Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
39 (2006), pp. 15–23

FIBREWISE ANR OF FIBREWISE MAPPING-SPACES

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(Received: February 27, 2006)

Dedicated to the memory of Professor Masahiro Sugawara

Abstract. In this paper, we prove that some fibrewise mapping-spaces $\mathrm{map}_B(X,Y)$ over a stratifiable space B are stratifiable spaces, and further, if Y is in $ANR_B(\mathcal{S}_B)$, then $\mathrm{map}_B(X,Y)$ is also in $ANR_B(\mathcal{S}_B)$, where the fibrewise class $\mathcal{S}_B = \{(X,p) \mid p : X \to B,\ X \text{ is stratifiable}\}$.

1. Introduction

In this paper, we assume that all spaces are topological spaces and all maps are continuous, and we will use the abbreviation *nbd(s)* for *neighborhood(s)*. Let $\mathcal{S}$ be the class of all stratifiable spaces, and $\mathcal{S}_B$ the fibrewise class $\{(X,p)|p : X \to B, X \in \mathcal{S}\}$ for a space $B \in \mathcal{S}$. For $(X,p) \in \mathcal{S}_B$, p is called a *stratifiable map*. For stratifiable spaces, see C.J.R.Borges [1]. For $ANR_B(\mathcal{S}_B)$ and $ANE_B(\mathcal{S}_B)$, see T.Miwa [7] [8], or Definition 1.2 in this paper. Throughout this paper, we assume that $B \in \mathcal{S}$ unless otherwise stated, and for undefined terminology in fibrewise topology, see [5].

In the previous paper [8], we studied some basic properties of $ANR_B(\mathcal{S}_B)$. In this paper, we prove that some fibrewise mapping-spaces $\mathrm{map}_B(X,Y)$ over B are stratifiable spaces, and if, further, Y is in $ANR_B(\mathcal{S}_B)$, then $\mathrm{map}_B(X,Y)$ is also in $ANR_B(\mathcal{S}_B)$.

First, we refer to some definitions. For a fibrewise space (X,p) where $p : X \to B$ is a map, we denote $X_b = p^{-1}(b)$ and $X_W = p^{-1}(W)$ for $b \in B$ and $W \subset B$.

Definition 1.1 ([5, Definition 1.16])**.** The fibrewise space (X,p) is *locally sliceable* if for each point $x \in X_b$ (where $b \in B$) there is a nbd W of b in B and a section $s : W \to X_W$ such that $s(b) = x$.

Definition 1.2 ([7, 8])**.** Let B be an arbitrary topological space. Then $Q_B = \{(X,p)|p : X \to B$ with a property $Q\}$ is said to be a *fibrewise class*. For $(X,p) \in$

2000 *Mathematics Subject Classification.* Primary 55R70 ; Secondary 54C35, 54C55, 54E20.

Key words and phrases. Absolute nbd retract (extensor) over B relative to a fibrewise class, fibrewise mapping-space, stratifiable space.

*) The first author was supported by Dalian University during his stay at Shimane University from January to February 2006.

Q_B and a closed subset A of X, a pair $((X,p),(A,p|A))$ will be called a Q_B-*pair*. In this case, we will simply say that (X,A) is a Q_B-pair.

(1) Let X be a space, A a subspace of X, (X,p) and (A,p') two fibrewise spaces. A fibrewise map $r : X \to A$ is said to be a *fibrewise retraction* of p to p' if it is a retraction. In this case, p' is said to be a *retract* of p. Further, if there are a nbd Y of A in X and a fibrewise retraction $r : Y \to A$, p' is said to be a *nbd retract* of p.

(2) Let $(X,p),(A,p')$ and (Z,q) be fibrewise spaces, $A \subset X$ and $p' = p|A$. For fibrewise maps $\psi : X \to Z$ and $\varphi : A \to Z$, ψ is said to be a *fibrewise extension* of φ if $\psi|A = \varphi$. Further, if there is a nbd U of A in X and a fibrewise map $\psi : U \to Z$ with $\psi|U = \varphi$, then ψ is said to be a *fibrewise nbd extension* of φ.

(3) Let (X,p) be a fibrewise space. (X,p) is said to be an *absolute* (resp. *nbd) retract* over B relative to Q_B if, for every $(Z,q) \in Q_B$ satisfying (Z,X) is a Q_B-pair, p is a (resp. nbd) retract of q. By $AR_B(Q_B)$ (resp. $ANR_B(Q_B)$) we denote all absolute (resp. nbd) retracts over B relative to Q_B.

(4) Let (Z,q) be a fibrewise space. (Z,q) is said to be an *absolute* (resp. *nbd) extensor* over B relative to Q_B if, for every fibrewise map $\varphi : A \to Z$, where A is a closed subspace of X with $(X,p) \in Q_B$, φ has a fibrewise (resp. nbd) extension to X (resp. a nbd U of A). By $AE_B(Q_B)$ (resp. $ANE_B(Q_B)$) we denote all absolute (resp. nbd) extensors over B relative to Q_B.

Definition 1.3. (1) For fibrewise spaces (X,p) and (Y,q), the fibrewise mapping-set $\mathrm{map}_B(X,Y)$ is the set $\bigcup_{b\in B} C(X_b,Y_b)$, where $C(X_b,Y_b)$ is the set of all maps of X_b to Y_b. Note that the projection $\pi : \bigcup_{b\in B} C(X_b,Y_b) \to B$ is defined by $\pi(f) = b$ for $f \in C(X_b,Y_b)$. (Note that we denote briefly the fibrewise mapping-set by $\mathrm{map}_B(X,Y)$ instead of $(\mathrm{map}_B(X,Y),\pi)$.)

(2) ([5;section 4]) Let (X,p) is a fibrewise space. By a *b-filter* on X we mean a pair $(b,\mathcal{F})$, where $b \in B$ and $\mathcal{F}$ is a filter on X such that b is a limit point of the filter $p(\mathcal{F})$ generated by $\{p(F)|F \in \mathcal{F}\}$ on B.

Given b-filter $(b,\mathcal{F})$ on X we describe a family Γ of members of $\mathcal{F}$ as a *fibrewise basis* for $\mathcal{F}$ if for each member M of $\mathcal{F}$ we have $X_W \cap E \subset M$ for some member $E \in \Gamma$ and some nbd W of b. In this situation we say that Γ *generates* $\mathcal{F}$. A *fibrewise subbasis* is a family which forms a fibrewise basis after finite intersections have been taken.

(3) ([5;section 9]) For fibrewise spaces (X,p), (Y,q), we define a topology to the fibrewise set $\mathrm{map}_B(X,Y)$. Let $N(K;V;W)$, where $K \subset X_W$ and $V \subset Y_W$, denote the set of maps $f : X_b \to Y_b$ $(b \in W)$ such that $f(K_b) \subset V_b$, where $K_b = K \cap X_b$ and $V_b = V \cap Y_b$. We describe such a subset $N(K;V;W)$ of $\mathrm{map}_B(X,Y)$ as *fibrewise compact-open* if W is open in B, $p|K : K \to W$ is proper and V is open in Y_W. For each map $f : X_b \to Y_b$ $(b \in B)$, the fibrewise compact-open sets to which f belongs form the fibrewise subbasis of a b-filter. Thus a fibrewise nbd system is defined and we call the fibrewise topology generated thereby the *fibrewise compact-open* topology.

Definition 1.4. (1) Let $\mathcal{U}$ be a family of subsets of a space X. We say that $\mathcal{U}$ is *closure preserving* (abbrev. *CP*) if, for every $\mathcal{U}' \subset \mathcal{U}$, $\overline{\cup\mathcal{U}'} = \cup\overline{\mathcal{U}'}$, where $\overline{\mathcal{U}'} = \{\overline{U}|U \in \mathcal{U}'\}$. For $A \subset X$, let $\mathcal{U}|A = \{U \cap A|U \in \mathcal{U}\}$. We say that $\mathcal{U}$ is *finite on compact sets* (abbrev. CF) in X if $\mathcal{U}|K$ is finite for each compact set K of X([9]). $\mathcal{U}$ is σ-CP-CF if it can be written as $\mathcal{U} = \bigcup_{n\in\mathbf{N}}\mathcal{U}_n$ such that each $\mathcal{U}_n$ is CP and CF in X.

(2) For a family $\mathcal{U}$ of subsets of a space X, $\mathcal{U}$ is called a *quasi-base* for X if, whenever $x \in X$ and U is a nbd of x, then there is $V \in \mathcal{U}$ such that $x \in \mathrm{Int}V \subset V \subset U$, where Int is the interior operator. (Note that a regular space with a σ-CP quasi base is originally called an M_2-space, but G.Gruenhage [3] proved that every stratifiable space is an M_2-space. Further note that every stratifiable space has a σ-CP quasi base consisting of closed sets.)

B.Guo proved in [4] the following theorem. $ANR(\mathcal{S})$ is the abbreviation of Absolute Nbd Retract for the class $\mathcal{S}$.

Theorem ([4, Theorme 2.1 and Corollary 2.5])**.**

(1) *Let X be a compact metric space and Y a stratifiable space which has a σ-CP-CF quasi-base consisting of closed sets. Then $C(X,Y)$ has a σ-CP quasi-base, hence it is stratifiable, where $C(X,Y)$ is the space of all continuous maps with the compact open topology.*

(2) *If, further, Y is an $ANR(\mathcal{S})$, then $C(X,Y)$ is an $ANR(\mathcal{S})$.*

We shall extend this theorem to fibrewise version as follows.

Theorem 1.1. *Let B be a stratifiable space and M a compact metric space, X a closed subset of $B\times M$ and Y a stratifiable space which has a σ-CP-CF quasi-base consisting of closed sets. Let (X,π) and (Y,q) be fibrewise spaces over B, where $\pi = \pi_B|X$ is the restriction of the projection $\pi_B : B\times M \to B$. Assume that (X,π) is locally sliceable, and (X,π) satisfies the following condition (*):*

(*) *If $C(\subset M)$ is compact and Int $C \neq \emptyset$, for $b(\in B)$ with $(b\times C)\cap X \neq \emptyset$, there is a nbd W of b and a section $s : W \to X_W$ such that $s(W) \subset (W\times C)\cap X$.*

Then we have:

(1) *The fibrewise mapping-space* $\mathrm{map}_B(X,Y)$ *with the fibrewise compact-open topology is a regular T_1-space having a σ-CP quasi-base, hence it is stratifiable.*

(2) *If, further, Y is in $ANR_B(\mathcal{S}_B)$, then* $\mathrm{map}_B(X,Y)$ *is also in $ANR(\mathcal{S}_B)$.*

In section 2, we shall prove this theorem and give an example in Example 2.1 which shows the necessity of local sliceability of (X,π), and give examples in Example 2.2 which show satisfy the condition (*) in Theorem 1.1.

2. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1. We use all notations of Theorem 1.1 and

$$N(K_1,\cdots,K_n;U_1,\cdots,U_n;W) = \bigcap\nolimits_{i=1}^{n} N(K_i;U_i;W).$$

Further, we exclusively use the following notations: Let $\mathcal{U} = \bigcup_{n\in\mathbf{N}}\mathcal{U}_n$ be a σ-CP-CF quasi-base for Y consisting of closed sets such that $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots$. Let $\mathcal{B} = \bigcup_{n\in\mathbf{N}}\mathcal{B}_n$ be a σ-CP quasi-base for B consisting of closed sets such that $\mathcal{B}_1 \subset$

$\mathcal{B}_2 \subset \cdots$. Further, we use the following: $Z = \mathrm{map}_B(X, Y)$, $\mathrm{Int}\mathcal{U} = \{\mathrm{Int}U | U \in \mathcal{U}\}$, $\mathrm{Int}\mathcal{B} = \{\mathrm{Int}V | V \in \mathcal{B}\}$, and $\mathcal{O}(Y)$ is the set of all open subsets of Y.

The space X in this theorem is a closed subset of $B \times M$ for the compact metric space (M, d). Using the compactness of M, M has a sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of finite closed covers of M such that for each $n \in \mathbf{N}$, $\{\mathrm{Int}A | A \in \mathcal{C}_n\}$ is an open cover of M and $\mathrm{mesh}\mathcal{C}_n \leq \frac{1}{n}$.

For each $W \in \mathcal{B}_n$ and each $C \in \mathcal{C}_n$ $(n \in \mathbf{N})$, let $K_{W\times C} = (W \times C) \cap X$. Then it is easy to see that $\pi | K_{W\times C} : K_{W\times C} \to W$ is a proper map.

We assume in the following Lemmas 2.1-2.5 that all conditions of Theorem 1.1 are satisfied. We shall start by proving the following lemma.

Lemma 2.1. *For each $W \in \mathcal{B}_n$ and each $C \in \mathcal{C}_n$ $(n \in \mathbf{N})$, the map $\pi | K_{W\times C} : K_{W\times C} \to W$ has the following property (**):*

(**) *For $x \in K_{W\times C}$, $\pi(x) = b$, there is a nbd W_b of b and a section $s : W_b \to X_{W_b}$ satisfying $s(b) = x$ and $s(W \cap W_b) \subset K_{W\times C}$.*

Proof. Since the fibrewise space (X, π) satisfies the condition (*), it is easy to see that $\pi | K_{W\times C}$ satisfies the condition (**). □

For each $n \in \mathbf{N}$, let $\mathcal{K}_n = \{K_{W\times C} | W \in \mathcal{B}_n, C \in \mathcal{C}_n\}$. For each $n \in \mathbf{N}$ and $(C_1, \cdots, C_m) \in (\mathcal{C}_n)^m$, let $\mathcal{N}^n_{(C_1,\cdots,C_m)}$ be a family of Z defined by

$$\{N(K_{W\times C_1}, \cdots, K_{W\times C_m}; U_1, \cdots, U_m; W) \mid \\ K_{W\times C_i} \in \mathcal{K}_n,\ U_i \in \mathcal{U}_n \ \text{ and } \ U_i \subset Y_W (\text{for } i = 1, \cdots, m)\}.$$

We shall show in Lemmas 2.2−2.4 that the family

$$\mathcal{N} = \bigcup_{n\in\mathbf{N}} \bigcup \{\mathcal{N}^n_{(C_1,\cdots,C_m)} \mid (C_1, \cdots, C_m) \in (\mathcal{C}_n)^m\}$$

is a quasi-base for Z consisting of closed subsets, and Z is a regular T_1-space. Further in Lemma 2.5, we shall prove that $\mathcal{N}^n_{(C_1,\cdots,C_m)}$ is CP for each $(C_1, \cdots, C_m) \in (\mathcal{C}_n)^m$.

After proving Lemmas 2.1−2.5, it is easily verified that Theorem 1.1 (1) is proved.

Lemma 2.2. *Let $f \in C(X_b, Y_b)$ $(b \in B)$ and O a nbd of f. Then there is an element $N(K_1, \cdots, K_m; G_1, \cdots, G_m; W)$ such that $W (\in \mathcal{B})$ is a nbd of b, each $\pi | K_i : K_i \to W$ is proper, $G_i \in \mathcal{U}$ $(i = 1, \cdots, m)$ and*

$$f \in N(K_1 \cap X_V, \cdots, K_m \cap X_V; Y_V \cap \mathrm{Int}G_1, \cdots, Y_V \cap \mathrm{Int}G_m; V) \\ \subset N(K_1, \cdots, K_m; G_1, \cdots, G_m; W) \subset O.$$

where $V = \mathrm{Int}W$, $X_V = \pi^{-1}(V)$ and $Y_V = q^{-1}(V)$.

Proof. From the fibrewise compact open topology, there is an open nbd W_1 of b, proper maps $p_i : L_i \to W_1$ (where $L_i \subset X_{W_1}$ for $i = 1, \cdots, m$) and open subsets $U_1, \cdots, U_m$ of Y_{W_1} such that $f \in N(L_1, \cdots, L_m; U_1, \cdots, U_m; W_1) \subset O$. Since $\mathcal{B}$ is a quasi-base of B, there is a nbd W of b such that $W \in \mathcal{B}$ and $W \subset W_1$. For each $i \in \{1, \cdots, m\}$, since $f(x) \in U_{i,b}$ $(= U_i \cap Y_b)$ for any $x \in L_{i,b} (= L_i \cap X_b)$, there is $G_i^x \in \mathcal{U}$ such that $f(x) \in \mathrm{Int}G_i^x \subset G_i^x \subset U_i \cap Y_W$, so $x \in f^{-1}(\mathrm{Int}G_i^x) \subset$

$f^{-1}(G_i^x) \subset f^{-1}(U_i \cap Y_W)$. By compactness of $L_{i,b}$, there are $x_i^1, \cdots, x_i^{n(i)} \in L_{i,b}$ such that, putting $G_{i,j} = G_i^{x_i^j}$,

$$L_{i,b} \subset \bigcup_{j=1}^{n(i)} f^{-1}(\mathrm{Int}G_{i,j}) = f^{-1}(\bigcup_{j=1}^{n(i)} \mathrm{Int}G_{i,j}) \subset f^{-1}(\bigcup_{j=1}^{n(i)} G_{i,j}) \subset f^{-1}(U_i \cap Y_W).$$

Since $\{f^{-1}(\mathrm{Int}G_{i,j}) \cup (X_W - X_b) \mid j = 1, \cdots, n(i)\}$ is a finite open (in X_W) cover of $L_i \cap X_W$, by the normality of $L_i \cap X_W$ there is an open (in X_W) cover $\{H_{i,1}, \cdots, H_{i,n(i)}\}$ of $L_i \cap X_W$ such that $\overline{H_{i,j}} \subset f^{-1}(\mathrm{Int}G_{i,j}) \cup (X_W - X_b)$. Let $V = \mathrm{Int}W$ and $K_{i,j} = \overline{H_{i,j}} \cap L_i$. Then it is clear that

$$\begin{aligned} f &\in N(K_{i,1} \cap X_V, \cdots, K_{i,n(i)} \cap X_V; Y_V \cap \mathrm{Int}G_{i,1}, \cdots, Y_V \cap \mathrm{Int}G_{i,n(i)}; V) \\ &\subset N(K_{i,1}, \cdots, K_{i,n(i)}; G_{i,1}, \cdots, G_{i,n(i)}; W) \\ &\subset N(L_i \cap X_W; U_i \cap Y_W; W) \subset N(L_i; U_i; W_1). \end{aligned}$$

Thus we have

$$\begin{aligned} f &\in \bigcap_{i=1}^{m} N(K_{i,1} \cap X_V, \cdots, K_{i,n(i)} \cap X_V; Y_V \cap \mathrm{Int}G_{i,1}, \cdots, Y_V \cap \mathrm{Int}G_{i,n(i)}; V) \\ &\subset \bigcap_{i=1}^{m} N(K_{i,1}, \cdots, K_{i,n(i)}; G_{i,1}, \cdots, G_{i,n(i)}; W) \\ &\subset N(L_1, \cdots, L_m; U_1 \cdots, U_m; W_1) \subset O. \end{aligned}$$

□

Lemma 2.3. *For $f \in C(X_b, Y_b)$ ($b \in B$) and a nbd $N(K; U; V)$ of f, where $U \in \mathcal{O}(Y)$, V is an open nbd of b in B and $\pi|K : K \to V$ is proper, there are $W \in \mathcal{B}_n$, $C_1, \cdots, C_m \in \mathcal{C}_n$ and $U_1, \cdots, U_m \in \mathcal{U}_n$ for some $n \in \mathbf{N}$ such that $N(K_{W\times C_1}, \cdots, K_{W\times C_m}; U_1, \cdots, U_m; W)$ is a nbd of f satisfying*

$$N(K_{W\times C_1}, \cdots, K_{W\times C_m}; U_1, \cdots, U_m; W) \subset N(K; U; V).$$

Proof. For the open nbd V of b, since B is stratifiable and $\mathcal{B}$ is a quasi-base of B, there is $W_1 \in \mathcal{B}$ such that $b \in \mathrm{Int}W_1 \subset W_1 \subset V$. Let $V_1 = \mathrm{Int}W_1$. Since $f(K_b) \subset U_b \subset U$ and $f(K_b)$ is compact where $K_b = K \cap X_b$ and $U_b = U \cap Y_b$, there are $U_1, \cdots, U_m \in \mathcal{U}$ such that

$$\begin{aligned} f(K_b) &\subset \mathrm{Int}U_1 \cup \cdots \cup \mathrm{Int}U_m \\ &\subset U_1 \cup \cdots \cup U_m \subset U \cap Y_{V_1}. \end{aligned}$$

Since $\{f^{-1}(Y_b \cap \mathrm{Int}U_j) | j = 1, \cdots, m\}$ is a finite open (in X_b) cover of K_b, there is an open (in X_{V_1}) cover $\{U_j^{'} | j = 1, \cdots, m\}$ of K_b such that $U_j^{'} \cap X_b = f^{-1}(Y_b \cap \mathrm{Int}U_j)$ ($j = 1, \cdots, m$). Since $\pi|K : K \to V$ is proper and $V_1 \subset V$, there is a nbd $W_2 (\in \mathcal{B})$ of b such that $W_2 \subset V_1$ and $K_{W_2} \subset U_1^{'} \cup \cdots \cup U_m^{'}$. Further, there is a closed (in K_{W_2}) cover $\{K_1, \cdots, K_m\}$ of K_{W_2} such that $K_j \subset U_j^{'}$ ($j = 1, \cdots, m$). Since $\pi|K_{W_2} : K_{W_2} \to W_2$ is proper, $\pi|K_j : K_j \to W_2$ is proper for $j = 1, \cdots, m$. Then

$$f \in N(K_1, \cdots, K_m; \mathrm{Int}U_1, \cdots, \mathrm{Int}U_m; W_2) \subset N(K_{W_2}; U_{W_2}; W_2) \subset N(K; U; V).$$

Since $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots$ and $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$, there is a sufficient large $n \in \mathbf{N}$ and a nbd $W(\in \mathcal{B}_n)$ of b and $C_{ij} \in \mathcal{C}_n$ $(i = 1, \cdots, m; j = 1, \cdots, l(i))$ such that $W \subset W_2$, $U_1, \cdots, U_m \in \mathcal{U}_n$ and for each $i = 1, \cdots, m$

$$K_i \cap X_W \subset (\bigcup_{j=1}^{l(i)} (W \times \mathrm{Int}C_{ij})) \cap X \subset (\bigcup_{j=1}^{l(i)} (W \times C_{ij})) \cap X \subset U_i',$$
$$K_b \cap (W \times \mathrm{Int}C_{ij}) \neq \emptyset (\text{for } j = 1, \cdots, l(i)).$$

By $K_{W \times C} = (W \times C) \cap X$, it is easily verified that

$$\begin{aligned} f &\in \bigcap_{i=1}^{m} N(K_{W\times C_{i1}}, \cdots, K_{W\times C_{il(i)}}; \overbrace{\mathrm{Int}U_i, \cdots, \mathrm{Int}U_i}^{l(i)}; W) \\ &\subset \bigcap_{i=1}^{m} N(K_{W\times C_{i1}}, \cdots, K_{W\times C_{il(i)}}; \overbrace{U_i, \cdots, U_i}^{l(i)}; W) \\ &\subset N(K_1, \cdots, K_m; U_1, \cdots, U_m; W_2) \subset N(K; U; V). \end{aligned}$$

This completes the proof of this lemma. □

Lemma 2.4. *In* $Z = \mathrm{map}_B(X, Y)$*, we have:*

1. *Each member of* $\mathcal{N} = \bigcup_{n\in\mathbf{N}} \bigcup \{\mathcal{N}^n_{(C_1,\cdots,C_m)} | (C_1, \cdots, C_m) \in (\mathcal{C}_n)^m\}$ *is closed in* Z.
2. Z *is a regular* T_1*-space.*

Proof. (1) To prove this, it is sufficient to show that, for $W \in \mathcal{B}_n$, $C \in \mathcal{C}_n$ and $U \in \mathcal{U}_n$, $U \subset Y_W$ for $n \in \mathbf{N}$, $N(K_{W\times C}; U; W)$ is closed in Z. Let $f \notin N(K_{W\times C}; U; W)$ and $f \in C(X_b, Y_b)$ for some $b \in B$. In the case $b \notin W$, we put $W_1 = B - W$. Then it is easy to see that W_1 is an open nbd of b, $\pi | X_{W_1} : X_{W_1} \to W_1$ is proper, $N(X_{W_1}; Y_{W_1}; W_1)$ is an open nbd of f satisfying $N(K_{W\times C}; U; W) \cap N(X_{W_1}; Y_{W_1}; W_1) = \emptyset$. In the other case that $b \in W$, there is $x \in K_{W\times C} \cap X_b$ such that $f(x) \notin U_b$. Since U is closed in Y, $Y - U (= V)$ is an open nbd of $f(x)$ in Y. From Lemma 2.1, for the point $x \in K_{W\times C} \cap X_b$, there is a nbd W_1 of b and a section $s : W_1 \to X_{W_1}$ satisfying that $s(b) = x$ and $s(W \cap W_1) \subset K_{W\times C}$. Let $L = s(W_1)$. Then it is easily verified that $\pi | L : L \to W_1$ is proper, $N(L; V_{W_1}; W_1)$ is a nbd of f and $N(K_{W\times C}; U; W) \cap N(L; V_{W_1}; W_1) = \emptyset$, where $V_{W_1} = V \cap Y_{W_1}$. Thus we can see that $N(K_{W\times C}; U; W)$ is closed in Z.

(2) Since (X, π) is locally sliceable and (Y, q) is fibrewise Hausdorff, from [5] Proposition 9.3 Z is fibrewise Hausdorff, therefore Z is T_1.

Finally, it is easily verified from (1) of this lemma and Lemmas 2.2-2.3 that Z is regular. □

Lemma 2.5. $\mathcal{N}^n_{(C_1,\cdots,C_m)}$ *is* CP *for each* $(C_1, \cdots, C_m) \in (\mathcal{C}_n)^m$. *Therefore,* $\mathcal{N}$ *is* σ*-*CP (*quasi-base for* Z *from* Lemmas 2.2-2.3).

Proof. We shall show that each $\mathcal{N}^n_{(C_1,\cdots,C_m)}$ is CP. First note by Lemma 2.4 that each member of $\mathcal{N}^n_{(C_1,\cdots,C_m)}$ is closed. Let $\mathcal{U}'$ be any subfamily of $(\mathcal{U}_n)^m$, $\mathcal{B}'$ any subfamily of $\mathcal{B}_n$, and

$$\mathcal{N}' = \{N(K_{W\times C_1}, \cdots, K_{W\times C_m}; U_1, \cdots, U_m; W) \mid W \in \mathcal{B}', (U_1, \cdots, U_m) \in \mathcal{U}', U_i \subset Y_W \ (i = 1, \cdots, m)\}$$

To prove that $\overline{\bigcup \mathcal{N}'} = \bigcup \mathcal{N}'$, let $g \in Z - \bigcup \mathcal{N}'$. Then, there is $b \in B$ such that $g \in C(X_b, Y_b)$. Let $\mathcal{B}_1' = \{W \in \mathcal{B}' | b \notin W\}$, $\mathcal{B}_2' = \{W \in \mathcal{B}' | b \in W\}$ and

$$\mathcal{N}_1' = \{N(K_{W\times C_1}, \cdots, K_{W\times C_m}; U_1, \cdots, U_m; W) \in \mathcal{N}' \mid W \in \mathcal{B}_1', U_i \subset Y_W (i = 1, \cdots, m)\},$$
$$\mathcal{N}_2' = \mathcal{N}' - \mathcal{N}_1'.$$

For each $k = 1, \cdots, m$, let $p_k : (\mathcal{U}_n)^m \to \mathcal{U}_n$ be the k-th projection and

$$\mathcal{U}'(k) = \{U \in p_k(\mathcal{U}') | g((b \times C_k) \cap X_b) \not\subset U_b\}.$$

We shall show the assertion in the following way: For each $\mathcal{N}_i'(i = 1, 2)$, we construct open nbds N_1 and N_2 of g such that $N_i \cap (\bigcup \mathcal{N}_i') = \emptyset$ $(i = 1, 2)$ respectively, so we can prove that $N = N_1 \cap N_2$ is an open nbd of g satisfying $N \cap (\bigcup \mathcal{N}') = \emptyset$.

First, we shall construct N_1. Since $\mathcal{B}_1' \subset \mathcal{B}_n$ and $\mathcal{B}_n$ is a closure preserving family consisting of closed sets, $b \in B - \bigcup \mathcal{B}_1' (= W_1)$ and W_1 is an open nbd of b. Then it is easy to see that $N(X_{W_1}; Y_{W_1}; W_1)$ is an open nbd of g satisfying $N(X_{W_1}; Y_{W_1}; W_1) \cap \bigcup \mathcal{N}_1' = \emptyset$. Thus we can take $N_1 = N(X_{W_1}; Y_{W_1}; W_1)$.

Next we shall construct N_2. In case $\mathcal{U}'(k) = \emptyset$ for some k, let $N_k(g) = Z$. In case $\mathcal{U}'(k) \neq \emptyset$, we can write

$$\{U \in \mathcal{U}'(k) | U \cap g((b \times C_k) \cap X_b) \neq \emptyset\} = \{U_{k1}, \cdots, U_{km(k)}\},$$

because $\mathcal{U}_n$ is CF and $g((b \times C_k) \cap X_b)$ is compact. Note from the construction $\mathcal{U}'(k)$ that $g((b\times C_k)\cap X_b) - U_{ki} \neq \emptyset$ for each $i = 1, \cdots, m(k)$. We can choose points $x_{ki} = (b, a_{ki}) \in b \times C_k$ for $i = 1, \cdots, m(k)$ such that $g(x_{ki}) \in g((b \times C_k) \cap X_b) - U_{ki}$. Then $V_{ki} = Y - \bigcup\{U \in \mathcal{U}_n | g(x_{ki}) \notin U\}$ is an open nbd of $g(x_{ki})$ in Y, because $\mathcal{U}_n$ is CP. Since (X, π) is locally sliceable, there is an open nbd V of b such that a section $s_i : V \to X_V$ satisfying $s_i(b) = x_{ki}$ for each $i = 1, \cdots, m(k)$. Let $K_{ki} = s_i(V)$ for each $i = 1, \cdots, m(k)$, then it is obvious that $\pi | K_{ki} : K_{ki} \to V$ is proper and $N_k(g) = N(K_{k1}, \cdots K_{km(k)}; V_{k1} \cap Y_V, \cdots, V_{km(k)} \cap Y_V; V)$ is an open nbd of g. Then we shall show that we can take $N_2 = \bigcap_{k=1}^m N_k(g)$ satisfying $N_2 \cap (\bigcup \mathcal{N}_2') = \emptyset$. For any

$$N = N(K_{W\times C_1}, \cdots, K_{W\times C_m}; U_1, \cdots, U_m; W) \in \mathcal{N}_2',$$

it is satisfied that $g \notin N$ by $g \in Z - \bigcup \mathcal{N}'$, therefore $g((b \times C_k) \cap X_b) \not\subset U_k$ for some $k \in \{1, \cdots, m\}$. Hence $g((b \times C_k) \cap X_b) \cap U_k = \emptyset$ or $U_k = U_{ki}$ for some $i \in \{1, \cdots, m(k)\}$, so $V_{ki} \cap U_k = \emptyset$. Since $x_{ki} \in K_{ki} \cap K_{W\times C_k}$ and $V_{ki} \cap U_k = \emptyset$, $N_k(g) \cap N(K_{W\times C_k}; U_k; W) = \emptyset$. Therefore $N_2 \cap N = \emptyset$. □

Now, we start the proof of Theorem 1.1 (2).

Proof of Theorem 1.1(2). We shall show that Z is in $ANE_B(\mathcal{S}_B)$. Let (E, r) be in $\mathcal{S}_B$, A a closed subset of E and $\varphi : A \to Z$ a fibrewise map. Since $Z = \mathrm{map}_B(X, Y)$, we can define a fibrewise function $\hat{\varphi} : A\times_B X \to Y$ by $\hat{\varphi}(a,x) = \varphi(a)(x)$ ($a \in A_b$, $x \in X_b$, $b \in B$), where $A\times_B X$ is the fibrewise product of fibrewise spaces A and X ([5;section 1]). Since $\pi : X \to B$ is proper, from [5;Corollary 9.14] $\hat{\varphi}$ is continuous. Further, since Y is in $ANR_B(\mathcal{S}_B)$, so from [8] Theorem 1.1(4), Y is in $ANE_B(\mathcal{S}_B)$. Therefore there is a nbd U of $A\times_B X$ in $E\times_B X$ and a fibrewise map $\hat{\Phi} : U \to Y$ such that $\hat{\Phi}|A\times_B X = \hat{\varphi}$. Since $\pi : X \to B$ is proper, it is easy to see that there is an open nbd V of A in E such that $V\times_B X \subset U$. (We use the same notation $\hat{\Phi} : V\times_B X \to Y$.) Then a fibrewise function $\Phi : V \to Z = \mathrm{map}_B(X,Y)$ defined by $\Phi(z)(x) = \hat{\Phi}(z,x)$ is continuous by [5;Proposition 9.7], and $\Phi|A = \varphi$. Thus Z is in $ANE_B(\mathcal{S}_B)$, so in $ANR_B(\mathcal{S}_B)$. □

In Theorem 1.1, "local sliceability" of (X,π) plays an essential role. For this, see the following example. This example is appeared in [5;p.68] and [6;Remark 1.7].

Example 2.1. Let I be the closed unit interval, $M = B = I$, and $x_0 = (0,1)$ be the point in $I \times I$. Let $X = Y = \{x_0\} \cup (I \times \{0\})$, $\pi : X \to B$ and $q : Y \to B$ be the natural projections to the first axis. Then it is clear that (X,π) is not locally sliceable. Take $f, g \in C(X_0, Y_0)$ satisfying $f(x_0) = x_0, f((0,0)) = (0,0)$ and $g(\{(0,0), x_0\}) = \{(0,0)\}$, where $X_0 = \pi^{-1}((0,0))$ and $Y_0 = q^{-1}((0,0))$. Then it is easy to see that f and g do not have separated nbds. Therefore $\mathrm{map}_B(X,Y)$ is not (fibrewise) Hausdorff.

In connection with this example, the following proposition is established in [5; p.68] Proposition 9.3.

Proposition. *Let X be locally sliceable over B. Then $map_B(X,Y)$ is fibrewise Hausdorff whenever Y is fibrewise Hausdorff.*

In Theorem 1.1, we want to remove the condition (*) for (X,π). But we cannot yet prove the same type theorem of Theorem 1.1 under an assumption which the condition (*) is removed, or cannot yet give an example in which the same type theorem of Theorem 1.1 does not hold under an assumption which the condition (*) is removed.

Now we give some examples of (X,π) which satisfy the condition (*). These examples show that the condition (*) does not seem to be very strange. We can easily show that the following examples satisfy the condition (*).

Example 2.2. Let I be the closed unit interval, and $M = B = I$.

(1) Let $X = \{(x,y)|(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 \le (\frac{1}{2})^2\}$ be the closed convex subset of $B \times M$, and $\pi : X \to B$ the natural projection.
(2) Let $X = \{(x,y)|(x-\frac{1}{2})^{\frac{2}{3}} + (y-\frac{1}{2})^{\frac{2}{3}} \le (\frac{1}{2})^{\frac{2}{3}}\}$ be the closed non-convex subset of $B \times M$, and $\pi : X \to B$ the natural projection.

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