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P-Regularity in Semigroups

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In this paper, firstly the concept of P-regularity, which is a generalization of both the concept of "orthodox" and the concept of "(special) involution" (see [7]), is introduced in the class of regular semigroups, and secondly the structure of P-regular semigroups is discussed.

§ 1. Introduction

Let S be a regular semigroup, and E_S the set of idempotemts of S. Let P be a subset of E_S such that $P^2 \subset E_S$ and $qP^1q \subset P$ for all $q \in P$, where P^1 is the adjunction of the identity 1 to P. Let $V(a)$ be the set of inverses of a for each $a \in S$. Let $Q_P(S) = \{a \in S$: There exists $a^* \in V(a)$ such that $aP^1a^* \subset P$ and $a^*P^1a \subset P\}$. Then, it is easy to see that $Q_P(S)$ satisfies the following:

(C.1.1) (1) $Q_P(S)$ is regular subsemigroup of S.
(2) $P \subset Q_P(S)$.
(3) Every (Green's) L-class [R-class] of $Q_P(S)$ contains at least one element of P.
(4) For any $a \in Q_P(S)$, there exists $a^* \in V(a)$ such that $a^* \in Q_P(S)$ and aP^1a^*, $a^*P^1a \subset P$.

Let T be a regular semigroup, and P a subset of E_T. If T satisfies

(C.1.2) (1) each L-class of T contains an element of P; and each R-class of T contains an element of P,
(2) $P^2 \subset E_T$, and
(3) for $q \in P$, $qP^1q \subset P$,

then T is called *a weakly P-regular semigroup*. For $a \in T$, an element $a^* \in T$ such that $a^* \in V(a)$, $aP^1a^* \subset P$ and $a^*P^1a \subset P$ is called *a P-inverse of a* (of course, such a P-inverse a^* not necessarily exists for a given $a \in T$), and (a, a^*) is called *a P-regular pair*. Let $V_P(a)$ be the set of all P-inverses of a.

If a weakly P-regular semigroup T further satisfies

(C.1.3) for any $a \in T$, $V_P(a) \neq \square$,

then T is called *a P-regular semigroup.*

The subsemigroup $Q_P(S)$ of S is a P-regular semigroup. Hence, it is obvious that if S is a weakly P-regular semigroup then $Q_P(S)$ is a full P-regular subsemigroup of S. It is easily seen as follows: It is already seen that $Q_P(S)$ is P-regular. Let $f \in E_S$. Then, there exist $p, q \in P$ such that $f \mathcal{L} p$, $f \mathcal{R} q$, where $\mathcal{L}$ and $\mathcal{R}$ are Green's L- and R-relations respectively. Now, $pq \in E_S$, and $pq \mathcal{L} q$, $pq \mathcal{R} p$. Hence, $qp = f$. Thus, $E_S \subset P^2$. This implies $P^2 = E_S$, and hence $E_S \subset Q_P(S)$.

Further, in this case $Q_P(S)$ is the greatest P-regular subsemigroup of S. If a regular semigroup S is [weakly] P-regular for a subset P of E_S, then the set P is called *a* [*weak*] *characteristic set* (abbrev., *a* [*weak*] *C-set*) in S. This concept of a C-set is a generalization of both the concept of a P-system (see [8]) in a regular semigroup and the concept of the set of projections (see [6]) in a regular *-semigroup. In Nordahl and Scheiblich [4], it has been firstly noted that every L-class [R-class] of a regular *-semigroup contains just one projection.

If $\{P_i : i \in I\} = \Omega$ is a set of [weak] C-sets in a regular semigroup S, then S is called [*weakly*] $\{P_i : i \in I\}$*-regular* or, sometimes simply, [*weakly*] Ω*-regular.*

EXAMPLE. Let S be a regular semigroup, and let $P \subset E_S$. Consider the following special cases:

(C.1.4) $(1)_l$ Each L-class of S contains just one element of P.
$(1)_r$ Each R-class of S contains just one element of P.
(2) $P = E_S$.

If S is P-regular, and P satisfies

I	(C.1.4), (2)
II	〃, $(1)_l$, $(1)_r$
III	〃, (2), $(1)_l$[(2), $(1)_r$]
IV	〃, (2), $(1)_l$, $(1)_r$

, then S is

I	orthodox
II	a regular *-semigroup having P as its projections (see [8])
III	a left inverse [a right inverse] semigroup (see [5])
IV	an inverse semigroup

respectively.

Further, it is easy to see that if S is a $\{P_1, P_2\}$-regular semigroup and if P_1 and P_2 satisfy (C.1.4), (2) and (C.1.4), $(1)_l$, $(1)_r$ respectively, then S is an orthodox *-semigroup having P_2 as its projections.

As was seen in Example above, the class of orthodox semigroups and the class of regular *-semigroups are contained in the class of P-regular semigroups. In this paper, we shall study the structure of P-regular semigroups, and also that of weakly P-regular semigroups.

§ 2. Basic properties

As was shown above, if S is a weakly P-regular semigroup, $Q_P(S)=\{a \in S: a$ has a P-inverse$\}$ is the greatest P-regular subsemigroup of S and contains E_S. This $Q_P(S)$ is called *the inner part of* S, and in particular it is denoted by $N_P(S)$.

THEOREM 2.1. *The inner part $N_P(S)$ of a weakly P-regular semigroup is P-regular. Accordingly, every P-regular semigroup can be obtained as the inner part of a weakly P-regular semigroup.*

PROOF. Obvious.

PROPOSITION 2.2. *For a P-regular semigroup S, the following two conditions are equivalent*:

(1) $P=E_S$.

(2) $P^2 \subset P$.

PROOF. Assume that $P^2 \subset P$. For $f \in E_S$, there exists $f^* \in V_P(f)$. Hence, $f^*=f^*fff^* \in P^2 \subset P \subset E_S$. Thus, $f=ff^*f^*f \in P$. Hence, $P=E_S$. Conversely, if $P=E_S$ then $P^2 \subset E_S=P$.

PROPOSITION 2.3. *Let f: $S \rightarrow T$ be a homomorphism of a [weakly] P-regular semigroup S onto a regular semigroup T. Let $\bar{P}=\{qf: q \in P\}$. Then, T is a [weakly] $\bar{P}$-regular semigroup.*

PROOF. Assume that S is weakly P-regular. Since $P \subset E_S$, $Pf \subset E_S f=E_T$. That is, $\bar{P} \subset E_T$. For $qf \in \bar{P}$, $qf\bar{P}^1 qf=(qf)(P^1 f)(qf) \subset (qP^1 q)f \subset Pf=\bar{P}$. For any $\bar{a}=af \in T$, there exist $p, q \in P$ such that $a\mathscr{L}p$ and $a\mathscr{R}q$. Then, $af\mathscr{L}pf$ and $af\mathscr{R}qf$, and $pf, qf \in \bar{P}$. Therefore, every L-class of T contains an element of $\bar{P}$; and every R-class of T contains an element of $\bar{P}$. Thus, T is weakly $\bar{P}$-regular. Next, assume that S is P-regular. Of course, T is weakly $\bar{P}$-regular as was shown above. Let $af \in T$. Then, there exists $a^* \in V_P(a)$. Now, $(af)(a^*f)=(aa^*)f \in \bar{P}$. Similarly, $(a^*f)(af)=(a^*a)f \in \bar{P}$. For any $qf \in \bar{P}$, $(af)(qf)(a^*f)=(aqa^*)f \in \bar{P}$ and $(a^*f)(qf)(af)=(a^*qa)f \in \bar{P}$. Since $a^*f \in V(af)$, $a^*f \in V_P(af)$.

PROPOSITION 2.4. *Let S be a regular semigroup, and P a subset of E_S such that $P \cap L \neq \square$ and $P \cap R \neq \square$ for each L-class L and R-class R of S. Then, the following*

(1)–(3) *are equivalent*:

(1) *S is weakly P-regular.*

(2) *For any $q \in P$, $qP^1q \subset P$; and if (a, a^*), (b, b^*) are regular pairs such that aP^1a^*, $a^*P^1a \subset P$ and bP^1b^*, $b^*P^1b \subset P$, then (ab, b^*a^*) is a regular pair such that $abP^1b^*a^* \subset P$ and $b^*a^*P^1ab \subset P$.*

(3) *For any $p, q \in P$, $pqp \in P$ and $qp \in V(pq)$.*

PROOF. (1)⇒(2) and (2)⇒(3) are obvious. (3)⇒(1): Since $pq \in V(qp)$ for $p, q \in P$, $pq(qp)pq = pq$. Hence, $(pq)^2 = pq$. Thus, $P^2 \subset E_S$. Since $pP^1p \subset P$ for $p \in P$, S is weakly P-regular.

PROPOSITION 2.5. *In a weakly P-regular semigroup,*

(1) $P^2 = E_S$,

(2) for $f \in E_S$, $f^* \in V_P(f)$ *implies* $f^* \in E_S$.

PROOF. Let $f^* \in V_P(f)$. Then, $f^* = f^*fff^* \in P^2 \subset E_S$. Hence, $f = ff^*f^*f \in P^2$. That is, $E_S = P^2$.

PROPOSITION 2.6. *Let S be a P-regular semigroup, and let $a \in S$. Let e, f be elements of P such that $e\mathscr{R}a\mathscr{L}f$. Then, there exists a unique $a^* \in V_P(a)$ such that $aa^* = e$ and $a^*a = f$.*

PROOF. Since S is a regular semigroup, there exists $a' \in V(a)$ such that $aa' = e$ and $a'a = f$. Since S is P-regular, there exists $\bar{a} \in V_P(a)$ such that $aP^1\bar{a} \subset P$ and $\bar{a}P^1a \subset P$. Now, $aP^1a' = aa'aa'aP^1a'a\bar{a}aa' \quad aa'aP\bar{a}aa' \subset ePe \subset P$. Similarly, $a'P^1a \subset P$. Hence, $a' \in V_P(a)$.

COROLLARY. *Let a, b be two elements of a P-regular semigroup S. Then, $a\mathscr{H}b$ (where $\mathscr{H}$ is Green's H-reglation) if and only if there exist $a' \in V_P(a)$ and $b' \in V_P(b)$ such that $aa' = bb'$ and $a'a = b'b$.*

§3. The semigroup of P-regular pairs

Let S be a P-regular semigroup. Let $S^* = \{(a, a^*) : a \in S, (a, a^*) \text{ is a } P\text{-regular pair}\}$. Then,

PROPOSITION 3.1. (1) *S^* is a regular *-semigroup under the binvry operation $\circ$ and the unary operation $\sharp$ defined by*

$$(a, a^*) \circ (b, b^*) = (ab, b^*a^*),$$

$$(a, a^*)^\sharp = (a^*, a).$$

(2) *The set of all projections of the regular *-semigroup S^* is $P^* = \{(p, p) : p \in P\}$.*

PROOF. (1): By Proposition 2.4, it is obvious that S^* is a semigroup. Since $(a, a^*) \in S^*$ implies $(a^*, a) \in S^*$, S^* is regular and $\#$ satisfies (i) $((a, a^*)^\#)^\# = (a, a^*)$, (ii) $((a, a^*) \circ (b, b^*))^\# = (b, b^*)^\# \circ (a, a^*)^\#$, and (iii) $(a, a^*) \circ (a, a^*)^\# \circ (a, a^*) = (a, a^*)$. Hence, S^* is a regular $*$-semigroup.

(2): Every element of P^* is obviously a projection. Conversely, let (a, a^*) be a projection. Then, $(a, a^*) = (a^*, a)$, and hence $a = a^*$ and a is an idempotent. Hence, $a = aa^* \in P$. That is, $(a, a^*) \in P^*$.

PROPOSITION 3.2. *If S is a band [an orthodox semigroup], then S^* is also a band [an orthodox semigroup].*

PROOF. Obvious.

Let f be the mapping of S^* to S defined by $(a, a^*)f = a$. Then, f is clearly a homomorphism, and $af^{-1} = \{(a, a^*): a^* \in V_P(a)\}$. Therefore, if we define af^{-1} by $\mathcal{V}_P(a)$, then $\bar{S}^* = \{\mathcal{V}_P(a): a \in S\}$ becomes a regular semigroup under the multiplication

$$\mathcal{V}_P(a)\mathcal{V}_P(b) = \mathcal{V}_P(ab) \qquad \text{for} \quad a, b \in S,$$

and is isomorphic to S.

From the results above,

THEOREM 3.3. *For any regular P-semigroup S, there exist a regular $*$-semigroup S^* and a homomorphism f of S^* onto S such that $P^*f = P$, where P^* is the set of projections of S^*. Accordingly, every P-regular semigroup is a homomorphic image of a regular $*$-semigroup.*

Let $\{P_i: i \in I\}$ be a family of C-sets in a regular semigroup S; hence, S is $\{P_i: i \in I\}$-regular. Consider a P_i-regular semigroup S_i for each i, where $S_i = S$ as a regular semigroup. Let $\prod\{S_i: i \in I\} = T$ be the direct product of P_i-regular semigroups S_i. Denote an element of T by $(x_i)_{i \in I}$ (the i-th coordinate is $x_i \in S_i$ for all $i \in I$).

Consider $\mathcal{S} = \{((x_i)_{i \in I}, (x_i^*)_{i \in I}): (x_i)_{i \in I} \in T$, and each x_i^* is a P_i-inverse of $x_i\}$. Then, $\mathcal{S}$ becomes a regular $*$-semigroup under the following binary operation $\circ$ and the unary operation $\#$:

$$((x_i)_{i \in I}, (x_i^*)_{i \in I}) \circ ((y_i)_{i \in I}, (y_i^*)_{i \in I})$$

$$= ((x_i y_i)_{i \in I}, (y_i^* x_i^*)_{i \in I}),$$

$$((x_i)_{i \in I}, (x_i^*)_{i \in I})^\# = ((x_i^*)_{i \in I}, (x_i)_{i \in I}).$$

Now, consider the mapping ϕ: $((x_i)_{i \in I}, (x_i^*)_{i \in I})\phi = (x_i)_{i \in I}$.

Then, ϕ is a surjective homomorphism of $\mathcal{S}$ onto $\prod\{S_i: i \in I\} = T$. Since $\psi: S \to T$ defined by $x\psi = (x_i)_{i \in I}$, where $x_i = x$ for all $i \in I$, is an injective homomorphism, we have the following result:

THEOREM 3.4. *If S is a $\{P_i: i \in I\}$-regular semigroup, then S is embedded in a* *homomorphic image of a regular *-semigroup.*

§4. Construction

As was shown in §3, every P-regular semigroup can be obtained as a homomorphic image of a regular *-semigroup. Hence, the problem of constructing all possible P-regular semigroups is reduced to the following two problems:

I. Description of all congruences on a given regular semigroup.
II. Construction of regular *-semigroups.

The first problem I is too routine to state here. We may only consider the usual description of congruences on general semigroups (for example, see [1]). The second problem II has been completely solved by the previous papers [6] and [7] of one of the authors (for the construction of fundamental regular *-semigroups, see also Imaoka [2]). In particular, it has been shown in [6] that every fundamental regular *-semigroup is obtained as a full regular *-subsemigroup of the Munn semigroup T_F over a fundamental regular warp F. Further, it has been also shown in [7] that a general regular *-semigroup can be obtained as a *-regular product of a fundamental regular *-semigroup and a certain special partial groupoid which is a union of groups.

Next, let S be a P-regular semigroup. Then, we have the following result which is a generalization of the description of the maximum idempotent-separating congruence for a regular *-semigroup (see [3], [6]).

PROPOSITION 4.1. *The maximum idempotent-separating congruence τ on S is given by*

(C.4.1) *$a\tau b$ if and only if there exist $a^* \in V_P(a)$ and $b^* \in V_P(b)$ such that $axa^* = bxb^*$ and $a^*xa = b^*xb$ for all $x \in P$.*

PROOF. Let us denote the given relation by μ. It is easy to see that μ is reflexive and symmetric. Suppose that $a\mu b$ and $b\mu c$. Then, there exist $a' \in V_P(a)$, b', $b^* \in V_P(b)$ and $c^* \in V_P(c)$ such that $a'xa = b'xb$, $axa' = bxb'$, $b^*xb = c^*xc$ and $bxb^* = cxc^*$ for all $x \in P$. By simple calculation, we have $aa'b = b$, $ab'b = a$, $bb'a = a$, $cc^*b = b$, $cb^*b = c$ and $bb^*c = c$. Let $\bar{a} = b^*ba'bb'$. Then, by using the results above it is easy to see that $\bar{a} \in V_P(a)$. Let $\bar{c} = b^*bc^*bb'$. Then, similarly $\bar{c} \in V_P(c)$. Now, it follows by simple calculation that $ax\bar{a} = cx\bar{c}$ and $\bar{a}xa = \bar{c}xc$ for $x \in P$. Hence, μ is an equivalance relation. It is easy to see that $a\mu b$ implies $ac\mu bc$ and $ca\mu cb$ for any $c \in S$. Thus, μ is a congruence. Next, we prove that μ is idempotent-separating. Suppose that $e, f \in E_S$ and $e\mu f$. Then, $exe' = fxf'$ and $e'xe = f'xf$ for some $e' \in V_P(e)$, $f' \in V_P(f)$ and for all $x \in P$. Hence, $f'(e'e)f = e'(e'e)e = e'e$. Now, $ef = ee'ef = ee'(e'e)ef = ef'(e'e)f = ef'e'ef = ee'e = e$. On the other hand, $e(ff')e' = fff'f' = ff'$. Hence, $ef = ef(ff')f'f = ee(ff')e'e = e(ff')e'f = ff'f = f$. Thus, $e = f$. Therefore, μ is idempotent-separating. Finally, let ρ be any idempotent-separating congruence on S. If $a\rho b$ then $a\mathscr{H}b$.

Then, by Corollary to Proposition 2.6 it follows that $aa'=bb'$ and $a'a=b'b$ for some $a'\in V_P(a)$ and $b'\in V_P(b)$. Then, $a'=a'aa'=a'bb'$, $a'ab'=b'bb'=b'$. Therefore, $a\rho b$ implies $a'ab'\rho a'bb'$. This shows that $b'\rho a'$. Since ρ is a congruence, $a\rho b$ and $b'\rho a'$ imply $axa'\rho bxb'$ and $a'xa\rho b'xb$ for $x\in P$. Since axa', bxb', $a'xa$ and $b'xb$ are idempotents, we have $axa'=bxb'$ and $a'xa=b'xb$ for $x\in P$. Hence, $\rho\subset\mu$. Consequently, μ is the maximum idempotent-separating congruence on S.

Let S be a fundamental P-regular semigroup. Then, S^* is a regular *-semigroup and the set of projections is $P^*=\{(p, p)\colon p\in P\}$. Hence, S^* is a P^*-regular semigroup. Let τ be the maximum idempotent-separating congruence on S^*, and assume that $\alpha\tau\beta$ for $\alpha, \beta\in S^*$. Let $\alpha=(a, a^*)$ and $\beta=(b, b^*)$. Then, for any $(p, p)\in P^*$, $(a, a^*)(p, p)\cdot(a^*, a)=(b, b^*)(p, p)(b^*, b)$ and $(a^*, a)(p, p)(a, a^*)=(b^*, b)(p, p)(b, b^*)$. Hence, $(apa^*, apa^*)=(bpb^*, bpb^*)$ and $(a^*pa, a^*pa)=(b^*pb, b^*pb)$. Therefore, $apa^*=bpb^*$ and $a^*pa=b^*pb$ for all $p\in P$. Since S is fundamental, it follows from Proposition 4.1 above that $a=b$ and $a^*=b^*$. Hence, $(a, a^*)=(b, b^*)$. Thus, S^* is fundamental. Since S is a homomorphic image of S^*, we have the following:

THEOREM 4.2. *A fundamental P-regular semigroup is a homomorphic image of a fundamental regular *-semigroup.*

Finally, we obtain the following from the results above:

THEOREM 4.3. *Let S be a fundamental regular *-semigroup, and P the set of projections of S. Let ξ be a congruence on S, and let $\bar{P}=\{\bar{p}\colon p\in P\}$, where $\bar{p}=p\xi$. Then, S/ξ $(=\bar{S})$ is a $\bar{P}$-regular semigroup. If τ is the maximum idempotent-separating congruence on $\bar{S}$, then $\bar{S}/\tau$ is a fundamental $\tilde{P}$-regular semigroup, where $\tilde{P}=\{\bar{p}\tau\colon \bar{p}\in\bar{P}\}$.*

Conversely, every fundamental P-regular semigroup can be obtained in this fashion.

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