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# $\mathscr{P}$-Cryptogroups

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A semigroup which is a band of groups is called a cryptogroup (see [4]). Let P be a C-set in a cryptogroup S. Then, S(P) is $\mathscr{P}$-regular (see [11]). In this case, we simply say that S(P) is a $\mathscr{P}$-cryptogroup. In this paper, the structure of $\mathscr{P}$-cryptogroups is investigated.

## §1. Introduction

Let $S$ be a regular semigroup, and $E_S$ the set of idempotents of $S$. Let $P$ be a subset of $E_S$ such that $P \cap L \neq \square$ and $P \cap R \neq \square$ for every $\mathscr{L}$-class $L$ and $\mathscr{R}$-class $R$ of $S$ (where $\mathscr{L}$ and $\mathscr{R}$ are Green's $L$-and $R$-relations respectively). If the pair $(S. P)$ of $S$ and $P$ satisfies

(C.1) (1) $P^2 \subset E_S$,

(2) $qPq \subset P$ for $q \in P$,

then we say that $S(P)$ is *weakly $\mathscr{P}$-regular*. If $(S, P)$ further satisfies

(3) for any $x \in S$, there exists $x^* \in V(x)$ (where $V(x)$ denotes the set of all inverses of $x$) such that $xP^1x^*$, $x^*P^1x \subset P$ (where $P^1$ is the adjunction of 1 to $P$),

then $S(P)$ is called *$\mathscr{P}$-regular*. In this case, $x^*$ above is called a *$\mathscr{P}$-inverse* of $x$, and the set of all $\mathscr{P}$-inverses of $x$ is denoted by $V_{\mathscr{P}}(x)$.

If $S(P)$ is $\mathscr{P}$-regular and if $V_{\mathscr{P}}(q) \subset P$ for every $q \in P$, then $S(P)$ is called *strongly $\mathscr{P}$-regular*.

In a regular semigroup $S$, a subset $P$ of $E_S$ is called *a full subset of $E_S$* if $P \cap L \neq \square$ and $P \cap R \neq \square$ for every $\mathscr{L}$-class $L$ and $\mathscr{R}$-class $R$ of $S$. Further, a full subset $P$ of $E_S$ is called *left* [*right*] *minimal* if $P \cap L$ [$P \cap R$] consists of a single element for every $\mathscr{L}$-class $L$ [$\mathscr{R}$-class $R$] of $S$. A full subset $P$ of $E_S$ is called a $C$-set in $S$ if it satisfies (1)–(3) of (C.1).

For example, if $S$ is a regular semigroup then $S(E_S)$ is $\mathscr{P}$-regular if and only if $S$ is orthodox. As another example, if $S$ is a regular semigroup with special involution # (that is, a regular $*$-semigroup having # as its special involution; see [8]) and if $Q$ is the set of all projections of $S$, then $S(Q)$ is $\mathscr{P}$-regulr and $Q$ is a both left and right minimal full subset of $E_S$. Conversely, if $S(Q)$ is a $\mathscr{P}$-regular semigroup and if $Q$ is a both left and right minimal full subset of $E_S$, then every element $x$ of $S$ has a unique $\mathscr{P}$-inverse $x^{\#}$, and $S$ becomes a regular $*$-semigroup having $Q$ as its projections under the special involution #

defined by "$x^{\sharp}$ = (the $\mathscr{P}$-inverse of $x$)". Hereafter, such a regular $*$-semigroup having $\sharp$ and $Q$ as its special involution and the projections respectively is denoted by $S(Q; \sharp)$. From the examples above, it is easy to see that the class of $\mathscr{P}$-regular semigroups contains both the class of orthodox semigroups and that of regular $*$-semigroups. The following shows a part of the connection between orthodox semigroups, inverse semigroups, regular $*$-semigroups and strongly $\mathscr{P}$-regular semigroups:

THEOREM 1.1. *Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Then*:

(1) *$P=E_S$ if and only if $P$ is closed with respect to the multiplication. Hence, in this case $S$ is orthodox.*

(2) *$S(P)$ is strongly $\mathscr{P}$-regular if and only if $pq \in P$ implies $qp \in P$ for every $p, q \in P$.*

(3) *$S(P)$ is a regular $*$-semigroup having $P$ as its projections if and only if $pq \in P$ implies $qp \in P$ and $pq=qp$ for $p, q \in P$.*

(4) *$S(P)$ is an inverse semigroup if and only if $pq=qp$ for $p, q \in P$.*

PROOF. (1) Obvious. (2): The "if" part: Let $p \in P$, and $q \in V_{\mathscr{P}}(P)$. Let $pq=u$ and $qp=v$. Then, $u, v \in P$. Since $uv \in E_S$, $uv\ \mathscr{R} u$ and $uv\ \mathscr{L}\ v$, we have $p=uv$. Similarly, $vu=q$. Since $uv \in P$, it follows that $vu \in P$. Hence, $q \in P$, that is, $S(P)$ is strongly $\mathscr{P}$-regular. The "only if" part: Let $qp \in P$ for $p, q \in P$. Then, every $\mathscr{P}$-inverse of $pq$ is contained in $P$. Hence, $qp \in P$ since $qp$ is a $\mathscr{P}$-inverse of $pq$.

(3): The "if" part: We need only to show that $P$ is a $p$-system (see [8]). Suppose that $p\ \mathscr{L}\ q$ for $p, q \in P$. Then, $pq=p \in P$. Therefore, $pq=qp$. Hence, $p=q$. Thus, each of $L \cap P$ and $R \cap P$ consists of a single element for every $\mathscr{L}$-class $L$ and $\mathscr{R}$-class $R$. This implies that $P$ is a $p$-system in $S$. (4): The "if" part: Let $p, q \in P$. Since $pqp \in P$, $pqp=ppq=pq \in P$. Therefore, $E_S=P^2 \subset P$, that is, $P=E_S$. Thus, $ef=fe$ for $e, f \in E_S$. That is, $S(P)$ is an inverse semigroup.

The "only if" part: For $p, q \in P$, $pq=pqp \in P$. Thus, $P^2 \subset P$, and hence $E_S=P$ by (1). Since $S(P)$ is an inverse semigroup, $pq=qp$ for $p, q \in E_S=P$.

Further, we have the following:

THEOREM 1.2. *Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Then, $S(P)$ is strongly $\mathscr{P}$-regular if and only if $p, q, h \in P$ and $q\ \mathscr{L}\ h\ \mathscr{R}\ p$ imply that there exists $u \in P$ such that $p\ \mathscr{L}\ u\ \mathscr{R}\ q$.*

PROOF. The "if" part: Let $p \in P$, and $p^*$ a $\mathscr{P}$-inverse of $p$. Let $pp^*=q$ and $p^*p=h$. Then, $q, h \in P$ and $q\ \mathscr{R}\ p\ \mathscr{L}\ h$. Hence, there exists $u \in P$ such that $q\ \mathscr{L}\ u\ \mathscr{R}\ h$. Now, $qh=p$ and $hq=u$. Since $p^*=hq=u \in P$, $S(P)$ is strongly $\mathscr{P}$-regular. The "only if" part: Let $p, q, h \in P$, and $q \mathscr{L}\ h \mathscr{R} p$. There exists $u \in V(h)$ such that $p\ \mathscr{L} u \mathscr{R} q$. Now, $hu=p$ and $uh=q$. Since $pq=h$ and $qp=u$ and since $pq \in P$, it follows that $qp \in P$. Then, $u \in P$.

The basic properties of a $\mathscr{P}$-regular semigroup and the structures of some special $\mathscr{P}$-regular semigroups have been studied in the previous papers [11] and [12]. A regular semigroup is called *a cryptogroup* if it is a band of groups (see [4]). In this paper, we shall investigate the structure of $\mathscr{P}$-regular cryptogroups (abbrev., $\mathscr{P}$-cryptogroups).

## §2. Fundamental properties

A completely regular semigroup $S$ is uniquely decomposed into a semilattice $\Lambda$ of completely simple subsemigroups $\{S_\lambda: \lambda\in\Lambda\}$. This decomposition is called *the structure decomposition* of $S$, and is denoted by $S\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$. In this case $\Lambda$ is unique up to isomorphism, and is called *the structure semilattice* of $S$.

It has been shown in [6] that an orthodox cryptogroup $S$ is isomorphic to the spined product (hence, of course a subdirect product) of $E_S$ and a Clifford semigroup $C$ (see [6]). That is, there exists a Clifford semigroup $C$ whose structure semilattice $\Lambda$ is the same as that of $E_S$, such that $S$ is isomorphic to the spined product $E_S \bowtie_\Lambda C$ of $E_S$ and $C$. That is, let $E_S\sim\Sigma\{E_\lambda: \lambda\in\Lambda\}$ and $C\sim\Sigma\{C_\lambda: \lambda\in\Lambda\}$ be the structure decompositions of $E_S$ and $C$. Then,

$E_S \bowtie_\Lambda C=\Sigma\{E_\lambda\times C_\lambda$ (direct product): $\lambda\in\Lambda\}$ (where $\Sigma$ means disjoint sum), and the multiplication is given by

$(e, a)\,(f, b)=(ef, ab)$,

and $S\cong E_S \bowtie_\Lambda C$.

It is obvious that any $\mathscr{P}$-regular semigroup is weakly $\mathscr{P}$-regular. Conversely,

Lemma 2.1. *For a cryptogroup $S$, $S(P)$ is $\mathscr{P}$-regular if and only if it is weakly $\mathscr{P}$-regular.*

Proof. The "only if" part is obvious. The "if" part: Let $S(P)$ be a band $\Lambda$ of groups $\{G_\lambda: \lambda\in\Lambda\}$. Of course, each $G_\lambda$ is an $\mathscr{H}$-class (where $\mathscr{H}$ denotes Green's $H$-relation) of $S(P)$. Let $e_\lambda$ be the identity of $G_\lambda$. Let $x\in H_{e_\lambda}$ (the $\mathscr{H}$-class containing $e_\lambda$; hence $H_{e_\lambda}=G_\lambda$). Then, there exist $p$, $q$ such that $pq=e_\lambda$. There exists $x^*\in V(x)\cap H_{qp}$. Now, $xx^*=p$ and $x^*x=q$. For any $h\in P$, $(xhx^*)^2=xhx^*$. There exist $G_\tau$, $G_\delta$ such that $h\in G_\tau$ and $x^*\in G_\delta$. Then, $xhx^*\in G_{\lambda\tau\delta}$, and $pqhqp\in G_{\lambda\tau\delta}$. Hence, $xhx^*=pqhqp\in P$. Similarly, $x^*hx\in P$. Thus, $x^*\in V_\mathscr{P}(x)$. Therefore, $S(P)$ is $\mathscr{P}$-regular.

Thus, for cryptogroups, weakly $\mathscr{P}$-regularity and $\mathscr{P}$-regularity are just the same. It is well-known that a regular semigroup is an inverse semigroup if and only if every element has a unique inverse. Similarly, the following is interest as a characterization of a regular *-semigroup:

Theorem 2.2. *A $\mathscr{P}$-regular semigroup $S(P)$ is a regular $*$-semigroup $S(P;\sharp)$ if and only if every element $x$ of $S(P)$ has a unique $\mathscr{P}$-inverse.*

Proof. The "only if" part: Suppose that $S(P)$ is a regular $*$-semigroup $S(P;\sharp)$. Then, it is easy to see that $x^\sharp$ is a unique $\mathscr{P}$-inverse of $x$ for any element $x\in S(P)$ (see [8]). The "if" part: Assume that every element $x$ of the $\mathscr{P}$-regular semigroup $S(P)$ has a unique $\mathscr{P}$-inverse $x^\sharp$. Suppose that a certain $\mathscr{L}$-class $L$ contains two different elements $p$, $q$ of $P$. Since $pq=p$ and $qp=q$, we have $pPq=pqPqp\subset pPp\subset P$ and $qPp=qpPpq\subset qPq\subset P$. Since $q\in V(P)$, $q$ is a $\mathscr{P}$-inverse of $p$, and hence $p=q$. This is a contradiction. Thus, each $\mathscr{L}$-class contains a unique element of $P$. Similarly, each $\mathscr{R}$-class contains a unique element of $P$. Therefore, $P$ is a both left and right minimal full subset of $E_S$, and accordingly $S(P)$ becomes a regular $*$-semigroup $S(P;\ \sharp)$.

## §3. Completely simple $\mathscr{P}$-regular semigroups

First, we have:

THEOREM 3.1. *Let B be a rectangular band, and P a full subset of B. Then, P is a C-set in B, and accordingly B(P) is $\mathscr{P}$-regular.*

PROOF. Since $qPq=\{q\}\subset P$ for $q\in P$, $B(P)$ is weakly $\mathscr{P}$-regular. Since $B$ is of course a crypptogroup, it is also $\mathscr{P}$-regular.

COROLLARY. *if B is a square band (see [8]), and P a both left and right minimal full subset of B. Then, B(P) is $\mathscr{P}$-regular, and it becomes a regular $*$-semigroup $B(P;\sharp)$ under the special involution $\sharp$ defined by $x^\sharp=$(the $\mathscr{P}$-inverse of $x$).*

Next, we shall investigate the completely simple (weakly) $\mathscr{P}$-regular semigroups. Let $S$ be a completely simple semigroup. Then we can assume that $S$ is a Rees $I\times J$ matrix semigroup over a group $G$ with sandwich matrix $Q$; that is, $S=M(G; I, J; Q)$ (see [1]). Let $Q=[p_{ji}]$ $(j\in J,\ i\in I)$.

LEMMA 3.2. *For a completely simple semigroup $S=M(G;\ I,\ J;\ Q)$ and for idempotents $[p_{ji}^{-1}]_{ij}$, $[p_{sk}^{-1}]_{ks}$, the following (1), (2) are equivalent:*

(1) $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}\in E_S$ and $[p_{sk}^{-1}]_{ks}[p_{ji}^{-1}]_{ij}\in E_S$.

(2) $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}[p_{ji}^{-1}]_{ij}=[p_{ji}^{-1}]_{ij}$.

PROOF. (1)⇒(2): It is obvious that $[p_{sk}^{-1}]_{ks}[p_{ji}^{-1}]_{ij}$ is an inverse of $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}$. Hence, $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}[p_{sk}^{-1}]_{ks}[p_{ji}^{-1}]_{ij}=[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}[p_{ji}^{-1}]_{ij}\in E_S$. Then, we have $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}[p_{ji}^{-1}]_{ij}=[p_{ji}^{-1}]_{ij}$. (2)⇒(1): Obvious.

By the result above, we have:

LEMMA 3.3. *Let S be a completely simple semigroup, and P a full subset of $E_S$. Then, the following* (1) *and* (2) *are equivalent*:

(1) $P^2\subset E_S$.

(2) *For any* $q\in P$, $qPq=\{q\}$.

*Further, $S(P)$ is $\mathscr{P}$-regular if and only if it satisfies one of* (1) *and* (2).

PROOF. The first part follows from Lemma 3.2. It is obvious that if $S(P)$ is $\mathscr{P}$-regular then $P$ satisfies (1) and (2). Conversely, suppose that $P$ satisfies (1) or (2). Then, $S(P)$ is weakly $\mathscr{P}$-regular. Since $S$ is a cryptogroup, $S(P)$ is $\mathscr{P}$-regular.

Suppose that $P$ is a $C$-set in $S=M(G;\ I,\ J;\ Q)$. Let $T=\{(i,j)\in I\times J\colon [p_{ji}^{-1}]_{ij}\in P\}$. Then, of course

(C.3) (1) for any $i\in I$, there exists $j\in J$ such that $(i, j)\in T$, and

(2) for any $j\in J$, there exists $i\in I$ such that $(i, j)\in T$.

Since $P$ is a $C$-set, $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}=[p_{si}^{-1}]_{is}$ for $(i,j)$, $(k, s)\in T$. Hence, $p_{ji}^{-1}\,p_{jk}^{-1}\,p_{sk}^{-1}=p_{si}^{-1}$, that is, $p_{ji}^{-1}\,p_{jk}=p_{si}^{-1}\,p_{sk}$.

Thus, $Q=[p_{uv}]$ satisfies the following:

(C.4) $p_{ji}^{-1}\,p_{jk}=p_{si}^{-1}\,p_{sk}$ for any $(i, j)$, $(k, s)\in T$.

Conversely, suppose that $T$ is a subset of $I\times J$ such that it satisfies (C.3). In this case, if $Q=[P_{uv}]$ satisfies (C.4), then $S(P)$ is weakly $\mathscr{P}$-regular, and hence $\mathscr{P}$-regular, with respect to $P=\{[p_{ji}^{-1}]_{ij}\colon (i, j)\in T\}$.

First, it is obvious that $P$ is a full subset of $E_S$. For any $[p_{ji}^{-1}]_{ij}$, $[p_{sk}^{-1}]_{ks}\in P$, $[p_{ji}^{-1}]_{ij}[p_{sk}^{-1}]_{ks}=[p_{ji}^{-1}\,p_{jk}^{-1}\,p_{sk}^{-1}]_{is}=[p_{si}^{-1}]_{is}$ (by (C.4))$\in E_S$. Hence, it follows from Lemma 3.3 that $S(P)$ is weakly $\mathscr{P}$-regular, and accordingly $\mathscr{P}$-regular. Thus, we have:

THEOREM 3.4. *Let $S=M(G; I, J; Q)$ be a completely simple semigroup, and $Q=[p_{uv}]$. Let $T$ be a subset of $I\times J$ such that*

(1) *$T$ satisfies (C.3), and*

(2) *$P=\{[p_{ji}^{-1}]_{ij}\colon (i, j)\in T\}$ satisfies (C.4),*

*then $S(P)$ is $\mathscr{P}$-regular. Further, every completely simple $\mathscr{P}$-regular semigroup is constructed in this fashion.*

Let $S(P)$ be a $\mathscr{P}$-regular semigroup. Let $T$ be a regular subsemigroup of $S$, and put $U=P\cap T$. Then, $T(U)$ is called *a $\mathscr{P}$-regular subsemigroup* of $S(P)$ if $T(U)$ is $\mathscr{P}$-regular. Let $S_1(P_1)$ and $S_2(P_2)$ be $\mathscr{P}$-regular semigroups, and $f\colon S_1(P_1)\to S_2(P_2)$ a homomorphism. Then, $f$ is called *a $\mathscr{P}$-homomorphism* if $P_1f=S_1f\cap P_2$. Let $S(P)$ be a $\mathscr{P}$-regular semigroup, and $\tau$ a congruence on $S(P)$. Let $x\tau=\bar{x}$ for $x\in S$, and $\bar{X}=\{\bar{x}\colon x\in X\}$ for a subset $X$ of $S$. Then, $\bar{S}(\bar{P})$ is a $\mathscr{P}$-regular semigroup, which we call *the factor $\mathscr{P}$-regular semigroup of $S(P)$ mod $\tau$* and denote by $S(P)/(\tau)_{\mathscr{P}}$. Hereafter, this congruence $\tau$ is especially called *a $\mathscr{P}$-congruence*. Hence, a congruence and a $\mathscr{P}$-congruence are essentially the same. A bijective $\mathscr{P}$-homomorphism is called a *$\mathscr{P}$-isomorphism*. Hereafter, a $\mathscr{P}$-regular band is simply called *a $\mathscr{P}$-band*. Let $E(P)$, $S(Q)$ be a rectangular $\mathscr{P}$-band and a completely simple $\mathscr{P}$-regular semigroup, and $E(P)\times S(Q)=T(U)$ the direct product of $E(P)$ and $S(Q)$, where $U=\{(p, q)\colon p\in P$ and $q\in Q\}$. Then, $T(U)$ is $\mathscr{P}$-regular. This $T(U)$ is called *a $\mathscr{P}$-direct product* of $E(P)$ and $S(Q)$ (for the general case, see §5). Let $V$ be a subdirect product of $E$ and $S$. Let $(e, x)\in V$. Then, there exists $(f, x^{-1})\in V$, where $x^{-1}$ is the group inverse of $x$ and $f\in E$. Then $(e, x)\,(f, x^{-1})=(ef, h)$, where $h=xx^{-1}$. Similarly, $(f, x^{-1})\,(e, x)=(fe, h)$. Hence, $(ef, h)\,(f, x^{-1})\,(fe, h)=(e, x^{-1})\in V$. Hence, $V$ is a completely simple semigroup. Let $K=\{(p, q)\in V\colon p\in P$ and $q\in Q\}$. If $V(K)$ is $\mathscr{P}$-regular, then $V(K)$ is a $\mathscr{P}$-regular subsemigroup of $E(P)\underset{\mathscr{P}}{\times}S(Q)=T(U)$, where $E(P)\underset{\mathscr{P}}{\times}S(Q)$ denotes the $\mathscr{P}$-direct product of $E(P)$ and $S(Q)$. This $V(K)$

is called *a $\mathscr{P}$-subdirect product* of $E(P)$ and $S(Q)$.

We shall show later the following: Any completely simple $\mathscr{P}$-regular semigroup $S(U)$ is $\mathscr{P}$-isomorphic to a $\mathscr{P}$-subdirect product of a rectangular $\mathscr{P}$-band $E(P)$ and a completely simple $*$-semigroup $T(Q; \sharp)$. Conversely, a $\mathscr{P}$-subdirect product $S(U)$ of a rectangular $\mathscr{P}$-band $E(P)$ and a completely simple $*$-semigroup $T(Q; \sharp)$ is a completely simple $\mathscr{P}$-regular semigroup.

EXAMPLES. 1. Let $S=M(G; I, J; Q)$ be a completely simple semigroup such that $Q=[p_{ji}]$, where $p_{ji}=1$ for all $(j, i)\in J\times I$. Then, $E_S=\{[1]_{ij}: (i, j)\in I\times J\}$. Let $T$ be a subset of $I\times J$, and assume that $T$ satisfies (C.3). Then, $P=\{[1]_{ij}: (i,j)\in T\}$ is a $C$-set in $S$, and $S(P)$ is $\mathscr{P}$-regular. In particular, $S(E_S)$ is $\mathscr{P}$-regular and is orthodox.

2. Let $S$ be a completely simple semigroup: $S=M(G; I, J; Q)$. Let $T$ be a subset of $I\times J$, and assume that $T$ satisfies (C.3). Further, assume that $Q=[p_{uv}]$ satisfies $p_{ji}=1$ for $(i, j)\in T$ and $p_{si}=p_{jk}^{-1}$ for $(i, j)$, $(k, s)\in T$. Put $P=\{[1]_{ij}: (i, j)\in T\}$. Then, $S(P)$ is $\mathscr{P}$-regular. In particular, consider the case where $I=J$ and $p_{ii}=1$ for all $(i, i)\in I\times I$ and $p_{it}=p_{ti}^{-1}$ for all $(i, t)\in I\times I$. Let $T=I\times I$, and $P=\{[1]_{ii}: (i, i)\in T\}$. Then, $T$ satisfies (C.3) and $S(P)$ is $\mathscr{P}$-regular. In fact, in this case $S(P)$ is a regular $*$-semigroup $S(P; \sharp)$. Further, it has been shown in [5] that every completely simple regular $*$-semigroup is constructed in this fashion.

## §4. $\mathscr{P}$-Bands

Let $B$ be a band, and $B\sim\Sigma\{B_\lambda: \lambda\in\Lambda\}$ the structure decomposition of $B$. Let $P\subset B$. If $B(P)$ is $\mathscr{P}$-regular, then $B_\lambda(P_\lambda)$, where $P_\lambda=B_\lambda\cap P$, is also $\mathscr{P}$-regular, that is, $P_\lambda$ is a full subset of $B_\lambda$. Conversely, let $P_\lambda$ be a full subset of $B_\lambda$ for all $\lambda\in\Lambda$. Then, $B_\lambda(P_\lambda)$ is $\mathscr{P}$-regular, but $P=\Sigma\{P_\lambda: \lambda\in\Lambda\}$ is not necessarily a $C$-set in $B$, and hence $B(P)$ is not necessarily $\mathscr{P}$-regular. However, we can construct the least $C$-set $Q_p$ containing $P$ as follows:

Let $p_1, p_2, \dots, p_n\in P$, and consider the element $p_1p_2\cdots p_{n-1}p_np_{n-1}\cdots p_2p_1$. Let $Q_p$ be all these elements, that is, $Q_p=\{p_1p_2\cdots p_{n-1}p_np_{n-1}\cdots p_2p_1$ ($n$ arbitrary): $p_i\in P$ for all $i=1, 2, \dots, n\}$. Then, clearly $qQ_pq\subset Q_p$ for any $q\in Q_p$. *Hence*, $Q_p$ is a $C$-set in $B$ and $Q_p\supset P$. It is obvious that any $C$-set (in $B$) containing $P$ contains $Q_p$. Therefore, $Q_p$ is the least $C$-set containing $P$. Of course, if $P$ itself is a $C$-set in $B$, then $Q_p=P$. Hence, we have:

THEOREM 4.1. *Let $B$ be a band, and $P$ a full subset of $B$.*
*Let $Q_p=\{p_1p_2\cdots p_{n-1}p_np_{n-1}\cdots p_2p_1$ ($n$ arbitrary): $p_i\in P$ for $i=1, 2, \dots, n\}$. Then, $Q_p$ is the least $C$-set containing $P$, and $B(Q_p)$ is $\mathscr{P}$-regular. Further, every $\mathscr{P}$-band is constructed in this fashion.*

Consider special kinds of bands, in particular the class of regular bands and that of normal bands. Let $B$ be a regular band, and define multiplication $\circ$ in $B$ as follows:

(C.5) $\qquad a\circ b=aba \qquad$ for $\quad a, b\in B$.

Then, $B(\circ)$ is also a band. Let $P$ be a full subset of $B$ (not of $B(\circ)$). Then, it is easy to see that $pPp \subset P$ if and only if $P(\circ)$ is a subband of $B(\circ)$. Hence, $P$ is a $C$-set in $B$ if and only if $P(\circ)$ is a subband of $B(\circ)$.

Therefore, we have:

THEOREM 4.2. *Let $B$ be a regular band, and $P$ a full subset of $B$. Then, $P$ is a $C$-set in $B$ if and only if $P(\circ)$ is a subband of $B(\circ)$.*

Next, let $B$ be a normal band. It is well-known that $B$ is a strong semilattice $\Lambda$ of rectangular bands $\{B_\lambda: \lambda \in \Lambda\}$. That is, there exists a transitive system $\{\phi^\alpha_\beta: \alpha \geqslant \beta, \alpha, \beta \in \Lambda\}$ of homomorphisms $\phi^\alpha_\beta: B_\alpha \to B_\beta$ such that the product of $a \in B_\lambda$ and $b \in B_\delta$ is given by $ab = (a\phi^\lambda_{\lambda\delta})(b\phi^\delta_{\lambda\delta})$ (see [10]). In this case, denote $B$ by $B = \mathscr{S}(B_\lambda; \Lambda; \phi^\alpha_\beta)$. Then we have:

THEOREM 4.3. *Let $P$ be a full subset of a normal band $B = \mathscr{S}(B_\lambda; \Lambda; \phi^\alpha_\beta)$. Let $P \cap B_\lambda = P_\lambda$ for each $\lambda \in \Lambda$. Then, $B(P)$ is $\mathscr{P}$-regular if and only if $P_\alpha \phi^\alpha_\beta \subset P_\beta$ for $\alpha, \beta \in \Lambda$ with $\alpha \geqslant \beta$.*

PROOF. The "if" part: Obvious. The "only if" part: Let $p \in P_\alpha$ and $\alpha \geqslant \beta$. Since $B(P)$ is $\mathscr{P}$-regular, $pP_\beta p \subset P_\beta$. Hence, $pqp = (p\phi^\alpha_\beta)q(p\phi^\alpha_\beta) = p\phi^\alpha_\beta \subset P_\beta$ for $q \in P_\beta$.

## §5. $\mathscr{P}$-Cryptogroups

Let $S(P)$ and $V(Q)$ be $\mathscr{P}$-regular semigroups. Consider the direct product $W$ of $S$ and $V$; that is, $W = S \times V$. Let $K = \{(p, q) \in S \times V: p \in P$ and $q \in Q\}$. Then, $W(K)$ is $\mathscr{P}$-regular. This $W(K)$ is called *the $\mathscr{P}$-direct product* of $S(P)$ and $V(Q)$, and denoted by $S(P) \times_{\mathscr{P}} V(Q)$. Let $T(P_T)$ be a $\mathscr{P}$-regular subsemigroup of $W(K) = S(P) \times_{\mathscr{P}} V(Q)$, where $P_T = T \cap K$. If the first and second projections of $T(P_T)$ to $S(P)$ and $V(Q)$ are surjective $\mathscr{P}$-homomorphisms, then $T(P_T)$ is called *a $\mathscr{P}$-subdirect product* of $S(P)$ and $V(Q)$. Now, we consider the special case where $S(P)$ and $V(Q)$ are $\mathscr{P}$-cryptogroups.

Let $A(P)$ and $B(Q)$ be $\mathscr{P}$-cryptogroups, and $A \sim \Sigma\{A_\lambda: \lambda \in \Lambda\}$ and $B \sim \Sigma\{B_\lambda: \lambda \in \Lambda\}$ be the structure decompositions of $A$ and $B$ respectively, and put $P_\lambda = P \cap A_\lambda$ and $Q_\lambda = Q \cap B_\lambda$ for $\lambda \in \Lambda$ (we assume that $A$ and $B$ have the same structure semilattice $\Lambda$). Then, each $A_\lambda(P_\lambda)$ $[B_\lambda(Q_\lambda)]$ is $\mathscr{P}$-regular. Let $S(U) = \Sigma\{A_\lambda(P_\lambda) \times_{\mathscr{P}} B_\lambda(Q_\lambda): \lambda \in \Lambda\}$, where $U = \Sigma\{P_\lambda \times Q_\lambda$ (cartesian product): $\lambda \in \Lambda\}$. Then, of course $S(U)$ is also a cryptogroup under the multiplication $(a, b)(c, d) = (ac, bd)$. Now, let $(e, f) \in P_\lambda \times Q_\lambda$ and $(h, t) \in P_\delta \times Q_\delta$. Then, it is easy to see that $(e, f)(h, t) \in E_S$ and $(e, f)(h, t)(e, f) \in U$. Hence, $S(U)$ is weakly $\mathscr{P}$-regular, and accordingly $\mathscr{P}$-regular. This $S(U)$ is called *$\mathscr{P}$-spined product* of $A(P)$ and $B(Q)$, and denoted by $A(P) \bowtie^{\mathscr{P}}_{\Lambda} B(Q)$. Now, let $T(V)$ be a $\mathscr{P}$-regular subsemigroup of $A(P) \bowtie^{\mathscr{P}}_{\Lambda} B(Q)$ such that

(C.6) (1) the first and second projections of $S(U) = A(P) \bowtie^{\mathscr{P}}_{\Lambda} B(Q)$ are surjective $\mathscr{P}$-homomorphisms of $T(V)$ onto $A(P)$ and $B(Q)$ respectively, and

(2) $(a, b) \in T(V)$ implies $(a^{-1}, b^{-1}) \in T(V)$, where $a^{-1}, b^{-1}$ are the group inverses of $a$, $b$ respectively,

then $T(V)$ is called *a $\mathscr{P}$-subspined product* of $A(P)$ and $B(Q)$, and denoted by $A(P)\underset{\Lambda}{\overset{\mathscr{P}}{\ltimes\!\!\rtimes}}B(Q)$, etc.

Now, let $S(P)$ be a $\mathscr{P}$-cryptogroup, and $S\sim\Sigma\{S_\lambda: \lambda\in\Lambda\}$ the structure decomposition of $S$. Let $S_\lambda\cap P=P_\lambda$ for each $\lambda\in\Lambda$. Then, $S_\lambda(P_\lambda)$ is a completely simple $\mathscr{P}$-regular semigroup. Now, $S(P)$ is a band of groups $\{G_\gamma: \gamma\in\Gamma\}$, where $\Gamma$ is a band and each $G_\gamma$ is an $\mathscr{H}$-class (where $\mathscr{H}$ is Green's $H$-relation). Let $\Gamma\sim\Sigma\{\Gamma_\lambda: \lambda\in\Lambda\}$ the structure decomposition of $\Gamma$. Let $\nu$ be the least strong $\mathscr{P}$-congruence on $S(P)$. This is given as follows (see [11]): Let $\nu$ be the transitive closure of the relation $\nu^\circ$ defined by $\nu^\circ=\{(a, b)\in S\times S: V_{\mathscr{P}}(a)\cap V_{\mathscr{P}}(b)\neq\square\}$. Then, it follows from [12] that $\nu$ is the least strong $\mathscr{P}$-congruence on $S(P)$ which makes $S(P)$ to a regular $*$-semigroup $S(P: \sharp)=S(P)/(\nu)_{\mathscr{P}}$, where $x\nu=\tilde{x}$ and $\tilde{X}=\{\tilde{x}: x\in X\}$ for any subset $X$ of $S(P)$. Now, $\tilde{S}(\tilde{P})=\cup\{\tilde{G}_\gamma: \gamma\in\Gamma\}$. Further, it follows from [11] that $x\ \nu\ y$ implies $x, y\in S_\lambda$ for some $\lambda\in\Lambda$. Since a homomorphic image of a completely simple semigroup is completely simple (see [3]), $S_\lambda/\nu$ is completely simple. Therefore, $\tilde{S}(\tilde{P})$ has the structure decomposition $\tilde{S}(\tilde{P})\sim\Sigma\{\tilde{S}_\lambda(\tilde{P}_\lambda): \lambda\in\Lambda\}$, and each $\tilde{S}_\lambda(\tilde{P})$ is a completely simple $*$-semigroup $\tilde{S}(\tilde{P}_\lambda; \sharp)$. Since $\tilde{S}(\tilde{P})=\cup\{\tilde{G}_\gamma: \gamma\in\Gamma\}$, $\tilde{S}(\tilde{P})$ is also a band of groups. Hence, $\tilde{S}(\tilde{P}; \sharp)$ is a $*$-cryptogroup (that is, a regular $*$-semigroup which is a cryptogroup). Next, define $\rho$ on $S(P)$ as follows: $x\ \rho\ y$ if and only if $x, y\in G_\gamma$ for some $\gamma\in\Gamma$. Let $e_\gamma$ be the identity of $G_\gamma$. Let $x\rho=\bar{x}$ and $\bar{X}=\{\bar{x}: x\in X\}$ for $X\subset S(P)$. Then, it is easy to see that $\bar{S}(\bar{P})=S(P)/(\rho)_{\mathscr{P}}$ is a $\mathscr{P}$-band, and $\bar{e}_\gamma\bar{e}_\delta=\bar{e}_{\gamma\delta}$. Hence, $\bar{S}(\bar{P})=\{\bar{e}_\gamma: \gamma\in\Gamma\}$ is isomorphic to $\Gamma$. Now, let $x, y\in S_\lambda(P_\lambda)$ and assume that $x(\rho\cap\nu)y$. Then, $x, y\in G_\delta$ for some $\delta\in\Gamma$. Since $xy^{-1}\nu\ yy^{-1}$, we have $xy^{-1}=e_\delta$, and hence $x=y$. Therefore, $f: S(P)\rightarrow\bar{S}(\bar{P})\underset{\Lambda}{\overset{\mathscr{P}}{\ltimes\!\!\rtimes}}\tilde{S}(\tilde{P}; \sharp)$ defined by $xf=(\bar{x}, \tilde{x})$ is a $\mathscr{P}$-isomorphism of $S(P)$ to $S(P)f=\{(\bar{x}, \tilde{x}): x\in S(P)\}\subset\bar{S}(\bar{P})\underset{\Lambda}{\overset{\mathscr{P}}{\ltimes\!\!\rtimes}}\tilde{S}(\tilde{P}; \sharp)$. Let $S(P)f=T(Q)$, where $Q=\{(\bar{p}, \tilde{p}): p\in P\}$. Then, it is easy to see that $T(Q)$ is a $\mathscr{P}$-regular subsemigroup of $\bar{S}(\bar{P})\underset{\Lambda}{\overset{\mathscr{P}}{\ltimes\!\!\rtimes}}\tilde{S}(\tilde{P}; \sharp)$ and is a $\mathscr{P}$-subspined product of $\bar{S}(\bar{P})$ and $\tilde{S}(\tilde{P}; \sharp)$. Conversely, let $E(P)$ be a $\mathscr{P}$-band, and $T(Q; \sharp)$ a $*$-cryptogroup. Then, $T(Q; \sharp)$ is a band $\Gamma$ of groups $\{T_\gamma: \gamma\in\Gamma\}$. Assume that $E(P)$ and $T(Q; \sharp)$ have the same structure semilattice $\Lambda$, and $E\sim\Sigma\{E_\lambda: \lambda\in\Lambda\}$ and $T\sim\Sigma\{T_\lambda: \lambda\in\Lambda\}$ the structure decompositions of $E$ and $T$ respectively, and put $P_\lambda=E_\lambda\cap P$ and $Q_\lambda=Q\cap T_\lambda$ for each $\lambda\in\Lambda$. Let $S(U)$ be a $\mathscr{P}$-subspined product of $E(P)$ and $T(Q; \sharp)$; that is, $S(U)=E(P)\underset{\Lambda}{\overset{\mathscr{P}}{\ltimes\!\!\rtimes}}T(Q; \sharp)$. Then, $S(U)$ is of course a $\mathscr{P}$-regular semigroup. For any $e\in E(P)$, there exists $a\in S(U)$ such that $(e, x)=a$ for some $x\in T_\gamma$. Now, let $S_{e,\gamma}=\{(e, x)\in E\times T_\gamma: (e, x)\in S(U)\}$. Let $(e, x), (e, y)\in S_{e,\gamma}$. Then, $(e, x)(e, y)=(e, xy)\in S(U)$. Hence, $(e, x)(e, y)\in S_{e,\gamma}$. Further, $e, x$ have group inverses $e^{-1}=e$ and $x^{-1}$ in $E$ and $T_\gamma$ respectively. Therefore, $(e, x^{-1})\in S(U)\cap S_{e,\gamma}$. Thus, $S_{e,\gamma}$ is a group. Hence, $S(U)=\Sigma\{S_{e,\gamma}: e\in E$ and $\gamma\in\Gamma\}$ such that $S_{e,\gamma}\neq\square$. Further, for $(e, a)\in S_{e,\gamma}$ and $(f, b)\in S_{f,\delta}$, $(e, a)(f, b)=(ef, ab)\in S_{ef,\gamma\delta}$, that is, $S_{e,\gamma}S_{f,\delta}\subset S_{ef,\gamma\delta}$. Therefore, $S(U)$ is a band of the groups $\{S_{e,\gamma}: e\in E$ and $\gamma\in\Gamma$ such that $S_{e,\gamma}\neq\square\}$. Thus, $S(U)$ is a $\mathscr{P}$-cryptogroup.

By the result above, we have:

THEOREM 5.1. *Every $\mathscr{P}$-cryptogroup is $\mathscr{P}$-isomorphic to a $\mathscr{P}$-subspined product $S(U)$*

*of a $\mathscr{P}$-band $E(P)$ and a $*$-cryptogroup $T(Q; \sharp)$. Conversely, any $\mathscr{P}$-subspined product $S(U)$ of a $\mathscr{P}$-band $E(P)$ and a $*$-cryptogroup $T(Q; \sharp)$ is a $\mathscr{P}$-cryptogroup.*

The structure of $*$-cryptogroups has been clarified in [9]. The theorem above is a generalization of the structure theorem (Theorem 4, [6]) for strictly inversive semigroups (that is, orthodox cryptogroups) to the class of $\mathscr{P}$-regular cryptogroups. In fact: Let $S$ be an orthodox cryptogroup, and $S \sim \Sigma\{S_\lambda: \lambda \in \Lambda\}$ the structure decomposition of $S$. Then, $E_S$ has the structure decomposition $E_S \sim \Sigma\{E_\lambda: \lambda \in \Lambda\}$, where $E_\lambda = S_\lambda \cap E_S$. Further, $S(E_S)$ and $S_\lambda(E_\lambda)$ are $\mathscr{P}$-regular. Now let $x\rho = \bar{x}$ and $x\nu = \tilde{x}$ for $x \in S$, and $\bar{X} = \{\bar{x}: x \in X\}$ and $\tilde{X} = \{\tilde{x}: x \in X\}$ for $X \subset S$. Then, $\bar{S}(\bar{E}_S) = S(E_S)/(\rho)_{\mathscr{P}}$ is isomorphic to the band $E_S$. On the other hand, the least strong $\mathscr{P}$-congruence $\nu$ on $S(E_S)$ is the least inverse semigroup congruence on $S$ (see [2], [7]), and hence $\tilde{S}(\tilde{E}_S) = S(E_S)/(\nu)_{\mathscr{P}}$ is a Clifford semigroup. Further, each $\tilde{S}_\lambda(\tilde{E}_\lambda)$ is a group. Let $T = \{(\bar{x}, \tilde{x}): x \in S\}$. Then, it follows from the result above that $S$ is isomorphic to $T = \bar{S}(\bar{E}_S) \overset{\mathscr{P}}{\underset{\Lambda}{\bowtie}} \tilde{S}(\tilde{E}_S)$. Now, let $T_\lambda = \{(\bar{x}, \tilde{x}): x \in S_\lambda\}$ for $\lambda \in \Lambda$. Let $(\bar{x}, \tilde{y}) \in \bar{S}_\lambda \times \tilde{S}_\lambda$. Then, $(\bar{x}, \tilde{y}) = (\overline{xx^{-1}yxx^{-1}}, \widetilde{xx^{-1}yxx^{-1}}) \in T_\lambda$. Therefore, $T_\lambda = \bar{S}_\lambda \times \tilde{S}_\lambda$. Hence, $T$ is the spined product $\bar{S} \underset{\Lambda}{\bowtie} \tilde{S}$ of $\bar{S}$ and $\tilde{S}$. Now, $\bar{S} \cong E_S$. Therefore, $S$ is isomorphic to the spined product of $E_S$ and the Clifford semigroup $\tilde{S}$. This is just the structure theorem for strictly inversive semigroups given by [6].

As a special case of the theorem above, if $S(P)$ is a completely simple $\mathscr{P}$-regular semigroup, then the structure semilattice of $S$ consists of a single element. Therefore, we have the following as a corollary to Theorem 5.1:

COROLLARY. *A completely simple $\mathscr{P}$-regular semigroup $S(P)$ is $\mathscr{P}$-isomorphic to a $\mathscr{P}$-subdirect product of a rectangular $\mathscr{P}$-band $E(Q)$ and a completely simple $*$-semigroup $T(K; \sharp)$.* Conversely, *a $\mathscr{P}$-subdirect product $S(P)$ of a rectangular $\mathscr{P}$-band $E(Q)$ and a completely simple $*$-semigroup $T(K; \sharp)$ is a completely simple $\mathscr{P}$-regular semigroup.*

## §6. Strongly $\mathscr{P}$-regular cryptogroups

Let $B$ be a band, and $P$ a $C$-set in $B$. Then, $B(P)$ is $\mathscr{P}$-regular. Let $\nu$ be the least strong $\mathscr{P}$-congruence on $B(P)$. Then, $\tilde{B}(\tilde{P}) = B(P)/(\nu)_{\mathscr{P}}$ is the regular $*$-semigroup having $\tilde{P}$ as the projections, where $\tilde{x} = x\nu$ and $\tilde{X} = \{\tilde{x}: x \in X\}$ for $X \subset B$. Let $Q_p = \{e \in B: \tilde{e} = \tilde{q}$ for some $q \in P\}$.

Then,

LEMMA 6.1. *$B(Q_p)$ is strongly $\mathscr{P}$-regular.*

PROOF. Let $e \in Q_p$. Then, there exists $q \in P$ such that $\tilde{q} = \tilde{e}$. Hence, $q\ \nu\ e$, and $q \in P$. Let $f \in V_{\mathscr{P}}(e)$ in $B(Q_p)$. Then, $\tilde{e}\tilde{P}\tilde{f} \subset \tilde{Q}_p = \tilde{P}$ and similary $\tilde{f}\tilde{P}\tilde{e} \subset \tilde{P}$. Further, $\tilde{f} \in V(\tilde{e})$. Hence, $\tilde{f} \in V_{\mathscr{P}}(\tilde{e})$ in $\tilde{B}(\tilde{P})$. Since $\tilde{B}(\tilde{P})$ is a regular $*$-semigroup, a $\mathscr{P}$-inverse of $\tilde{e}$ ($= \tilde{q}$) is unique, and hence $\tilde{f} = \tilde{q} = \tilde{e}$. Therefore, $f \in Q_p$. Further, $\widetilde{eQ_pe} = \tilde{e}\tilde{P}\tilde{e} = \widetilde{qPq} \subset \tilde{P}$, and accordingly $eQ_pe \subset Q_p$. Thus, $Q_p$ is a $C$-set. Therefore, $B(Q_p)$ is strongly $\mathscr{P}$-regular.

Conversely,

LEMMA 6.2. *Let $B(U)$ be a strongly $\mathscr{P}$-regular semigroup, and $\nu$ the least strong $\mathscr{P}$-congruence on $B(U)$. Let $\tilde{B}(\tilde{U})=B(U)/(\nu)_{\mathscr{P}}$ where $x\nu=\tilde{x}$ and $\tilde{X}=\{\tilde{x}: x\in X\}$ for $X\subset B$. Then, $U=\{x\in B: \tilde{x}=\tilde{u}$ for some $u\in U\}$.*

PROOF. This follows from [11].

Thus, we have:

THEOREM 6.3. *Let $B$ be a band, and $P$ a C-set in $B$. Let $\nu$ be the least strong $\mathscr{P}$-congruence on the $\mathscr{P}$-band $B(P)$. Let $x\nu=\tilde{x}$ and $\tilde{X}=\{\tilde{x}: x\in X\}$ for $X\subset B$. Let $Q_p=\{x\in B: \tilde{x}=\tilde{q}$ for some $q\in P\}$.*

*Then, $B(Q_p)$ is strongly $\mathscr{P}$-regular. Every C-set $U$ such that $B(U)$ is strongly $\mathscr{P}$-regular is obtained in this fashion.*

Next, let $S$ be a cryptogroup. Assume that $S(P)$ is strongly $\mathscr{P}$-regular. In this case, each $\mathscr{H}$-class is a group. Of course, $S$ is a band $\Gamma$ of groups $\{G_\gamma: \gamma\in\Gamma\}$. Let $e_\gamma$ be the identity of $G_\gamma$ for each $\gamma\in\Gamma$. Then, $S(P)/(\mathscr{H})_{\mathscr{P}}=\bar{S}(\bar{P})$, where $x\mathscr{H}=\bar{x}$ and $\bar{X}=\{\bar{x}: x\in X\}$ for $X\subset S$, is isomorphic to $\Gamma$; an isomorphism $g$ is given by $\bar{x}g=\gamma$ if $x\in G_\gamma$. Let $\Lambda=\{\bar{p}g: p\in P\}$. Then, clearly $\Gamma(\Lambda)$ is $\mathscr{P}$-isomorphic to $\bar{S}(\bar{P})$. Now, let $\bar{p}\mathscr{L}\bar{u}\mathscr{R}\bar{q}$, where $p$, $u$, $q\in P$. Since $S(P)$ is strongly $\mathscr{P}$-regular, $\bar{p}\bar{u}=\bar{p}$ and $\bar{u}\bar{p}=\bar{u}$. Hence, $pu\,\mathscr{H}\,p$ and $up\,\mathscr{H}\,u$, and accordingly $p\mathscr{L}\,u$. Similarly, $u\,\mathscr{R}\,q$. Hence, there exists $v\in P$ such that $p\,\mathscr{R}\,v\,\mathscr{L}\,q$. Therefore, $\bar{p}\,\mathscr{R}\,\bar{v}\,\mathscr{L}\,\bar{q}$. Similarly, $\bar{p}\,\mathscr{R}\,\bar{u}\,\mathscr{L}\,\bar{q}$ implies that $\bar{p}\,\mathscr{L}\,\bar{v}\,\mathscr{R}\,\bar{q}$ for some $v\in P$. Thus, $\bar{S}(\bar{P})$ is strongly $\mathscr{P}$-regular, and hence $\Gamma(\Lambda)$ is strongly $\mathscr{P}$-regular.

From the results above, we easily obtain the following:

THEOREM 6.4. *Let $S(P)$ be a strongly $\mathscr{P}$-regular semigroup. Then, $S(P)$ is $\mathscr{P}$-isomorphic to a $\mathscr{P}$-subspined product of a strongly $\mathscr{P}$-regular band $T(Q)$ and a $*$-cryptogroup $W(U; \sharp)$.*

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