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Bound state solution of a One-Dimensional Three-Body System

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I solve the one-dimensional three-body bound state problem interacting pairwise δ function. I find symmetric conditions on the masses and the interaction strengths for the existence of a simple type of exact solutions. I find also a symmetric form of the Fourier transform of the bound state wave function.

Several years ago, McGuire[1)] showed that for N particles of equal mass and with equal strength, interacting with a pairwise δ function in one dimension, the bound state problem is solvable exactly. This model had been very useful and given rise to a great deal of interest[2)]. For the problem of unequal masses, Kiang and Niégawa[3)] solved for the case of $N=3$ and found two conditions on the masses and the coupling strengths for the existence of a simple type of exact solution. At first sight their conditions are not symmetric between three particles. So I treat the same problem in a somewhat different manners from theirs and find symmetric conditions. I have verified that these conditions are equivalent to those of them. I also find symmetric forms of the Fourier transform of the bound state wave function.

The Hamiltonian for three particles in the c.m. system interacting with a pairwise δ function in one dimension is

$$H=K-\sum_{j>i=1}^{3} g_{ij}\delta(x_i-x_j) \tag{1}$$

with

$$K=\sum_{i=1}^{3} p_i^2/2m_i-P^2/2M. \tag{2}$$

In Eq. (2) $P=\sum_{i=1}^{3} p_i$ is the total momentum and $M=\sum_{i=1}^{3} m_i$. For the sake of the later convenience we use three sets of canonical variables;

$$x_a=x_1-(m_2x_2+m_3x_3)/(m_2+m_3),$$

$$x_{23}=x_2-x_3, \tag{3a}$$

$$x_b = x_2 - (m_3 x_3 + m_1 x_1)/(m_3 + m_1),$$
$$x_{31} = x_3 - x_1, \tag{3b}$$
$$x_c = x_3 - (m_1 x_1 + m_2 x_2)/(m_1 + m_2),$$
$$x_{12} = x_1 - x_2. \tag{3c}$$

In Eq. (3), a, b and c designate three channels[4)] and, for example, a means $1+(2, 3)$ where (i, j) is the bound state of particles i and j. I assume that all coupling constants g_{ij} are positive in order to have single bound state for each channels. I also define canonical conjugate momenta to Eq. (3) as follows;

$$p_a = -i\partial/\partial x_a = m_a[p_1/m_1 - (p_2+p_3)/(m_2+m_3)],$$
$$p_{23} = -i\partial/\partial x_{23} = m_{23}(p_2/m_2 - p_3/m_3), \tag{4a}$$

and the cyclic permutations of (a, b, c) and $(1, 2, 3)$ of Eq. (4a). In Eq. (4a), $m_{ij} = m_i m_j/(m_i + m_j)$ and $m_a = m_1(m_2+m_3)/M$. There exist following linear relations among these three sets of canonical variables;

$$x_a = -x_b m_{23}/m_3 - x_{31}m_{31}/m_a = -x_c m_{23}/m_2 + x_{12}m_{12}/m_a,$$
$$x_{23} = x_b - x_{31}m_{31}/m_3 = -x_c - x_{12}m_{12}/m_2, \tag{8a}$$

and the cyclic permutations of the above relations. There exist also linear relations between canonical momenta;

$$p_a = -p_b m_{31}/m_3 - p_{31} = -p_c m_{12}/m_2 + p_{12},$$
$$p_{23} = p_b m_{23}/m_b - p_{31}m_{23}/m_3 = -p_c m_{23}/m_c - p_{12}m_{23}/m_2, \tag{9a}$$

and the cyclic permutations of them. By the use of (4a) etc., the kinetic energy K defined by (2) is rewritten as follows;

$$K = p_a^2/2m_a + p_{23}^2/2m_{23} = p_b^2/2m_b + p_{31}^2/2m_{31} = p_c^2/2m_c + p_{12}^2/2m_{12}. \tag{10}$$

Next I define the Hamiltonian of the partial system by

$$H_a = K - g_{23}\delta(x_{23}), \tag{11a}$$

and the cyclic permutation of the above equation. For this Hamiltonian the particle 1 is free and the channel momentum p_a becomes a constant of motion. As is well known this Hamiltonian allows a bound state with the eigenvalue $E_{23} = -g_{23}^2 m_{23}/2$, and the eigenfunction

$$\psi_B(x_{23}) = N_a e^{-g_{23}m_{23}|x_{23}|}. \tag{12}$$

In the followings I show that the product of the eigenfunctions of the partial system becomes the eigenfunction of H, given by Eq. (1), provided two conditions (15) below are satisfied. I define

$$\psi_B(x_a, x_{23}) \equiv N\, e^{-g_{23}m_{23}|x_{23}|} e^{-g_{31}m_{31}|x_{31}|} e^{-g_{12}m_{12}|x_{12}|}. \tag{13}$$

Operating K to $\psi_B(x_a, x_{23})$, I can freely choose an appropriate form of K given by Eq. (10), according to each factor of Eq. (13). The calculation is straightforward and the resuslt is

$$\begin{aligned} K\, \psi_B(x_a, x_{23}) = & [-g_{23}^2 m_{23}/2 - g_{31}^2 m_{31}/2 - g_{12}^2 m_{12}/2 + g_{23}\delta(x_{23}) + g_{31}\delta(x_{31}) \\ & + g_{12}\delta(x_{12})]\psi_B(x_a, x_{23}) + \{g_{23}\varepsilon(x_{23})[g_{31}m_{31}m_{23}\varepsilon(x_{31})/m_3 \\ & + g_{12}m_{12}m_{23}\varepsilon(x_{12})/m_2] + g_{31}\varepsilon(x_{31})g_{12}m_{12}m_{31}\varepsilon(x_{12})/m_1\}\psi_B(x_a, x_{23}). \end{aligned} \tag{14}$$

In Eq. (14) $\varepsilon(x)$ is the sign function. In order that $\psi_B(x_a, x_{23})$ becomes eigenfunction of H, the term in the curly bracket of the r.h.s. of Eq. (14) must become a constant. These conditions are easily found and given by

$$g_{23}/(m_2+m_3) = g_{31}/(m_3+m_1) = g_{12}/(m_1+m_2) \equiv 1/\kappa. \tag{15}$$

If the conditions (15) are satisfied the value of the curly bracket is shown to be $-m_1m_2m_3/\kappa^2$. Therefore $\psi_B(x_a, x_{23})$ satisfy

$$(H-E)\psi_B(x_a, x_{23}) = 0, \tag{16}$$

and E is given by

$$E = -(m_1+m_2)\,(m_2+m_3)\,(m_3+m_1)/2\kappa^2. \tag{17}$$

I have verified that the conditions (15) are equivalent to Eqs. (2.24) and (2.25) in the reference 3).

It should be noted that, for N particles of equal mass and with coupling strength, McGuire's Solution[1)] has the similar form to Eq. (13). But, it turned out that it is impossible to get an exact solution of similar form for the case of unequal masses and $N=4$.

Finally I calculate the Fourier transform of Eq. (13) thereby use the Dirac's bra-cket symbols. I set $\psi_B(x_a, x_{23}) = (x_a x_{23}|B)$, and denote the eigenvalues of various momenta with the corresponding capital letters. I define as follows;

$$\begin{aligned} \phi_B(P_a, P_{23}) = & \iint dx_a dx_{23} (P_a P_{23}|x_a a_{23})\,(x_a x_{23}|B) \\ = & \frac{N}{2\pi}\int dx_{23}\int dx_a e^{-i(P_a x_a + P_{23}x_{23})} e^{-g_{23}m_{23}|x_{23}|} e^{-g_{31}m_{31}|x_a + x_{23}m_{23}/m_3|} \\ & \times e^{-g_{12}m_{12}|x_a - x_{23}m_{23}/m_2|} \equiv \frac{N}{2\pi} I. \end{aligned} \tag{18}$$

In Eq. (18) I make use of the cyclic permutations of Eq. (8a). The calculation of I is rather tedious and I write the result only.

$$I=4m_1m_2m_3\kappa^{-2}[(m_2^2+m_3^2+m_1m_2+m_3m_1-m_2m_3)P_a^2/(m_2+m_3)-2(m_2-m_3)P_aP_{23}$$
$$+(m_2+m_3)P_{23}^2+(m_1+m_2)(m_2+m_3)(m_3+m_1)(m_1m_2+m_2m_3+m_3m_1)/\kappa^2]$$
$$\times(P_a^2+g_{23}^2m_1^2)^{-1}[(P_{23}-P_am_{23}/m_3)^2+g_{31}^2m_2^2]^{-1}$$
$$\times[(P_{23}+P_am_{23}/m_2)^2+g_{12}^2m_3^2]^{-1}. \tag{19}$$

By making use of the cyclic permutations of Eq. (9a) with capital letters, the last two factors of I can be rewritten as $(P_b^2+g_{31}^2m_2^2)^{-1}(P_c^2+g_{12}^2m_3^2)^{-1}$ and so the denominator of I is symmetric between three particles. Notice that the free bras satisfy $(P_aP_{23}|=(P_bP_{31}|=(P_cP_{12}|$. It is interesting that the quadratic form of P_a and P_{23} in the first bracket of I is form invariant under the transformations (9a) etc. with corresponding capital letters. Namely, I can easily show that

$$(m_2^2+m_3^2+m_1m_2+m_3m_1-m_2m_3)P_a^2/(m_2+m_3)-2(m_2-m_3)P_aP_{23}+(m_2+m_3)P_{23}^2$$
$$=(m_3^2+m_1^2+m_2m_3+m_1m_2-m_3m_1)P_b^2/(m_3+m_1)-2(m_3-m_1)P_bP_{31}+(m_3+m_1)P_{31}^2$$
$$=(m_1^2+m_2^2+m_3m_1+m_2m_3-m_1m_2)P_c^2/(m_1+m_2)-2(m_1-m_2)P_cP_{12}+(m_1+m_2)P_{12}^2. \tag{20}$$

At last, by the use of Eqs. (18) and (19) with $P_a=P_{23}=0$, the normalization constant in Eq. (13) is determined;

$$N=[m_1m_2m_3\kappa^{-2}(m_1+m_2)(m_2+m_3)(m_3+m_1)(m_1m_2+m_2m_3+m_3m_1)^{-1}]^{1/2}. \tag{21}$$

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References

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