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Free Products and Amalgamation of Generalized Inverse $*$-Semigroups

Dedicated to Professor Hisao Tominaga on his 60th birthday

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The author showed that the class of [left, right] normal bands has the stong amalgamation property [4]. In his paper [1], T. E. Hall proved that the class of [left, right] generalized inverse semigroups has the strong amalgamation property by using the concept of a bundled semilattice which is firstly introduced in [7]. By similar argument, we show that the classes of normal $*$-bands and generalized inverse $*$-semigroups have the strong amalgamation property. Also we give the structures of normal $*$-bands and generalized inverse $*$-semigroups and the free products in the variety of normal $*$-bands.

§ 1. Introduction

Let $\mathscr{A}$ be a class of algebras. For a family of algebras $\{A_i: i \in I\}$ from $\mathscr{A}$, each having an algebra $U \in \mathscr{A}$ as a subalgebra, the list $(\{A_i: i \in I\}; U)$ is called an *amalgam* from $\mathscr{A}$. We say that an amalgam $(\{A_i: i \in I\}; U)$ is *strongly embeddable* (or is *strongly embedded*) in an algebra B if there are $\mathscr{A}$-monomorphisms $\phi_i: A_i \to B$, $i \in I$, such that

(i) $\phi_i | U = \phi_j | U$ for all $i, j \in I$,

(ii) $A_i\phi_i \cap A_j\phi_j = U\phi_i$ for all $i, j \in I$ with $i \neq j$,

where $\phi_i | U$ denotes the restriction of ϕ_i to U. We say that a class $\mathscr{A}$ of algebras has *the strong amalgamation property* if every amalgam from $\mathscr{A}$ is strongly embeddable in an algebra from $\mathscr{A}$. If $A_i = A_j$ for all $i, j \in I$, $(\{A_i: i \in I\}; U)$ is called a *special amalgam* from $\mathscr{A}$. We say that $\mathscr{A}$ has *the special amalgamation property* if each special amalgam from $\mathscr{A}$ is strongly embeddable in an algebra from $\mathscr{A}$. It is well-known (see [9]) that in a class of algebras closed under isomorphisms and the formation of the union of any ascending chain of algebras, each amalgamation property follows from the case in which $|I| = 2$. If $|I| = 2$, we write an amalgam $(\{A_1, A_2\}; U)$ by $(A_1, A_2; U)$.

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies

(i) $(x^*)^* = x$,

(ii) $(xy)^* = y^*x^*$,

(iii) $xx^*x = x$.

Let S and T be regular $*$-semigroups. A homomorphism $\phi: S \to T$ is called a *$*$-homomorphism* if $x^*\phi = (x\phi)^*$ for all $x \in S$. A relation θ on S is called a *$*$-relation* if $(x, y) \in \theta$ implies $(x^*, y^*) \in \theta$. An idempotent e in S is called a *projection* if $e^* = e$. By $P(S)$ and $E(S)$, we denote the sets of projections and idempotents, respectively, of S. The following result is used frequently throughout this paper.

RESULT 1. ((i) due to [10] and (ii) due to [8]). *Let S be a regular $*$-semigroup. Then we have*

(i) *each $\mathscr{L}$-class and each $\mathscr{R}$-class contain one and only one projection,*

(ii) *$E(S) = P(S)^2$. More precisely, for any idempotent e, there exist projections f and g such that $e\mathscr{R}f$, $e\mathscr{L}g$ and $e = fg$.*

A regular $*$-semigroup is called an *orthodox $*$-semigroup* [*$*$-band*] if it is an orthodox semigroup [band]. An orthodox $*$-semigroup [$*$-band] S is called a *generralized inverse $*$-semigroup* [*normal $*$-band*] if $E(S)$ [S itself]satisfies the identity $axya = ayxa$. In section 2, we determine the structures of normal $*$-bands and generalized inverse $*$-semigroups by using a transitive system of mappings for the set of projections, which are slightly different from [11] and [13].

The author gave useful forms of the free products in the variety of [left, right] normal bands [3]. In section 3, we give the free products in the varieties of normal $*$-bands.

For an amalgam $(E_1, E_2; U)$ of semilattices, a semilattice E is called a *bundled semilattice* of it if $(E_1, E_2; U)$ is strongly embedded in E by monomorphisms ϕ_i: $E_i \to E$, $i = 1, 2$, say, such that for $e_i \in E_i$ and $e_j \in E_j$ with $i \neq j$, if $e_i\phi_i \leq e_j\phi_j$ (in E) then $e_i \leq u$ (in E_i) and $u \leq e_j$ (in E_j) for some $u \in U$. Let $(S_1, S_2; U)$ be an amalgam of inverse semigroups. Then the amalgamated product $E(S_1)*_{E(U)}E(S_2)$ in the class of semilattices is a bundled semilattice of $(E(S_1), E(S_2); E(U))$ [7], but $E(S_1)*_{E(U)}E(S_2)$ does not embed in the amalgamated product $S_1 *_U S_2$ in the class of inverse semigroups, in general [1]. In [7], the author showed that if the following result is true, the class of [left, right] generalized inverse semigroups has the strong amalgamation property. And Hall has proved the following.

RESULT 2 (due to [1]). *Any amalgam $(S, T; U)$ of inverse semigroups is strongly embedded in an inverse semigroup A such that $E(A)$ is a bundled semilattice of the amalgam $(E(S), E(T); E(U))$ of semilattices.*

In section 4, we show that the varieties of normal $*$-bands and generalized inverse $*$-semigroups have the strong amalgamation property.

The notation and terminology are those of [2] and [12], unless otherwise stated.

§ 2. Structures

Let Δ be a semilattice and P a disjoint union of subsets P_α, $\alpha \in \Delta$. For α, $\beta \in \Delta$

with $\alpha \geq \beta$, let $\tau_{\alpha,\beta}: P_\alpha \to P_\beta$ be a mapping. The family of mappings $\{\tau_{\alpha,\beta}: \alpha \geq \beta, \alpha, \beta \in \Delta\}$ is called a *transitive system of mappings for* $P = \bigcup_{\alpha \in \Delta} P_\alpha$ if it satisfies

(i) $\tau_{\alpha,\alpha}$ is the identity mapping of P_α for all $\alpha \in \Delta$,

(ii) $\tau_{\alpha,\beta}\tau_{\beta,\gamma} = \tau_{\alpha,\gamma}$ for $\alpha, \beta, \gamma \in \Delta$ with $\alpha \geq \beta \geq \gamma$.

THEOREM 1. *Let Δ be a semilattice and $\{\tau_{\alpha,\beta}: \alpha \geq \beta, \alpha, \beta \in \Delta\}$ a transitive system of mappings for $P = \bigcup_{\alpha \in \Delta} P_\alpha$. Let $B = \{(e, \alpha, f): e, f \in P_\alpha, \alpha \in \Delta\}$ and define its multiplication and unary operation by*

$$(e, \alpha, f)(g, \beta, h) = (e\tau_{\alpha,\alpha\beta}, \alpha\beta, h\tau_{\beta,\alpha\beta}),$$

$$(e, \alpha, f)^* = (f, \alpha, e),$$

Then B becomes a normal $*$-band with $P(B) = \{(e, \alpha, e): \alpha \in \Delta\}$ and its structure semilattice is Δ. Hereafter, we denote it by $B \sim \mathscr{T}(\Delta, P; \{\tau_{\alpha,\beta}\})$.

Conversely, every normal $*$-band is obtained by this fashion.

PROOF. We can easily obtain the first half of the theorem. Let E be a normal $*$-band and $E \equiv \sum \{E_\alpha: \alpha \in \Delta\}$ its structure decomposition. For any $\alpha \in \Delta$, let $P_\alpha = P(E) \cap E_\alpha$ and $P = \bigcup_{\alpha \in \Delta} P_\alpha$. It is obvious $P = P(E)$. For any $\alpha, \beta \in \Delta$ with $\alpha \geq \beta$, define $\tau_{\alpha,\beta}$ as follows: let e be any element of P_α, and let a and b be any elements of E_β. Since E is a normal $*$-band, it is clear that $ea\mathscr{R}eb$. By Result 1 (i), there exists one and only one element $f \in P_\beta$ such that $f\mathscr{R}ea$. We define it by $e\tau_{\alpha,\beta} = f$. Then we can easily see that $\{\tau_{\alpha,\beta}: \alpha \geq \beta, \alpha, \beta \in \Delta\}$ is a transitive system for $P = \bigcup_{\alpha \in \Delta} P_\alpha$. Let $B \sim \mathscr{T}(\Delta, P; \{\tau_{\alpha,\beta}\})$ and define a mapping $\phi: E \to B$ by $a\phi = (aa^*, \alpha, a^*a)$ for $a \in E_\alpha$. By Result 1 (ii), ϕ is a bijection. Let $a \in E_\alpha$, $b \in E_\beta$. By the similar argument above, there are uniquely $e, f \in P_{\alpha\beta}$ such that $e\mathscr{R}(aa^*)x$ and $f\mathscr{L}x(b^*b)$ for any $x \in E_{\alpha\beta}$. Then

$$\begin{aligned}(ab)\phi &= ((aa^*)(a^*abb^*))((a^*ab^*b)(b^*b))\phi \\ &= ((eu)(uf))\phi \quad \text{for some } u \in P_{\alpha\beta}, \\ &= (ef)\phi \quad \text{since } E_{\alpha\beta} \text{ is rectangular,} \\ &= (e, \alpha\beta, f) \\ &= ((aa^*)\tau_{\alpha,\alpha\beta}, \alpha\beta, (b^*b)\tau_{\beta,\alpha\beta}) \\ &= (aa^*, \alpha, a^*a)(bb^*, \beta, b^*b) \\ &= (a\phi)(b\phi).\end{aligned}$$

Thus ϕ is an $*$-isomorphism, and hence we have the theorem.

Next, we shall determine the structure of a generalized inverse $*$-semigroup. If Γ is an inverse semigroup and $E(\Gamma) = \Delta$, we denote it by $\Gamma(\Delta)$.

THEOREM 2. *Let $\Gamma(\Delta)$ be an inverse semigroup, P a set of disjoint union of subsets P_α, $\alpha\in\Delta$, and let $\{\tau_{\alpha,\beta}: \alpha\geq\beta, \alpha, \beta\in\Delta\}$ be a transitive system of mappings for $P=\bigcup_{\alpha\in\Delta} P_\alpha$. Let*

$$S=\{(e, \alpha, f): \alpha\in\Gamma, e\in P_{\alpha\alpha^{-1}}, f\in P_{\alpha^{-1}\alpha}\},$$

and its multiplication and unary operation are defined by

$$(e, \alpha, f)(g, \beta, h)=(e\tau_{\alpha\alpha^{-1},\alpha\beta(\alpha\beta)^{-1}}, \alpha\beta, h\tau_{\beta^{-1}\beta,(\alpha\beta)^{-1}\alpha\beta}),$$

$$(e, \alpha, f)^*=(f, \alpha^{-1}, e).$$

Then S is a generalized inverse $*$-semigroup with $P(S)=\{(e, \alpha, e): \alpha\in\Delta, e\in P_\alpha\}$ and its structure inverse semigroup is $\Gamma(\Delta)$. Hereafter, we denote it by $S\sim\mathscr{T}(\Gamma(\Delta), P, \{\tau_{\alpha,\beta}\})$.

Conversely, every generalized inverse $*$-semigroup is obtained by this fashion.

PROOF. By Theorem 1 above and Lemma 5 and 6 in [12], we can easily obtain the theorem.

§3. Free products

In this section, we shall describe the free product of normal $*$-bands. Let $\{B_i: i\in I\}$ be a family of normal $*$-bands. By Theorem 1, $B_i\sim\mathscr{T}(\Delta_i, P_i, \{\tau^i_{\alpha,\beta}\})$ for all $i\in I$. To construct the free product of the B_i in the variety of normal $*$-bands, we can assume without loss of generality that $B_i\cap B_j=\Box$, $\Delta_i\cap\Delta_j=\Box$ and $P_i\cap P_j=\Box$ if $i\neq j$. Let $\Delta=\{(\alpha_i)_{i\in I}: \alpha_i\in\Delta_i^{(1)}, i\in I$, only finitely many but at least one α_i are different from 1$\}$, where $\Delta_i^{(1)}$ denotes $\Delta_i\cup\{1\}$ obtained by adjoining an identity 1 to Δ_i whether or not it already has an identity. It is well-known [5] that such Δ is the free product of the Δ_i in the variety of semilattices. For convenience, we write simply (α_i) instead of $(\alpha_i)_{i\in I}$. Let

$$B=\{(a, (\alpha_i), b): a\in P_{\alpha_j} \text{ and } b\in P_{\alpha_k} \text{ for some } \alpha_j\neq 1 \text{ and } \alpha_k\neq 1 \text{ of } (\alpha_i)\in\Delta\}.$$

If $a\in P_{\alpha_j}$ and $b\in P_{\alpha_k}$, we sometimes denote $(a, (\alpha_i), b)$ by $(a, (\alpha_i; j, k), b)$. Define a multiplication and a unary operation on B by

$$(a, (\alpha_i; j, k), b)(c, (\beta_i; m, n), d)=(a\tau^j_{\alpha_j,\alpha_j\beta_j}, (\alpha_i\beta_i; j, n), d\tau^n_{\beta_n,\alpha_n\beta_n}),$$

$$(a, (\alpha_i; j, k), b)^*=(b, (\alpha_i; k, j), a).$$

THEOREM 3. *B is the free product of the B_i in the variety of normal $*$-bands.*

PROOF. It is clear that B is a normal $*$-band. For each $i\in I$, define a mapping $\phi_i: B_i\to B$ by

$$(e, \alpha_i, f)\phi_i = (e, (\hat{\alpha}_i), f) \quad \text{if} \quad e, f \in P_{\alpha_i}, \ a_i \in \Delta_i,$$

where $(\hat{\alpha}_i)$ denotes the element of Δ with the i-th entry equal to α_i and others equal to 1. It is obvious that each ϕ_i is a *-monomorphism, $i \in I$, and that $B = \langle \cup \{B_i\phi_i : i \in I\}\rangle$, the *-subsemigroup generated by $\cup \{B_i\phi_i : i \in I\}$.

Let E be any normal *-band and $\psi_i : B_i \to E$, $i \in I$, any *-homomorphisms. Define a mapping $\mu: B \to E$ as follows:

For any $(a, (\alpha_i; j, k), b) \in B$ such that only $\alpha_j, \alpha_k, \alpha_{t_1}, \alpha_{t_2}, \dots, \alpha_{t_n}$ are not equal to 1,

$$(a, (\alpha_i; j, k), b)\mu = ((a, \alpha_j, c))\psi_j)((e_1, \alpha_{t_1}, f_1)\psi_{t_1}) \cdots\cdots((e_n, \alpha_{t_n}, f_n)\psi_{t_n})((d, \alpha_k, b)\psi_k),$$

where $c \in P_{\alpha_j}$, $d \in P_{\alpha_k}$ and $e_{t_s}, f_{t_s} \in P_{\alpha_{t_s}}$ for $s = 1, 2, \dots, n$.
It is clear that μ is a *-homomorphism satisfying $\phi_i\mu = \psi_i$, $i \in I$. Hence B together with the ϕ_i is the free product of the B_i in the variety of normal *-bands.

By the construction of the free product in the variety of normal *-bands, we can easily have the following

COROLLARY 4. *Let $\{B_i : i \in I\}$ be a family of normal *-bands, and let B together with the *-monomorphisms ϕ_i be the free product of the B_i in the variety of normal *-bands. If F_i is a *-subband of B_i, $i \in I$, then $\langle \cup \{F_i\phi_i : i \in I\}\rangle$ is the free product of the F_i in the variety of normal *-bands.*

§4. Amalgamation

Firstly, we shall show that the variety of normal *-bands has the strong amalgamation property. Let $(B_1, B_2; U)$ be an amalgam of normal *-bands. By Theorem 1,

$$B_i \sim \mathcal{T}(\Delta_i, P_i; \{\tau^i_{\alpha,\beta}\}), \quad P_i = \bigcup_{\alpha \in \Delta_i} P^i_\alpha, \quad i = 1, 2,$$

$$U \sim \mathcal{T}(\Lambda, Q; \{\tau_{\alpha,\beta}\}), \quad Q = \bigcup_{\alpha \in \Lambda} Q_\alpha.$$

We can assume without loss of generality that $\Delta_1 \cap \Delta_2 = \Lambda$, $P^1_\alpha \cap P^2_\alpha = Q_\alpha$ and $\tau^1_{\alpha,\beta} | Q_\alpha = \tau^2_{\alpha,\beta} | Q_\alpha = \tau_{\alpha,\beta}$ for all $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$. Let $P = P_1 \cup P_2$.

Take any bundled semilattice Ω of the amalgam $(\Delta_1, \Delta_2; \Lambda)$, and assume that Δ_1 and Δ_2 are subsemilattices of Ω such that $\Delta_1 \cap \Delta_2 = \Lambda$. Let $\Gamma = \{\alpha \in \Omega : \alpha \leq \beta$ for some $\beta \in \Delta_1 \cup \Delta_2\}$. It is obvious that Γ is an ideal of Ω and it is also a bundled semilattice of $(\Delta_1, \Delta_2; \Lambda)$. Let

$$B = \{(a, \alpha, b) \in P \times \Gamma \times P : a \in P^i_\beta, \ b \in P^j_\gamma, \ \beta \geq \alpha, \ \gamma \geq \alpha\},$$

and define a multiplication and a unary operation of B by

$$(a, \alpha, b)(c, \beta, d)=(a, \alpha\beta, d),$$

$$(a, \alpha, b)^{*}=(b, \alpha, a).$$

It is clear that B is a normal $*$-band whose structure semilattice is Γ. Define a relation θ_1 on B by $\theta_1=\{((x\tau^i_{\xi,\gamma_1}, \alpha, y\tau^j_{\eta,\gamma_2}), (x\tau^m_{\xi,\delta_1}, \alpha, y\tau^n_{\eta,\delta_2})): \alpha\in\Gamma,\ i, j, m, n\in\{1, 2\}, \xi\in\Delta_i\cap\Delta_m, \eta\in\Delta_j\cap\Delta_n, x\in P^i_\xi\cap P^m_\xi, y\in P^j_\eta\cap P^n_\eta, \xi\geq\gamma_1\geq\alpha, \eta\geq\gamma_2\geq\alpha, \xi\geq\delta_1\geq\alpha$ and $\eta\geq\delta_2\geq\alpha\}$. We can easily see that θ_1 is a reflexive, symmetric and compatible $*$-relation. Let $\theta=\theta_1^t$, the transitive closure of θ_1. Then θ is a $*$-congruence and B/θ is a normal $*$-band whose structure semilattice is Γ.

LEMMA 5. *Let $\alpha\in\Delta_i$, $i=1, 2$, $\beta\in\Gamma$, $a, b\in P^i_\alpha$, $c\in P^m_{\gamma_1}$ and $d\in P^n_{\gamma_2}$. If $(a, \alpha, b)\theta(c, \beta, d)$, then $\alpha=\beta$ and the following conditions are satisfied.*

(i) *there exist $\mu\in\Delta_i\cap\Delta_m$, $u\in P^i_\mu\cap P^m_\mu$ and $\nu\in\Delta_m$ such that $\mu\geq\nu\geq\alpha$ (in Γ), $\gamma_1\geq\nu$ (in Δ_m), $a=u\tau^i_{\mu,\alpha}$ and $c\tau^m_{\gamma_1,\nu}=u\tau^m_{\mu,\nu}$,*

(ii) *there exist $\varepsilon\in\Delta_i\cap\Delta_n$, $v\in P^i_\varepsilon\cap P^n_\varepsilon$ and $\delta\in\Delta_n$ such that $\varepsilon\geq\delta\geq\alpha$ (in Γ), $\gamma_2\geq\delta$ (in Δ_n), $b=v\tau^i_{\varepsilon,\alpha}$ and $d\tau^n_{\gamma_2,\delta}=v\tau^n_{\varepsilon,\delta}$.*

PROOF. We prove by induction. Assume that (c, β, d) satisfies the statement of the lemma and let $(c, \beta, d)\theta_1(e, \omega, f)$, $e\in P^j_{\rho_1}$, $f\in P^k_{\rho_2}$. Firstly, we prove (i). By the definition of θ_1, $\alpha=\beta=\omega$ and there exist $\xi\in\Delta_m\cap\Delta_j$ and $x\in P^m_\xi\cap P^j_\xi$ such that $\xi\geq\gamma_1$, $\xi\geq\rho_1$, $c=x\tau^m_{\xi,\gamma_1}$ and $e=x\tau^j_{\xi,\rho_1}$.

Case 1: $i=m=j$ (see Fig. 1). Since $x\in P^i_\xi$, we have

$$x\tau^i_{\xi,\alpha}=((x\tau^i_{\xi,\gamma_1})\tau^i_{\gamma_1,\nu})\tau^i_{\nu,\alpha}=(c\tau^i_{\gamma_1,\nu})\tau^i_{\nu,\alpha}$$

$$=(u\tau^i_{\mu,\nu})\tau^i_{\nu,\alpha}=a,$$

$$e\tau^i_{\rho_1,\rho_1\nu}=(x\tau^i_{\xi,\rho_1})\tau^i_{\rho_1,\rho_1\nu}=x\tau^i_{\xi,\rho_1\nu}.$$

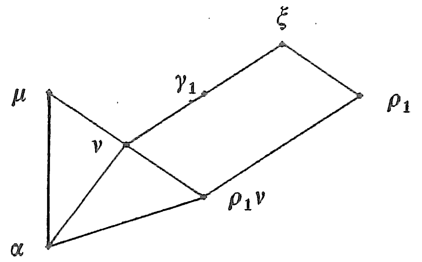

Fig. 1

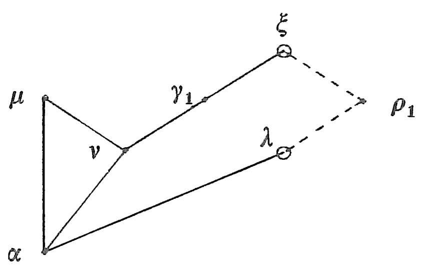

Fig. 2

Case 2: $i=m\neq j$ (see Fig. 2). Since $\rho_1 \geq \alpha$ (in Γ) and Γ is a bundled semilattice of $(\Delta_1, \Delta_2; \Lambda)$, there exists $\lambda \in \Lambda$ such that $\rho_1 \geq \lambda$ (in Δ_j) and $\lambda \geq \alpha$ (in Δ_i). Then

$$e\tau^j_{\rho_1,\lambda} = (x\tau^j_{\xi,\rho_1})\tau^j_{\rho_1,\lambda} = x\tau^j_{\xi,\lambda}.$$

By the same calculation in the case 1, we have $a = x\tau^i_{\xi,\alpha}$.

Case 3: $i\neq m=j$ (see Fig. 3). Since $\nu \geq \alpha$ (in Γ) and Γ is a boundled semilattice, there exists $\lambda \in \Lambda$ such that $\nu \geq \lambda$ (in Δ_m) and $\lambda \geq \alpha$ (in Δ_i). Then

$$x\tau^m_{\xi,\lambda} = ((x\tau^m_{\xi,\gamma_1})\tau^m_{\gamma_1,\nu})\tau^m_{\nu,\lambda} = (c\tau^m_{\gamma_1,\nu})\tau^m_{\nu,\lambda}$$

$$= (u\tau^m_{\mu,\nu})\tau^m_{\nu,\lambda} = u\tau^m_{\mu,\lambda} = u\tau_{\mu,\lambda} \in Q_\lambda,$$

since $u \in Q_\mu$ and $\mu, \nu \in \Lambda$. So we have

$$(u\tau_{\mu,\lambda})\tau^i_{\lambda,\alpha} = u\tau^i_{\lambda,\alpha} = a,$$

$$e\tau^m_{\rho_1,\rho_1\lambda} = (x\tau^m_{\xi,\rho_1})\tau^m_{\rho_1,\rho_1\lambda} = x\tau^m_{\xi,\rho_1\lambda}$$

$$= (x\tau^m_{\xi,\lambda})\tau^m_{\lambda,\rho_1\lambda} = (u\tau_{\mu,\lambda})\tau^m_{\lambda,\rho_1\lambda}.$$

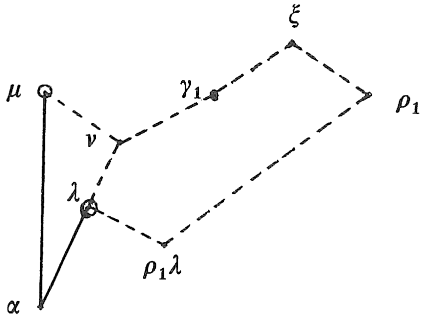

Fig. 3

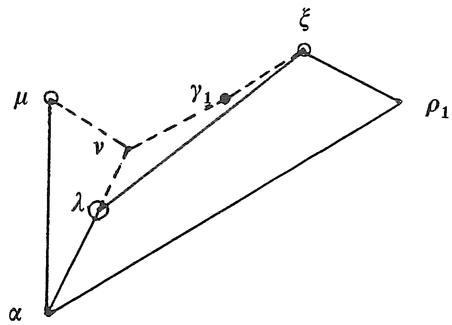

Fig. 4

Case 4: $i\neq m\neq j\ (=i)$ (see Fig. 4). Since $\nu \geq \alpha$ (in Γ) and Γ is a bundled semilattice, there exists $\lambda \in \Lambda$ such that $\nu \geq \lambda$ (in Δ_m) and $\lambda \geq \alpha$ (in Δ_i). Then

$$x\tau^m_{\xi,\alpha} = (x\tau_{\xi,\lambda})\tau^i_{\lambda,\alpha} = (((x\tau^m_{\xi,\gamma_1})\tau^m_{\gamma_1,\nu})\tau^m_{\nu,\lambda})\tau^i_{\lambda,\alpha}$$

$$= ((c\tau^m_{\gamma_1,\nu})\tau^m_{\nu,\lambda})\tau^i_{\lambda,\alpha} = ((u\tau^m_{\mu,\nu})\tau^m_{\nu,\lambda})\tau^i_{\lambda,\alpha}$$

$$= ((u\tau_{\mu,\lambda})\tau^i_{\lambda,\alpha} = u\tau^i_{\mu,\alpha} = a,$$

$$e\tau^i_{\rho_1,\alpha} = (x\tau^i_{\xi,\rho_1})\tau^i_{\rho_1,\alpha} = x\tau^i_{\xi,\alpha}$$

Thus we have (i). Similarly, we have (ii), and hence we obtain the lemma.

Let ϕ_i: $B_i \to B/\theta$, $i=1, 2$, be mappings defined by

$$(a, \alpha, b)\phi_i=(a, \alpha, b)\theta \qquad \text{for} \quad a, b \in P_\alpha^i, \quad \alpha \in \Delta_i.$$

By using Lemma 5, we can easily obtain that every ϕ_i is a $*$-monomorphism such that $\phi_1 | U=\phi_2 | U$ and $B_1\phi_1 \cap B_2\phi_2=U\phi_1$. And so $(B_1, B_2; U)$ is strongly embedded in B/θ whose structure semilattice is Γ. By the similar method of [6], we can embed E/θ into a normal $*$-band whose structure semilattice is Ω. Now we have the following theorem.

THEOREM 6. *Let $(B_1, B_2; U)$ be an amalgam of normal $*$-bands, and let*

$$B_i \sim \mathscr{T}(\Delta_i, P_i; \{\tau_{\alpha,\beta}^i\}), \quad P_i=\bigcup_{\alpha \in \Delta_i} P_\alpha^i, \quad i=1, 2,$$

$$U \sim \mathscr{T}(\Lambda, Q; \{\tau_{\alpha,\beta}\}), \quad Q=\bigcup_{\alpha \in \Lambda} Q_\alpha.$$

Let Ω be a bundled semilattice of the amalgam $(\Delta_1, \Delta_2; \Lambda)$. Then the amalgam $(B_1, B_2; U)$ is strongly embedded in a normal $*$-band E whose structure semilattice is Ω and $P_\alpha^i \subset R_\alpha$ for every $\alpha \in \Delta_i$, where $E \sim \mathscr{T}(\Omega, R; \{\tau'_{\alpha,\beta}\})$, $R=\bigcup_{\alpha \in \Omega} R_\alpha$. Therefore, the variety of normal $*$-bands has the strong amalgamation property.

REMARK 1. If the assumption that Ω is a bundled semilattice of $(\Delta_1, \Delta_2; \Lambda)$ is weakened to that $(\Delta_1, \Delta_2; \Lambda)$ is strongly embedded in Ω, the theorem above is not true (see [6]).

REMARK 2. We have another proof of that the variety of normal $*$-bands has the strong amalgamation property. Let $(B_1, B_2; U)$ be an amalgam of normal $*$-bands. Let $B=B_1 * B_2$, the free product of B_1 and B_2 in the variety of normal $*$-bands. We use the notation above. We denote $(\alpha_i)_{i \in \{1,2\}}$ in $\Delta_1 * \Delta_2$ by (α_1, α_2). Define a relation θ on B as follows:

For $\boldsymbol{a}, \boldsymbol{b} \in B$, $\boldsymbol{a}\,\theta_0 \boldsymbol{b}$ if and only if

$$\boldsymbol{a}=\boldsymbol{x}(u, (\sigma, 1), v)\boldsymbol{y},$$

$$\boldsymbol{b}=\boldsymbol{x}(u, (1, \sigma), v)\boldsymbol{y},$$

for some $\boldsymbol{x}, \boldsymbol{y} \in B^1$, $\sigma \in \Lambda$ and $u, v \in Q_\sigma$. Let $\theta_1=\theta_0 \cup \theta_0^{-1} \cup \iota$ and let $\theta=\theta_1^t$. Then B/θ is the free product of B_1 and B_2 amalgamating U in the variety of normal $*$-bands and its structure semilattice is the free product of Δ_1 and Δ_2 amalgamating Λ in the variety of semilattices.

A $*$-band is called *regular* if it satisfies the identity $axaya=axya$. The variety of regular $*$-bands has the special amalgamation property, but it does not have the strong amalgamation property.

COROLLARY 7. *Let $\mathscr{A}$ be the variety of $*$-bands defined by an identity $P=Q$. Then $\mathscr{A}$ has the strong amalgamation property if and only if $\mathscr{A}$ is one of the following varieties:*

(i) *one element semigroups;*
(ii) *semilattices;*
(iii) *rectangular $*$-bands;*
(iv) *normal $*$-bands.*

Next, we shall show that the variety of generalized inverse $*$-semigroups has the strong amalgamation property. Let $(S_1, S_2; V)$ be an amalgam of generalized inverse $*$-semigroups. By Theorem 2,

$$S_i \sim \mathscr{T}(\Gamma_i(\Delta_i), P_i, \{\tau^i_{\alpha,\beta}\}), \quad P_i = \bigcup_{\alpha\in\Delta_i} P^i_\alpha, \quad i=1, 2,$$

$$V \sim \mathscr{T}(\Xi(\Lambda), Q; \{\tau_{\alpha,\beta}\}), \quad Q = \bigcup_{\alpha\in\Lambda} Q_\alpha,$$

such that $\Gamma_1 \cap \Gamma_2 = \Xi$, $\Delta_1 \cap \Delta_2 = \Lambda$, $P^1_\alpha \cap P^2_\alpha = Q_\alpha$ and $\tau^1_{\alpha,\beta} | Q_\alpha = \tau^2_{\alpha,\beta} | Q_\alpha = \tau_{\alpha,\beta}$ for all $\alpha, \beta \in \Lambda$. Let $B_i \sim \mathscr{T}(\Delta_i, P_i; \{\tau^i_{\alpha,\beta}\})$, $i=1, 2$, and $U \sim \mathscr{T}(\Lambda, Q; \{\tau_{\alpha,\beta}\})$. Then $(B_1, B_2; U)$ is an amalgam of normal $*$-bands.

It follows from Result 2 that the amalgam $(\Gamma_1, \Gamma_2; \Xi)$ is strongly embedded in an inverse semigroup $\sum(\Omega)$ such that Ω is a bundled semilattice of $(\Delta_1, \Delta_2; \Lambda)$. By Theorem 6, the amalgam $(B_1, B_2; U)$ is strongly embedded in a normal $*$-band B whose structure semilattice is Ω. Let $B \sim \mathscr{T}(\Omega, P; \{\nu_{\alpha,\beta}\})$, and consider the generalized inverse $*$-semigroup $\mathscr{T}(\sum(\Omega), P, \{\nu_{\alpha,\beta}\}) \sim T$, say. Then it is clear that the amalgam $(S_1, S_2; V)$ is strongly embedded in T. Thus we have the main theorem.

THEOREM 8. *The variety of generalized inverse $*$-semigroups has the strong amalgamation property.*

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