1.4) and (1.6) under the interrelations (1.1) and (1.5). In investigating left loops, we shall often use the associated ternary systems instead of the muultiplications

A left I. P. left loop (G, μ) is said to be *homogeneous* if the associated ternary operation η is a homogeneous system in the sense of [1], i.e., if η satisfies the following equality (1.7) instead of (1.4):

(1.7) $$\eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)).$$

Note that this equality is equivalent to

(1.8) $$\eta(e, x, \eta(u, v, w)) = \eta(\eta(e, x, u), \eta(e, x, v), \eta(e, x, w)),$$

or to the condition

(H) Every left inner map $L_{x,y} = L_{xy}^{-1} L_x L_y$ is an automorphism of (G, μ).

In the preceding paper [3] we have introduced the concept of the canonical connection of analytic loops by means of the associated ternary operation satisfying (1.2), (1.3) and (1.4), and investigated the condition for an analytic geodesic loop to be changed for a homogeneous one without changing the unit and the syatem of geodesics (Theorem 3 [3]). Here, we note that all discussions in [3] on the canonical connections of analytic loops are available for our analytic left loops because it has not used in [3] that any right translation of the loops is bijective. This paper aims at determining all of the geodesic homogeneous left loops on $\boldsymbol{R}^n$ which is in projective relation with the abelian Lie group $(\boldsymbol{R}^n, +)$ by applying the results on analytic ternary operations in [3] (cf. Theorem 1).

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§2. Projective transformations of left loops

In the followings, we are concerned with analytic left loops. An *analytic* left loop

(G, μ) is a left loop on an analytic manifold G such that the following two mappings are analytic:

$$\mu\colon G\times G \longrightarrow G;\ \mu(x, y)=xy,$$
$$L\colon G\times G \longrightarrow G;\ L(x, y)=L_x^{-1}y.$$

REMARK. Analytic left loops are local loops around their units.

In the same manner as in [3] we introduce the canonical connections on analytic left loops as follows: Let (G, μ) be an analytic left loop. For any analytic vector fields X and Y on G, we set (cf. [2] and [3])

$$(\nabla_X Y)_x = X_x Y - \eta(x, X_x, Y_x), \quad x\in G, \tag{2.1}$$

where η is the ternary operation associated with (G, μ). We call the linear connection on G defined by (2.1) the *canonical connection* of the left loop (G, μ). The left loop (G, μ) is said to be *geodesic* if, for any $x\in G$, the geodesic local loop μ_x of the canonical connection centered at x satisfies

$$\mu_x(y, z) = \eta(x, y, z)$$

as far as μ_x is defined (cf. [3]). From the discussions and the results in §2 of [3] we can conclude that the following facts are valid for any geodesic left loop (G, μ): Let X_0 be a tangent vector to G at the unit e of (G, μ) and consider the analytic vector field X on G given by

$$X_x = \eta(e, x, X_0), \quad x\in G.$$

Then the integral curve $c(t)$, $t\in \boldsymbol{R}$, of the vector field X through $e=c(0)$ is a geodesic of the cannonical connection ∇ and it satisfies

$$L_{c(t+s)}^{-1}\, L_{c(t)}\, L_{c(s)} = 1_G \text{ (the identity map)} \tag{2.2}$$

for any $t, s\in \boldsymbol{R}$. This implies that any geodesic left loop has the left inverse property and that its canonical connection is always complete.

Now, let (G, μ) and $(\tilde G, \tilde\mu)$ be two analytic left loops on connected analytic manifolds G and $\tilde G$, respectively. Assume that these left loops are geodesic. Then, an analytic diffeomorphism ψ of G onto $\tilde G$ is said to be *geodesic preserving* if ψ sends every geodesic of the canonical connection ∇ of (G, μ) to a geodesic of the canonical connection $\tilde\nabla$ of $(\tilde G, \tilde\mu)$. If both of (G, μ) and $(\tilde G, \tilde\mu)$ are homogeneous, the geodesic preserving diffeomorphism ψ will be called a *projective transformation* of the left loops provided that ψ satisfies the followings:

(P.1) $\tilde\eta(\psi x, \psi y, \psi\eta(u, v, w))$

$$= \psi\eta(\psi^{-1}\tilde\eta(\psi x, \psi y, \psi u), \psi^{-1}\tilde\eta(\psi x, \psi y, \psi v), \psi^{-1}\tilde\eta(\psi x, \psi y, \psi w)),$$

(P.2) $$\psi\eta(x, y, \psi^{-1}\tilde{\eta}(\psi u, \psi v, \psi w)) = \tilde{\eta}(\psi\eta(x, y, u), \psi\eta(x, y, v), \psi\eta(x, y, w))$$

for any $x, y, u, v, w \in G$, where η (resp. $\tilde{\eta}$) is the homogeneous system associated with the homogeneous left loop (G, μ) (*resp.* $(\tilde{G}, \tilde{\mu})$). It is clear that any analytic isomorphism ψ of (G, μ) onto $(\tilde{G}, \tilde{\mu})$ is a projective transformation of left loops. For an analytic diffeomorphism ψ of G onto $\tilde{G}$, let ∇' be the linear connection on G induced from $\tilde{\nabla}$ under ψ, that is,

(2.3) $$\nabla'_X Y = \psi_*^{-1}\tilde{\nabla}_{\psi_* X}\psi_* Y$$

for any vector fields X and Y on G, where ψ_* denotes the differential of ψ.

Proposition 1. *Let (G, μ) and $(\tilde{G}, \tilde{\mu})$ be geodesic homogeneous left loops on connected analytic manifolds G and $\tilde{G}$, respectively, and ψ a projective transformation of (G, μ) onto $(\tilde{G}, \tilde{\mu})$. For the canonical connection ∇ of (G, μ) and the linear connection ∇' on G induced from the canonical connection of $(\tilde{G}, \tilde{\mu})$ under ψ, the* (1, 2)*-tensor fields $T = \nabla - \nabla'$ and $-T = \nabla' - \nabla$ on G are affine homogeneous structures* (cf. [3]) *of ∇ and ∇' respectively.*

Proof. Let (G, μ') be the left loop on G induced from $(\tilde{G}, \tilde{\mu})$ under ψ, i.e., $\mu' = \psi^{-1}\cdot\tilde{\mu}\cdot\psi\times\psi$. Then (G, μ') is a geodesic homogeneous left loop whose canonical connection is ∇'. The identity map of G is a projective transformation of (G, μ) onto (G, μ'), that is, the systems of geodesics of ∇ and ∇' are coincident and the associated homogenous systems η and η' of (G, μ) and (G, μ'), respectively, satisfy the following relations;

(2.4) $$\eta'(x, y, \eta(u, v, w)) = \eta(\eta'(x, y, u), \eta'(x, y, v), \eta'(x, y, w)),$$

(2.5) $$\eta(x, y, \eta'(u, v, w)) = \eta'(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w)).$$

The relation (2.4) shows that any displacement $\eta'(x, y)$ of η' is an automorphism of η. Hence, the torsion tensor S of ∇;

(2.6) $$S_x(X_x, Y_x) = \eta(x, X_x, Y_x) - \eta(x, Y_x, X_x), \quad x \in G,$$

(cf. [2]) satisfies the equation

(2.7) $$S_y(\eta'(x, y, X_x), \eta'(x, y, Y_x)) = \eta'(x, y, S_x(X_x, Y_x)), \quad x, y \in G,$$

for any vector fields X and Y on G. By Lemma in [2] we can see that the equation (2.7) implies $\nabla' S = 0$. In the same way, $\nabla S' = 0$ follows from (2.5), where S' denotes the torsion tensor of the connection ∇'. As noted in §1 we can apply Theorem 3 in [3] for geodesic left loops. Since (G, μ) and (G, μ') are assumed to be homogeneous, this theorem assures that $T = \nabla - \nabla'$ (resp. $-T = \nabla' - \nabla$) is an affine homogeneous structure of ∇ (resp. ∇') if and only if $\nabla' S = 0$ (resp. $\nabla S' = 0$). Thus the proof is completed. q.e.d.

Now, applying Theorem 5 in [3] to geodesic left loops, we have

COROLLARY. *Under the same assumption as in Proposition* 1, *the* (1, 2)-*tensor field* T *satisfies the equations* $\nabla T=0$, $\nabla' T=0$ *and*

$$T(X, X)=0 \tag{2.8}$$

$$T(X, S(Y, Z))=S(T(X, Y), Z)+S(Y, T(X, Z)) \tag{2.9}$$

$$T(X, R(Y, Z)W)=R(T(X, Y), Z)W+R(Y, T(X, Z))W + R(Y, Z)T(X, W) \tag{2.10}$$

$$T(X, T(Y, Z))=T(T(X, Y), Z)+T(Y, T(X, Z)) \tag{2.11}$$

$$R(X, Y)T(Z, W)=T(R(X, Y)Z, W)+T(Z, R(X, Y)W) \tag{2.12}$$

for any vector fields X, Y, Z, W *on* G.

§3. Projective relation in R^n

We restrict ourselves to investigating geodesic homogeneous left loops which are images of the abelian Lie group R^n under projective transformations of left loops. The isomorphism classes of such left loops are characterized by their tangent Lie triple algebras, all of which will be found in the sequel.

In what follows we denote the real affine n-space by $V=R^n$. The abelian Lie group $R^n=(V, +)$ can be regarded as a geodesic homogeneous (left) loop with the associated homogeneous system

$$\eta_0(x, y, z)=y-x+z \tag{3.1}$$

whose canonical connection ∇^0 is the natural flat connection of the affine space V with the torsion $S^0=0$ and the curvature $R^0=0$. Let (V, μ) be a geodesic homogeneous left loop on V. It will be called to be *in projective relation* with $(V, +)$ if the identity map of V is a projective transformation of $(V, +)$ onto (V, μ). Let $(G, \tilde{\mu})$ be a geodesic homogeneous left loop such that there exists a projective transformation of $(V, +)$ onto $(\tilde{G}, \tilde{\mu})$. Then, we can choose a unique geodesic homogeneous left loop (V, μ) on V with the unit $0\in V$ which is isomorphic to $(G, \tilde{\mu})$ under the projective transformation, say ψ. In fact, the multiplication μ is given on V by

$$\mu(x, y)=\psi^{-1}\tilde{\eta}(\tilde{0}, \psi x, \psi y), \quad x, y\in V, \tag{3.2}$$

where $\tilde{0}=\psi(0)$ and $\tilde{\eta}$ is the homogeneous system associated with $(G, \tilde{\mu})$. The left loop (V, μ) obtained above is in projective relation with $(V, +)$. Therefore, to find an isomorphism class of those geodesic homogeneous left loops which are images of $(V, +)$ under projective transformations of left loops, it is sufficient to find a left loop (V, μ) on V which is in projective relation with $(V, +)$ and which has the unit 0 as above.

Hereafter, we identify the vector space V with the tangent space of the affine space V at 0.

PROPOSITION 2. *Let (V, μ) be a geodesic homogeneous loop on V in projective relation with the abelian Lie group $\boldsymbol{R}^n=(V, +)$. Let ∇^0 denote the natural flat linear connection on $\boldsymbol{R}^n$. For the canonical connection ∇ of (V, μ), the* (1, 2)*-tensor field $T=\nabla^0-\nabla$ has the constant components (C^i_{jk}) with respect to the canonical coordinates in $\boldsymbol{R}^n$. The bracket operation given by $[x, y]=T_0(x, y)$ makes V a real Lie algebra $\mathfrak{L}_\mu=(V; T_0)$ with the structure constrants (C^i_{jk}), where T_0 denotes the skew-symmetric* (1, 2)*-tensor on the vector space V induced from T at the origin* 0.

PROOF. In Corollary to Proposition 1 we replace G, μ, μ' with V, $+$, μ, respectively. Then, the equation $\nabla^0 T=0$ means that the tensor field T has the constant components on $\boldsymbol{R}^n$. Since the torsion and the curvature of ∇^0 vanish on $\boldsymbol{R}^n$, the equalities (2.8)–(2.12) evaluated at the origin of $\boldsymbol{R}^n$ are reduced to

(3.3) $$T_0(x, x)=0$$

(3.4) $$T_0(x, T_0(y, z))=T_0(T_0(x, y), z)+T_0(y, T_0(x, z))$$

for $x, y, z\in V$. Hence the bracket $[x, y]=T_0(x, y)$ makes V a Lie algebra. q.e.d.

PROPOSITION 3. *Let (V, μ) and $(V, \tilde{\mu})$ be geodesic homogeneous left loops on V which are in projective relation with $\boldsymbol{R}^n=(V, +)$ and $\mathfrak{L}_\mu$ (resp. $\mathfrak{L}_{\tilde{\mu}}$) be the Lie algebra corresponding to (V, μ) (resp. $(V, \tilde{\mu})$) by* Proposition 2. *The left loops (V, μ) and $(V, \tilde{\mu})$ are isomorphic if and only if the corresponding Lie algebras $\mathfrak{L}_\mu$ and $\mathfrak{L}_{\tilde{\mu}}$ are isomorphic.*

PROOF. Let ∇ and $\tilde{\nabla}$ denote the canonical connections of (V, μ) and $(V, \tilde{\mu})$, respectively. Set $T=\nabla^0-\nabla$ and $\tilde{T}=\nabla^0-\tilde{\nabla}$. Denote $\mathfrak{G}_\mu$ (resp. $\mathfrak{G}_{\tilde{\mu}}$) the tangent Lie triple algebra of (V, μ) (resp. $(V, \tilde{\mu})$) with the bilinear operation xy (resp. $(xy)^\sim$) and the trilinear operation $[x\ y\ z]$ (resp. $[x\ y\ z]^\sim$). We apply Theorem 3 in [3] to these left loops. Then, the operations of the tangent Lie triple algebras are obtained respectively as follows;

(3.5) $$xy=2T_0(x, y), \quad [x\ y\ z]=-T_0(T_0(x, y), z)$$

(3.6) $$(xy)^\sim=2\tilde{T}_0(x, y), \quad [x\ y\ z]^\sim=-\tilde{T}_0(\tilde{T}_0(x, y), z).$$

By Theorem 2.2 in [1–II], the geodesic homogeneous left loops (V, μ) and $(V, \tilde{\mu})$ are isomorphic if and only if their tangent Lie triple algebras $\mathfrak{G}_\mu$ and $\mathfrak{G}_{\tilde{\mu}}$ are isomorphic. It follows from (3.5) and (3.6) that the tangent Lie triple algebras are isomorphic if and only if the corresponding Lie algebras are isomorphic. Thus the proof is completed.

q.e.d.

Let $\mathfrak{L}=(V; [\ ,\ ])$ be an n-dimensional real Lie algebra with the undelying vector space V. For any $x\in V$ we write $A(x)=\exp\ \mathrm{ad}\,x\in GL(V)$, where ad is the adjoint operation of the Lie algebra $\mathfrak{L}$.

PROPOSITION 4. *The binary operation* μ *on* V *given by*

$$\mu(x, y) = x + A(x)y \tag{3.7}$$

forms an analytic left loop with the unit 0. *The ternary operation associated with* (V, μ) *is given by*

$$\eta(x, y, z) = x + \mu(y - x, z - x). \tag{3.8}$$

Moreover, (V, μ) *is a geodesic homogeneous left loop which is in projective relation with the abelian Lie group* $\boldsymbol{R}^n = (V, +)$ *and the corresponding Lie algebra* $\mathfrak{L}_\mu$ *in* Proposition 2 *is coincident with the given Lie algebra* $\mathfrak{L}$.

PROOF. It is easy to check that (V, μ) is a left loop with the unit 0. We can also show that any geodesic of the canonical connection ∇ of (V, μ) is a straight line in V and that each left translation L_x of (V, μ) induces the parllel displacement of vectors along the straight line $c(t) = tx$, $t \in \boldsymbol{R}$, with respect to the connection ∇. Moreover, a direct calculation shows (3.8) for the associated ternary operation η. Since η satisfies (1.8) in §1, (V, μ) is homogeneous. In fact, left inner mappings of (V, μ) are given by

$$L_{x,y} = A(-\mu(x, y))A(x)A(y),$$

which are automorphisms of the Lie algebra $\mathfrak{L}$. This shows that every $L_{x,y}$ is an automorphism of the left loop (V, μ). Let η_0 be the homogeneous system of $(V, +)$ given by (3.1). Then, the equalities (2.4) and (2.5) for η and η_0 are easily checked. Thus, (V, μ) is in projective relation with $(V, +)$. Since

$$\eta(x, X_x, Y_x) = \operatorname{ad} X_x Y_x$$

holds for any $x \in V$ and $X_x, Y_x \in T_x(V)$ $(= V)$, the canonical connection ∇ of (V, μ) is given by

$$(\nabla_X Y)_x = X_x Y - [X_x, Y_x] \tag{3.9}$$

and the (1, 2)-tensor $T = \nabla^0 - \nabla$ has its value at the origin

$$T_0(X, Y) = [X, Y], \tag{3.10}$$

that is, $\mathfrak{L}_\mu = (V; T_0) = \mathfrak{L}$. q.e.d.

§4. Main theorems

Now we state our main theorems:

THEOREM 1. *There exists a one-to-one correspondence between the isomorphism* *classes of geodesic homogeneous left loops which are images of the abelian Lie group $\boldsymbol{R}^n$ under projective transformations of left loops and the isomorphism classes of real Lie algebras of dimension n.*

PROOF. Let V denote the affine space of dimension n. Assume that there exists a projective transformation ψ of the Lie group $\boldsymbol{R}^n=(V, +)$ onto a geodesic homogeneous left loop $(G, \tilde{\mu})$. Then, by Proposition 2, there corresponds an n-dimensional real Lie algebra $\mathfrak{L}_\mu$ to the left loop (V, μ) obtained from $(G, \tilde{\mu})$ by (3.2). Proposition 3 assures that this correspondence is one-to-one. Conversely, for any real Lie algebra $\mathfrak{L}$ of dimension n, we can construct a geodesic homogeneous left loop (V, μ) on V by (3.7). By Proposition 4, (V, μ) is in projective relation with $\boldsymbol{R}^n=(V, +)$ and the Lie algebra corresponding to (V, μ) by Proposition 2 is coincident with the given Lie algebra $\mathfrak{L}$. Therefore, the theorem is proved. q.e.d.

Now, we check the tangent Lie triple algebra $\mathfrak{G}$ of the geodesic homogeneous left loop (V, μ) considered in §3. From (3.5) and (3.10) we obtain the tangent Lie triple algebra $\mathfrak{G}=(V; xy, [x\ y\ z])$ as follows:

$$xy=2[x, y], \quad [x\ y\ z]=-[[x, y], z], \tag{4.1}$$

where $[\ ,\ ]$ denotes the Lie bracket of the Lie algebra $\mathfrak{L}_\mu=(V; [\ ,\ ])$.

PROPOSITION 5. *A linear subspace H of V (as the tangent space to V at the origin) is an ideal of the tangent Lie triple algebra of the geodesic homogeneous left loop (V, μ) if and only if $\mathfrak{G}=(H; [\ ,\ ])$ is an ideal of the Lie algebra $\mathfrak{L}_\mu=(V; [\ ,\ ])$.*

PROOF. By definition, a subsystem $\mathfrak{H}$ of the Lie triple algebra (=general Lie triple system) $\mathfrak{G}=(V; xy, [x\ y\ z])$ is an ideal of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{H}\subset\mathfrak{H}$ and $[\mathfrak{G}\ \mathfrak{H}\ \mathfrak{G}]\subset\mathfrak{H}$ (cf. [4]). Hence, if $\mathfrak{H}=(H; hg, [h\ g\ f])$ is an ideal of $\mathfrak{G}$, then by (4.1) we get $[x, h]=\frac{1}{2}xh\in H$ for any $x\in V$ and $h\in H$, i.e., $\mathfrak{H}=(H; [\ ,\ ])$ is an ideal of the Lie algebra $\mathfrak{L}_\mu=(V; [\ ,\ ])$. The converse is also shown by using (4.1). q.e.d.

A *normal subsystem* ([1–III]) of a homogeneous system (G, η) is a subsystem $(H, \eta|_H)$ satisfying

$$\eta(xH, yH, zH)=\eta(x, y, z)H$$

for any $x, y, z\in G$, where

$$xH=\eta(H, x, H), \quad x\in G.$$

Let (G, μ) be a homogeneous left loop with the associated homogeneous system (G, η). Aleft subloop $(H, \mu|_H)$ of (G, μ) is *normal* if the homogeneous system associated with $(H, \mu|_H)$ is a normal subsystem of the homogeneous system (G, η).

For normal left subloops of left loops on V in projective relation with $\boldsymbol{R}^n=(V, +)$, we have the following;

THEOREM 2. *Let* (V, μ) *be a geodesic homogeneous left loop on the affine space* $V = \boldsymbol{R}^n$ *with the tangent Lie triple algebra* $\mathfrak{G} = (V;\ xy,\ [x\ y\ z])$, *and let* $(H, \mu|_H)$ *be an analytic left subloop of* (V, μ). *Assume that* (V, μ) *is in projective relation with the abelian Lie group* $\boldsymbol{R}^n = (V, +)$. *Then,* $(H, \mu|_H)$ *is a normal left subloop of* (V, μ) *if and only if its tangent Lie triple algebra* $\mathfrak{H} = (H;\ hg,\ [h\ g\ f])$ *is an ideal of the Lie triple algebra* $\mathfrak{G}$.

PROOF. Since (V, μ) is assumed to be in projective relation with $\boldsymbol{R}^n = (V, +)$, without loss of generality, we can assume that (V, μ) is given by (3.7) for some n-dimensional real Lie algebra $\mathfrak{L} = (V; [\ ,\ ])$ (cf. Proof of Theorem 1). Then, any analytic left subloop $(H, \mu|_H)$ has an affine subspace H of V as its undelying submanifold. We identify H with its tangent space at the unit 0 (vector subspace of V). Suppose that $\mathfrak{H} = (H; hg, [h\,g\,f])$ is an ideal of the tangent Lie triple algebra $\mathfrak{G}$. By Proposition 5, $\mathfrak{H} = (H; [\ ,\ ])$ is an ideal of the Lie algebra $\mathfrak{L} = (V; [\ ,\ ])$. Since the associated homogeneous system (G, η) is given by (3.8), we get

$$\begin{aligned}\mu(x+H, y+H) &= \{x+u+A(x+u)\ (y+v) \mid u, v \in H\}\\ &= \mu(x, y)+H\end{aligned}$$

and

$$\begin{aligned}xH = \eta(H, x, H) &= \{x+A(x-u)\ (v-u) \mid u, v \in H\}\\ &= x+H.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\eta(xH, yH, zH) &= x+H+\mu(y-x+H, z-x+H)\\ &= \eta(x, y, z)+H\\ &= \eta(x, y, z)H,\end{aligned}$$

that is, $(H, \mu|_H)$ is a normal left subloop of (V, μ).

Conversely, let $(H, \mu|_H)$ be a nomal left subloop of (V, μ). Then, the homogeneous system $(H, \eta|_H)$ associated with $(H, \mu|_H)$ is a normal subsystem of (V, μ). Since H is closed in V, we can apply Theorem 3 in [1–III], which asserts that the tangent Lie triple algebra $\mathfrak{H} = (H; hg, [h\ g\ f])$ of $(H, \mu|_H)$ is an ideal of the Lie triple algebra $\mathfrak{G} = (V; xy, [x\ y\ z])$.

q.e.d.

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