Mem. Fac. Sci. Shimane Univ.
21, pp. 21–30 Dec. 20, 1987

Equivariant Homotopy Groups of Classical Groups

Dedicated to Professor Masahiro Sugawara on his 60th birthday

Hiromichi MATSUNAGA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 5, 1987)

In [4] we have studied the surjectivity of the forgetful homomorphism $f(G, X)$: $K_G(X) \to K(X)$. The homomorphism gives informations about lifting actions on *stable* vector bundles. One of the purpose of this paper is to study lifting actions on vector bundles and give actions explicitly for geometrical uses, for example, equivariant Hopf constructions and a lifting problem for other spaces than the spheres.

In section 1 we shall give a criterion for the existence of lifting actions which is obtained by G. Bredon's work [2]. Section 2 consists of results obtained by J. Folkman's theorems [3], and Proposition 3 in [5]. Moreover we shall prove the equivariance for representatives of of generators of the groups $\pi_3(SO(4))$ and $\pi_7(SO(8))$. In section 3 we shall prove the equivariance of Bott maps [1], which present us various constructions of equivariant maps. In the last section we shall apply results in preceding sections and obtain a non existence theorem, equivariant Hopf constructions and a lifting property on complex plane bundles over the complex projective plane.

§1. Bredon's exact sequence

In [2] G. Bredon has given an exact sequence for S^1 actions. The techinque used there is also applicable to S^3 actions. For use later, we reconstruct the exact sequence explicitly. For $i=1$ or 3, let μ: $S^i \times X \to X$ be an S^i action with a fixed point x_0 which we shall take as the base point. Let d be 2 or 4 according to $i=1$ or 3. ρ_i denotes the standard representation of S^i and θ the trivial one dimensional representation. As in [2] S_r^{dk+r} denotes the $dk+r$ dimensional sphere with the S^i action which is given by the representation $k\rho_i \oplus (r+1)\theta$. [, X] denotes the set of equivariant, base point preserving homotopy classes of equivariant maps. ψ: $[S_r^{dk+r}, X] \to \pi_{dk+r}(X)$ denotes the forgetful map, and β: $[S_r^{d(k+1)+r}, X] \to [S_r^{dk+r}, X]$ the map induced from the inclusion map $S_r^{dk+r} \subset S_r^{d(k+1)+r}$. Moreover we define a map α: $\pi_{dk+r+1}(X) \to [S_r^{d(k+1)+r}, X]$ as follows. Let f: $(S_r^{dk+r} * e, S_r^{dk+r}) \to (X, x_0)$ be a map, where e denotes the unit element of the group S^i. Define a map $\tilde{f}$: $S_r^{d(k+1)+r} = S_r^{dk+r} * S^i \to X$ by

$$\tilde{f}((1-t)x+tg) = \mu(g)f((1-t)g^{-1}x+te) \qquad \text{for} \quad 0 \leqq t \leqq 1, \quad x \in X, \quad g \in G,$$

and set $\alpha([f]) = [\tilde{f}]$, where [ ] denotes an equivalence class. Since the set $[S_r^{dk+r}, X]$

has a natural group structure, by a routine we have

PROPOSITION 1. *There exists the following exact sequence:*

$$\cdots \xrightarrow{\beta} [S_r^{dk+r+1}, X] \xrightarrow{\psi} \pi_{dk+r+1}(X) \xrightarrow{\alpha} [S_r^{d(k+1)+r}, X] \xrightarrow{\beta} \cdots$$

§ 2. Constructions of equivariant maps

In this section we give some constructions of equivariant maps in the case of classical groups $SO(n)$, $U(n)$ and $Sp(n)$.

(1) A theorem induced from Folkman's theorems.

Let I_k be the ideal generated by the monomial $(x-1)^k$ in the representation ring $R(S^1)=Z[x, x^{-1}]$. Set $(e^{2\pi it}-1)^k=\sum_j e^{2\pi ib(j)t}-\sum_j e^{2\pi ia(j)t}$ for $0\leqq t\leqq 1$, and let $T(g)$ and $S(g)$ be $2^{k-1}\times 2^{k-1}$ diagonal matrices with entries $e^{2\pi ib(j)t}$ and $e^{2\pi ia(j)t}$ for $1\leqq j\leqq 2^{k-1}$ respectively, where g is $e^{2\pi it}$. Let $f_1\colon S^1\to SU(n)\subset U(n)$, $n=2^{k-1}$, be the map defined by

$$f_1(e^{2\pi it})=\mathrm{Diag}\,(e^{2\pi i(b(j)-a(j))t}).$$

Since

$$\sum_{l=0}^{p}\binom{k}{2}2l-\sum_{l=1}^{p}\binom{k}{2l-1}(2l-1)=k\left(\sum_{l=0}^{p}\binom{k-1}{2l-1}-\sum_{l=1}^{p}\binom{k-1}{2l-2}\right)$$

$$=0 \text{ for } p=\left[\frac{k}{2}\right],$$

$\det\,\mathrm{Diag}\,(e^{2\pi i(b(j)-a(j))t})=1$. Then $f_1\simeq 0$. Therefore we have an equivariant extension $f_2\colon S^1*S^1\to U(n)$, where S^1 action on $U(n)$ is given by

$$U(n)\ni A \longrightarrow T(g)AS(g)^{-1}\in U(n) \qquad \text{for} \quad g\in S^1.$$

Let m be an arbitrary integer. We consider the restriction homomorphism of representation rings $Z(S^1)\to Z(Z_m)$ and use Proposition 3.3 in [3] to obtain that $\deg f_2\equiv 0 \bmod m$ and accordingly $\deg f_2=0$. Thus we have an equivariant extension $f_3\colon S^1*S^1*S^1=S_0^5\to U(n)$. If we continue this process, it follows from Theorem 3.1 in [3] that

PROPOSITION 2. *There exists an equivariant map $f_k\colon S_0^{2k-1}\to U(n)$ of degree* 1.

REMARK. By §4 in [3], we have similar results for $SO(n)$ and $Sp(n)$.

(2) A result obtained from Proposition 4 in [5].

Let $D(t)$ be the 2×2 matrix $\begin{pmatrix}\cos 2\pi t & -\sin 2\pi t\\ \sin 2\pi t & \cos 2\pi t\end{pmatrix}$ and S_l be the $2l\times 2l$ matrix with l times of $D(t)$ on the diagonal. Define an S^1 action on $SO(2l)$ by

$$SO(4k+2) \ni A \longrightarrow \mu(g)(A) = S_l(g) A S_l(g)^{-1} \quad \text{for} \quad g \in S^1.$$

By Proposition 1 in §1, we have a commutative diagram

$$\begin{array}{ccccccc}
\pi_{4k-1}(SO(4k)) & \xleftarrow{\psi} & [S^{4k-1}_{4k-3}, SO(4k)] & \xrightarrow{\beta} & \pi_{4k-3}(U(2k)) & \xrightarrow{\psi} & \pi_{4k-3}(SO(4k)) \\
\cong \downarrow i_* & & \cong \downarrow i^* & & \cong \downarrow i_* & & \cong \downarrow i_* \\
\pi_{4k-1}(SO(4k+2)) & \xleftarrow{\psi} & [S^{4k-1}_{4k-3}, SO(4k+2)] & \xrightarrow{\beta} & \pi_{4k-3}(U(2k+1)) & \xrightarrow{\psi} & \pi_{4k-3}(SO(4k+2)),
\end{array}$$

where $\cong$ denotes the obvious isomorphisms. Then we have

PROPOSITION 3. $i_*\psi([S^{4k-1}_{4k-3}, SO(4k)) \supset 2\pi_{4k-1}(SO(4k+2))$ *for* $k \geqq 1$.

PROOF. Since $\pi_{4k-2}(U(2k)) = \pi_{4k-2}(U(2k+1)) = 0$ and $\pi_{4k-2}(SO(4k)) = \pi_{4k+2}(SO(4k+2)) = 0$, $i_*: [S^{4k-1}_{4k-3}, SO(4k)] \to [S^{4k-1}_{4k-3}, SO(4k+2)]$ is an isomorphism. Then by Proposition 4 in [5], we obtain the result in Proposition 3.

(3) Lower dimensional cases.

Let $S_{l,k}$ and $S'_{l,k}$ be the $(2l+k)\times(2l+k)$ matrices $D(t)^l \times I_k$ and $I_k \times D(t)^l$ respectively, where I_k denotes the unit matrix of degree k. Now we consider equivariant homotopy sets $[S^3, SO(4)]$ and $[S^7, SO(8)]$ with suitable actions on the spaces. The following maps are known as representatives for generators of $\pi_3(SO(4))$:

$$\sigma_3\colon \quad S^3 \longrightarrow SO(4) \quad \text{given by} \quad \sigma_3(q)x = qx \qquad \text{for} \quad q, x \in S^3 = Sp(1),$$

$$\sigma'_3\colon \quad S^3 \longrightarrow SO(4) \quad \text{given by} \quad \sigma'_3(q)x = x\bar{q} \qquad \text{for} \quad q, x \in S^3 = Sp(1).$$

More explicitly for $q = q_0 + q_1 i + q_2 j + q_3 k$,

$$\sigma_3(q) = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix}, \quad \sigma'_3(q) = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}.$$

Now we consider q as a column vector ${}^t(q_0, q_1, q_2, q_3)$. Then

$$\sigma_3(S'_{1,2}q) = S'_{1,2}\sigma_3(q){}^tS'_{1,2},$$

$$\sigma_3(S_{2,0}q) = S_{2,0}\sigma_3(q)I_4,$$

$$\sigma'_3(S'_{1,2}q) = S'_{1,2}\sigma'_3(q){}^tS'_{1,2},$$

$$\sigma'_3(S_{2,0}q) = \hat{S}_{2,0}\sigma'_3(q)I_4 \quad \text{where} \quad \hat{S}_{2,0} = \begin{pmatrix} {}^tD & 0 \\ 0 & D \end{pmatrix}.$$

Now representatives σ_7, σ'_7 for generators of $\pi_7(SO(8))$ are given by

$\sigma_7((q, r))(x, y)=(q, r)(x, y)=(qx-\bar{y}r, yq+r\bar{x})$,

$\sigma_7'((q, r))(x, y)=(x, y)\overline{(q, r)}=(x\bar{q}+\bar{r}y, -rx+yq)$ for Cayley numbers (q, r), (x, y) with $\|(q, r)\|=\|(x, y)\|=1$. Therefore we have

$$\sigma_7(q, r)=\begin{pmatrix} \sigma_3(q) & -{}^t\sigma_3'(r)C \\ \sigma_3(r)C & {}^t\sigma_3'(q) \end{pmatrix}, \quad \sigma_7'(q, r)=\begin{pmatrix} \sigma_3'(q) & {}^t\sigma_3(r) \\ -\sigma_3(r) & {}^t\sigma_3'(q) \end{pmatrix},$$

where $C=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, and ${}^tA$ denotes the transposed matrix of A.

Then

$$\sigma_7(S_{2,4}'(q, r))=S_{2,4}'\sigma_7(q, r){}^tS_{2,4}',$$

$$\sigma_7(S_{4,0}(q, r))=S_{4,0}\sigma_7(q, r){}^t(S_{1,6}')^2,$$

$$\sigma_7'(S_{2,4}'(q, r))=S_{2,4}'\sigma_7'(q, r){}^tS_{2,4}',$$

$$\sigma_7'(S_{4,0}(q, r))=\begin{pmatrix} \hat{D} & 0 \\ 0 & S_{2,0} \end{pmatrix}\sigma_7'(q, r)\begin{pmatrix} I_4 & 0 \\ 0 & {}^t(S_{1,6})^2 \end{pmatrix}, \text{ where } \hat{D}=\begin{pmatrix} {}^tD(t) & 0 \\ 0 & D(t) \end{pmatrix}.$$

Now we consider $S^3=Sp(1)$ actions. By $(q'q, r)(x, y)=(q'qx-\bar{y}r, yq'q+r\bar{x})$,

it follows that $\sigma_7(q'q, r)(x, y)=\begin{pmatrix} \sigma_3(q')\sigma_3(q) & -{}^t\sigma_3'(r)C \\ \sigma_3(r)C & {}^t\sigma_3'(q){}^t\sigma_3'(q') \end{pmatrix}$

$$=\begin{pmatrix} \sigma_3(q') & 0 \\ 0 & I_4 \end{pmatrix}\sigma_7(q, r)\begin{pmatrix} I_4 & 0 \\ 0 & {}^t\sigma_3'(q') \end{pmatrix}, \quad \text{for } q'\overline{yq'}r=\bar{y}r.$$

By $(q'q, q'r)(x, y)=(q'qx-\bar{y}q'r, yq'q+q'r\bar{x})$,

$$\sigma_7(q'q, q'r)=\begin{pmatrix} \sigma_3(q')\sigma_3(q) & -{}^t\sigma_3'(r){}^t\sigma_3'(q')C \\ \sigma_3(q')\sigma_3(r)C & {}^t\sigma_3'(q){}^t\sigma_3'(q') \end{pmatrix}$$

$$=\begin{pmatrix} \sigma_3(q') & 0 \\ 0 & \sigma_3(q') \end{pmatrix}\begin{pmatrix} \sigma_3(q) & -{}^t\sigma_3'(r)C \\ \sigma_3(r)C & {}^t\sigma_3'(q) \end{pmatrix}\begin{pmatrix} I_4 & 0 \\ 0 & C{}^t\sigma_3'(q'){}^t\sigma_3(q')C \end{pmatrix},$$

and by $(x, y)\overline{(q'q, r)}=(x\bar{q}\bar{q}'+\bar{r}y, -rx+yq'q)$, for $q'\bar{q}'\bar{y}q'q=yq'q$, it follows that

$$\sigma_7'(q'q, r)=\begin{pmatrix} \sigma_3'(q')\sigma_3'(q) & {}^t\sigma_3(r) \\ -\sigma_3(r) & {}^t\sigma_3'(q){}^t\sigma_3'(q') \end{pmatrix}$$

$$=\begin{pmatrix} \sigma_3'(q') & 0 \\ 0 & I_4 \end{pmatrix}\begin{pmatrix} \sigma_3'(q) & {}^t\sigma_3(r) \\ -\sigma_3(r) & {}^t\sigma_3'(q) \end{pmatrix}\begin{pmatrix} I_4 & 0 \\ 0 & {}^t\sigma_3'(q') \end{pmatrix}, \quad \text{for } \bar{r}yq'\bar{q}'=\bar{r}y.$$

By $(x, y)(\overline{q'q, q'r})=(x\overline{qq'}+\bar{r}\bar{q}'y, -q'rx+yq'q)$, it follows that

$$\sigma'_7(q'q, q'r)=\begin{pmatrix} \sigma'_3(q')\sigma'_3(q) & {}^t\sigma_3(r){}^t\sigma_3(q') \\ -\sigma_3(q')\sigma_3(r) & {}^t\sigma'_3(q){}^t\sigma'_3(q') \end{pmatrix}$$

$$=\begin{pmatrix} \sigma'_3(q') & 0 \\ 0 & \sigma_3(q') \end{pmatrix}\begin{pmatrix} \sigma'_3(q) & {}^t\sigma_3(r) \\ -\sigma_3(r) & {}^t\sigma'_3(q) \end{pmatrix}\begin{pmatrix} I_4 & 0 \\ 0 & {}^t\sigma'_3(q'){}^t\sigma_3(q') \end{pmatrix}$$

for $\bar{r}\bar{q}'yq'\bar{q}'=\bar{r}\bar{q}'y$ and $q'\bar{q}'yq'q=yq'q$.

NOTE. Let G be a compact Lie group and S^n be a G-sphere. Let D: $G\to SO(k)$ be a homomorphism. Suppose that a map χ: $S^n\to SO(2k)$ satisfies

$$\chi(gx)=\begin{pmatrix} D(g) & 0 \\ 0 & I_k \end{pmatrix}\chi(x)\begin{pmatrix} I_k & 0 \\ 0 & {}^tD(g) \end{pmatrix} \quad \text{for} \quad g\in G.$$

Since $\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}\begin{pmatrix} I_k & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}=\begin{pmatrix} D & 0 \\ 0 & I_k \end{pmatrix}$, the map $\chi'=\chi\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$ satisfies $\chi'(gx)$ $=\begin{pmatrix} D(g) & 0 \\ 0 & I_k \end{pmatrix}\chi'(x)\begin{pmatrix} {}^tD(g) & 0 \\ 0 & I_k \end{pmatrix}$, and obviously χ' is homotopic to χ.

§ 3. Equivariance of Bott maps

(1) Unitary groups $U(n)$

Let W_n be the standard complex $U(n)$ module and V_2^0 be a 2-dimensional real module. We choose basis for W_n and V_2^0. Then the map λ_C: $U(n)\to G_n(C^{2n})$ in (4.5) of [1] can be described as follows:

$$\lambda_C(A, \phi)=\begin{pmatrix} (\cos^2\phi/2)I_n & (\sin\phi/2\cos\phi/2)\bar{A} \\ (\sin\phi/2\cos\phi/2){}^tA & (\sin^2\phi/2)I_n \end{pmatrix} \quad \text{for} \quad \begin{matrix} A\in U(n) \\ 0\leqq\phi\leqq\pi \end{matrix}$$

$$=\begin{pmatrix} (\cos^2\phi/2)I_n & -(\sin\phi/2\cos\phi/2)I_n \\ -(\sin\phi/2\cos\phi/2)I_n & (\sin^2\phi/2)I_n \end{pmatrix} \quad \text{for} \quad \begin{matrix} A\in U(n) \\ \pi\leqq\phi\leqq 2\pi \end{matrix}.$$

Further,

$$\lambda_C(SAS^{-1}, \phi)=\begin{pmatrix} \bar{S} & 0 \\ 0 & S \end{pmatrix}\lambda_C(A, \phi)\begin{pmatrix} \bar{S}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix},$$

and the map f_C: $G_n(C^{2n})\to\Omega U(2n)$ is given by

$$f_C(P, \theta)=Pe^{i\theta}+(1-P)e^{-i\theta}.$$

Hence

$$f_C(\lambda_C(SAS^{-1}, \phi), \theta)=\begin{pmatrix}\bar{S} & 0\\ 0 & \bar{S}\end{pmatrix} f_C(\lambda_C(A, \phi), \theta)\begin{pmatrix}\bar{S} & 0\\ 0 & \bar{S}\end{pmatrix}^{-1}.$$

Thus we have proved

PROPOSITION 4. *Let* χ: $S^k\to U(n)$ *be an equivariant map of type* (S, S). *Then the map*: $E^2S^k\to U(2n)$ *which corresponds to* $\Omega f_C\circ\lambda\circ\chi$ *is an equivariant map of type* $\left(\begin{pmatrix}\bar{S} & 0\\ 0 & \bar{S}\end{pmatrix}, \begin{pmatrix}\bar{S} & 0\\ 0 & \bar{S}\end{pmatrix}\right)$, *where* E^2 *denotes the double suspension.*

REMARK. If the fixed point set of S^k is an m-sphere S^m for some $m\geqq 1$, then we obtain a homomorphism b: $[S^k, U(n)]\to[E^2S^k, U(2n)]$.

(2) Orthogonal groups $O(n)$.

According to the notations in [1], the map $\varepsilon_R^H\cdot\lambda_R$: $O(n)\to\Omega G_n(H^{2n})$, say λ, is given by

$$\lambda(A, \phi)=\begin{pmatrix}(\cos^2\phi/2)I_n & (\sin\phi)A\\ (\sin\phi)^tA & (\sin^2\phi/2)I_n\end{pmatrix} \quad \text{for} \quad \begin{matrix}A\in 0(n)\\ 0\leqq\phi\leqq\pi\end{matrix}.$$

We have $\lambda(TAS^{-1}, \phi)=\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}\lambda(A, \phi)\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}^{-1}$ for T, $S\in O(n)$. Further we use the following maps given in §6 of [1],

$$f_{1,\theta}=f_1(\ ,\theta):\quad G_n(H^{2n})\ni P\longrightarrow u=Pe^{i\theta/2}+(1-P)e^{-i\theta/2}\in U(4n),$$

$$\hat{f}_{2,\theta}=f_2(\ ,\Theta):\quad U(4n)\ni u\longrightarrow g=ue_r^{j\theta/2}u^{-1}\in SO(8n),$$

$f_{3,x}=f_3(\ , x)$: $SO(8n)\ni g\to ge_r^{ix}g^{-1}$, where e_r denotes the right multiplication. Since $\varepsilon_H^R\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}$ commutes with $e_r^{j\theta}/2$ and e_r^{ix}, we have

$$f_{3,x}\hat{f}_{2,\theta}f_{1,\theta}\lambda(TAS^{-1}, \phi)=\varepsilon_H^R\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}f_{3,x}\hat{f}_{2,\theta}f_{1,\theta}\lambda(A, \phi)\varepsilon_H^R\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}^{-1}.$$

Thus we have proved

PROPOSITION 5. *Let* χ: $S^k\to O(n)$ *be an equivariant map of type* (T, S). *Then The map* $E^4S^k\to SO(8n)$ *which corresponds to* $\Omega^3f_3\circ\Omega^2\hat{f}_2\circ\Omega f_1\circ\lambda$ *is an equivariant map of type* $\left(\varepsilon_H^R\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}, \varepsilon_H^R\begin{pmatrix}T & 0\\ 0 & S\end{pmatrix}\right)$.

Next we have

PROPOSITION 6. *Let $7 \leqq k < n$ and suppose that the forgetful map ψ: $[S^k, O(n)] \to \pi_k(O(n))$ is epic. Then ψ: $[E^4S^k, SO(8n)] \to \pi_{k+4}(SO(8n))$ is epic mod torsion.*

PROOF. It is known that ε_H^R: $\pi_{4k+3}(Sp(2n)) \to \pi_{4k+3}(SO(8n))$ is isomorphic for even k and image $\varepsilon_H^R \supset 4\pi_{4k+3}(SO(8n))$ for odd k. Then the proposition is obtained by the commutative diagram

$$\begin{array}{ccc}
\Omega^4 Sp(2n) & \xrightarrow{\varepsilon_H^R} & \Omega^4 SO(8n) \\
\uparrow & & \uparrow{\scriptstyle \Omega^3 f_3} \\
\Omega^3(Sp(2n)/U(2n)) & & \Omega^3(SO(8n)/U(4n)) \\
\uparrow & & \uparrow{\scriptstyle \Omega^2 f_2} \\
\Omega^2(U(2n)/O(2n)) & & \Omega^2(U(4n)/Sp(2n)) \\
\uparrow & & \uparrow{\scriptstyle \Omega^2 f_1} \\
G_n(R^{2n}) & \xrightarrow{\varepsilon_H^R} & G_n(H^{2n}). \\
\uparrow{\scriptstyle \lambda_R} & \nearrow{\scriptstyle \lambda} & \\
O(n) & &
\end{array}$$

§4. Applications

(1) Non existence

Let $S^{(k)}$ be the $8k \times 8k$ matrix with k-times of $\begin{pmatrix} S & 0 \\ 0 & I_4 \end{pmatrix}$ on the diagonal, where S is the matrix $S_{2,0} = \begin{pmatrix} {}^tD(t) & 0 \\ 0 & D(t) \end{pmatrix}$, (3) in §2. We define an action of S^1 on the group $SO(8k)$ by

$$SO(8k) \ni A \longrightarrow S^{(k)}A(S^{(k)})^{-1}.$$

Then we have

PROPOSITION 7. *Let $k \geqq 2$. Then the group $\psi([S^{8k-1}_{4k-1}, SO(8k)]$ is a torsion group* mod (τ) *in $\pi_{8k-1}(SO(k))$, where τ is the class of the characteristic map of the tangent bundle of S^{8k}.*

PROOF. By (3) in §2, ψ: $[S^7_3, SO(8)] \to \pi_7(SO(8))$ is an epimorphism mod (τ). Hence it follows from (2) of §3 that ψ: $[S^{11}_7, SO(64)] \to \pi_{11}(SO(64))$ is an epimorphism mod torsion. By Proposition 1, we have a commutative diagram

$$
\begin{array}{ccccc}
\pi_{10}(SO(64))=0 & & \pi_8(SO(64))=Z_2 & & \\
\downarrow & & \downarrow & & \\
[S_7^{11}, SO(64)] & \xrightarrow{\beta} & [S_7^9, SO(64)] & \xrightarrow{\beta} & \pi_7(SO(32)\times U(16)) \\
\downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\
\pi_{11}(SO(64)) & i_* & \pi_9(SO(64))=Z_2 & i_* & \pi_7(SO(64)) \\
 & & & & \\
\pi_{10}(SO(16))=0 & & \pi_8(SO(16))=Z_2 & & \\
\downarrow & & \downarrow & & \\
[S_7^{11}, SO(16)] & \xrightarrow{\beta} & [S_7^9, SO(16)] & \xrightarrow{\beta} & \pi_7(SO(8)\times U(4)) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_{11}(SO(16)) & & \pi_9(SO(16))=Z_2 & & \pi_7(SO(16))
\end{array}
$$

Let i_{0*}: $\pi_7(SO(8))\to\pi_7(SO(32))$ be the epimorphism which is induced from the inclusion map $SO(8)\subset SO(32)$, and ι_7 be the generator of the stable group $\pi_7(U(n))$, $n\geqq 4$. Then there exists an element $x\in[S_7^{11}, SO(64)]$ such that $\beta^2(x)=2i_{0*}(\sigma_7)-\iota_7$ and $\psi(x)$ is a non zero multiple of the generator of $\pi_{11}(SO(64))$. Similarly there exists an element $x_1\in[S_7^{11}, SO(16)]$ such that $\beta^2(x_1)=2\sigma_7-\iota_7$ in $\pi_7(SO(8)\times U(4))$. Since β^2: $[S_7^{11}, SO(64)]\to\mathrm{Ker}\,\psi\subset\pi_7(SO(32)\times U(16))$ is an isomorphism mod torsion, by the commutative diagram

$$
\begin{array}{ccc}
[S_7^{11}, SO(64)] & \xrightarrow{\psi} & \pi_{11}(SO(64)) \\
\uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} \\
[S_7^{11}, SO(16)] & \xrightarrow{\psi} & \pi_{11}(SO(16)),
\end{array}
$$

$\psi(x_1)$ is non zero multiple of the generator of $\pi_{11}(SO(16))$. Therefore ψ: $[S_7^{11}, SO(16)]\to\pi_{11}(SO(16))$ is an isomorphism mod torsion. Now let $k\geqq 2$ be even and $N=(k-2)/2$. By Proposition 6 and the commutitive diagram

$$
\begin{array}{ccccccc}
[S^1*S^1*S^{4k-1}, SO(16N)] & & & \xrightarrow{\beta^2} & \pi_{4k-1}(SO(8N)\times U(4N)) & & \\
\uparrow{\scriptstyle i_*} & \searrow{\scriptstyle\psi} & & & \uparrow{\scriptstyle i_*} & \searrow{\scriptstyle\psi} & \\
 & & \pi_{4k+3}(SO(16N)) & & & & \pi_{4k-1}(SO(16N)) \\
[S^1*S^1*S^{4k-1}, SO(8k)] & & \uparrow{\scriptstyle i_*} & \xrightarrow{\beta^2} & \pi_{4k-3}(SO(4k)\times(2k)) & & \\
 & \searrow{\scriptstyle\psi} & & & & \searrow{\scriptstyle\psi} & \\
 & & \pi_{4k+3}(SO(8k)) & & & & \pi_{4k-1}(SO(8k)),
\end{array}
$$

we obtain the result in Proposition 7 for the case where k is even. For odd k, by a similar argument, we can complete the proof.

(2) Equivariant Hopf constructions

Let G be a compact Lie group and μ: $G\times S^k\to S^k$ an action, and χ: $S^k\to SO(n)$ an equivariant map of type (T, S), where T, S: $G\to SO(n)$ are homomorphisms. Then the map f: $S^k\times S^{n-1}\to S^{n-1}$ defined by $f(x, y)=\chi(x)y$ for $x\in S^k$, $y\in S^{n-1}$ is also equivariant with respect to obvious actions. Therefore the Hopf construction $G(f)$: $S^k * S^{n-1}\to ES^{n-1}=S^n$ is an equivariant map. Suppose that the fixed point set of S^k is an m-sphere S^m for some $m\geqq 1$ and $T=S$. Then the set $[S^k, SO(n)]$ admits a group structure and the map

$$J_G\colon [S^k, SO(n)]\ni[\chi]\longmapsto [G(f)]\in[S^{k+n}, S^n]$$

is a homomorphism, i.e. an equivariant J-homomorphism.

Example. By (3) in §3 we have an equivariant J-homomorphism J_{S^1}: $[S^7_3, SO(8)]\to[S^{15}, S^8]$. Consider the commutative diagram

$$\begin{array}{ccc} [S^7_3, SO(8)] & \xrightarrow{J_G} & [S^{15}, S^8] \\ \downarrow \psi & & \downarrow \psi \\ \pi_7(SO(8)) & \xrightarrow{J} & \pi_{15}(S^8), \end{array} \quad G=S^1,$$

where J is the usual J-homomorphism. Since σ'_7 is in the ψ image, $\psi([S^{15}, S^8])$ includes the element of Hopf invariant one in $\pi_{15}(S^8)$.

(3) Lifting actions on complex plane bundles over the complex projective plane. Let CP^n be the n dimensional complex projective space. We have a cofibration $CP^1\overset{i}{\subset} CP^2\xrightarrow{q} S^4$. The map q is given by

$$q([z_1, z_2, z_3])=(2\bar{z}_3 z_1, 2\bar{z}_3 z_2, 1-2|z_3|^2) \qquad \text{for} \quad [z_1, z_2, z_3]\in CP^2.$$

We consider the S^1 action $S'_{1,2}$ on S^3, (3) in §2. Then we have the S^1 action on S^4 given by (trivial one)$\oplus\rho_{S^1}$. Here we quote the note (3) in §2. It is easy to see that the action admits a lifting on CP^2. Then we have

Proposition 8. *For any complex plane bundle E, the bundle $E\oplus\underline{C}$ admits a lifting action.*

Proof. The first Chern class C_1 $(E\oplus(\det E)^{-1})=0$. Then we have a complex plane bundle $E_1\to S^4$ such that $E\oplus(\det E)^{-1}$ is isomorphic to $q*E_1\oplus\underline{C}$. Then we have an isomorphism

$$E\oplus(\det E)^{-1}\oplus(\det E)\cong q*E_1\oplus(\det E)\oplus\underline{C},$$

and hence

$$E\oplus\underline{C}\cong q*E_1\oplus(\det E),$$

where the right hand side admits a lifting. Hence we have the result of Proposition 8.

NOTE. Considering the bundle $E\otimes(\det E)^{-1}$, it is easy to see that if the first Chern class $C_1(E)$ is even then the bundle E admits a lifting. In the case C_1 odd, I do not know whether there exists such a bundle that can not admit any lifting or not.

References

[1] R. Bott, Quelques remarques sur les théorèmes de periodicité, Bull. Soc. math. France, **87** (1959), 293–310.

[2] G. Bredon, Homotopical properties of fixed point sets of circle group actions I, Amer. J. Math. **91** (1969), 874–888.

[3] J. Folkman, Equivariant maps of spheres into the classical groups, Mem. of Amer. Math. Soc., 1971.

[4] H. Matsunaga and H. Minami, Forgetful homomorphisms in equivariant K-theory, Publ. Res. Inst. Math. Sci. Kyoto Univ. **22** (1) (1986), 143–150.

[5] H. Matsunaga, Note on bundles over spheres with group actions, Mem. Fac. Sci. Shimane Univ. **20** (1986), 21–24.