Mem. Fac. Sci. Shimane Univ.,
21, pp. 17–20 Dec. 20, 1987

Topological Properties of Some Locally Convex Linear Topological Spaces

Dedicated to Professor Yoshihiro Tashiro on his 60th birthday

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(Received September 5, 1987)

In this paper, we introduce a topology $\mathscr{T}$ in a real vector space L such that $(L, \mathscr{T})$ is a locally convex linear topological space and $\mathscr{T}$ is the strongest locally convex topology contained in the finite topology. Furthermore, we show that $(L, \mathscr{T})$ is an M_1-space and an AR (stratifiable)-space.

§ 1. Introduction

An *affine space* is a real vector space with any topology that induces the Euclidean topology on its finite dimensional flats (cf. [4; pp. 416, Definition 4.1]). Clearly, every real linear topological space is an affine space. But the converse is not true. In fact, there is a real vector space with the finite topology such that it is neither a linear topological space nor a locally convex space ([4; pp. 416, 4.3]), where the *finite topology* in a real vector space is the weak topology determined by the Euclidean topology on each finite dimensional linear subspace (cf. [4; pp. 416, Definition 4.2]). (Note that a countable dimensional linear space with the finite topology is a locally convex linear topological space [3; Lemma 4.4].)

In this paper, we introduce a topology $\mathscr{T}$ in a real vector space L such that $(L, \mathscr{T})$ is a locally convex linear topological space and $\mathscr{T}$ is the strongest locally convex topology contained in the finite topology. As for topological properties of vector spaces, all vector spaces with the finite topology are completely and perfectly normal, furthermore, they are always paracompact, and every subsets (not only the closed subsets) is also paracompact (cf. [4; pp. 416]). In this paper, we show that $(L, \mathscr{T})$ in the above is an M_1-space and an AR (stratifiable)-space.

Throughout this paper, N denotes the set of all natural numbers. For M_1-spaces and stratifiable spaces, see [2] and [1], respectively. For AR and linear topological spaces, see [5] and [6], respectively. Every terminology is referred to [4] or [6], unless otherwise stated.

§2. Construction of a locally convex topology and theorems

Throughout this paper, we exclusively use the following notation: L is a real vector space with a basis $\mathscr{B}=\{u_\alpha: \alpha\in\Lambda\}$.

Let $\mathscr{E}_n$ be all n-dimensional linear subspaces of L generated by n-elements of $\mathscr{B}$ (i.e. $\mathscr{E}_n=\{\langle u_{\alpha_1},\dots, u_{\alpha_n}\rangle: u_{\alpha_i}\in\mathscr{B}$ for $i=1,\dots, n\}$). Now, we begin with a construction of a local base (i.e. a base for the neighborhood system of 0 [6; pp. 34]). The proofs of Lemmas are given in the later sections.

CONSTRUCTION 2.1. For each $\alpha\in\Lambda$, pick up an $n_\alpha\in N$. Let $U_1=\cup\{\{tu_\alpha: |t|<1/n_\alpha\}: \alpha\in\Lambda\}$. By using induction, if U_{n-1} has been defined for $n\geqq 2$, let $U_n=\cup\{\mathrm{conv}(E\cap U_{n-1}): E\in\mathscr{E}_n\}$, where conv A is the convex hull of A. Let $U(n_\alpha:\alpha\in\Lambda)=\cup\{U_n: n\in N\}$ and $\mathscr{U}$ be all $U(n_\alpha: \alpha\in\Lambda)$.

Then it is obvious that $U(n_\alpha: \alpha\in\Lambda)$ is convex. Furthermore $\mathscr{U}$ satisfies the local base conditions as follows:

LEMMA 2.2. *$\mathscr{U}$ satisfies the following.*

(i) *For U and V in $\mathscr{U}$ there is $W\in\mathscr{U}$ with $W\subset U\cap V$;*

(ii) *for U in $\mathscr{U}$ there is $V\in\mathscr{U}$ with $V+V\subset U$;*

(iii) *for U in $\mathscr{U}$ there is $V\in\mathscr{U}$ with $aV\subset U$ for each scalar a with $|a|\leqq 1$;*

(iv) *for x in L and U in $\mathscr{U}$ there is a scalar a with $x\in aU$.*

By using this lemma and [6; Theorem 5.1], $\mathscr{T}=\{W$: For each $x\in W$, there is $U\in\mathscr{U}$ with $x+U\subset W\}$ is a vector topology (i.e. $(L, \mathscr{T})$ is a linear topological space) and $\mathscr{U}$ is a local base for this topology $\mathscr{T}$. Furthermore $\mathscr{T}$ satisfies the following.

LEMMA 2.3. *$\mathscr{T}$ is the strongest locally convex topology contained in the finite topology.*

By Lemmas 2.2 and 2.3, we have

THEOREM 2.4. *$(L, \mathscr{T})$ is a locally convex linear topological space, and $\mathscr{T}$ is the strongest locally convex topology contained in the finite topology.*

In connection with this theorem, the following question naturally arises.

QUESTION 2.5. Is there a topology $\mathscr{T}'$ contained in the finite topology such that $\mathscr{T}'$ is not locally convex, but $(L, \mathscr{T}')$ is a linear topological space?

Next, in order to prove that $(L, \mathscr{T})$ is an M_1-space, we use the following notation. For each $E\in\mathscr{E}_n$, since E is homeomorphic to n-dimensional Euclidean space, E has a countable dense subset $\{x_m(E): m\in N\}$. (In particular, without loss of generality, we may suppose that, for each $m\in N$, the number of non-zero coordinates of $x_m(E)$ is just $n=\dim E$.) From now on, we use this notation. For $x=a_1u_{\alpha_1}+\cdots+a_nu_{\alpha_n}$, let $l(x)=\min\{|a_i|: a_i\neq 0$ for $i=1,\dots, n\}$. For each $n\in N$, let $\mathscr{U}_n=\{U(n_\alpha: \alpha\in\Lambda)$:

$n_\alpha \geqq n$ for any $\alpha \in \Lambda\}$. Then, for each $E \in \mathscr{U}_n$, let $\mathscr{U}_n^m(E) = \{x_m(E) + U: U \in \mathscr{U}_k\}$, where $k = \min\{j: 1/j < l(x_m(E))\}$, and $\mathscr{U}_n^m = \cup \{\mathscr{U}_n^m(E): E \in \mathscr{E}_n\}$.

Then, the following lemmas are obtained.

LEMMA 2.6. $\cup\{\mathscr{U}_n^m: n, m \in N\}$ *is a base for* $(L, \mathscr{T})$.

LEMMA 2.7. *For each* $m, n \in N$, $\mathscr{U}_n^m$ *is closure preserving.*

Thus, since it is obvious that $(L, \mathscr{T})$ is regular, by Lemmas 2.6 and 2.7 we have

THEOREM 2.8. $(L, \mathscr{T})$ *is an* M_1*-space.*

Since $(L, \mathscr{T})$ is a locally convex linear topological space, by [1; Theorem 4.3] we have

THEOREM 2.9. $(L, \mathscr{T})$ *is an* AR (stratifiable)*-space.*

§3. Proofs of Lemmas 2.2 and 2.3

PROOF OF LEMMA 2.2. (i): For any U and V in $\mathscr{U}$, let $U = U(n_\alpha: \alpha \in \Lambda)$ and $V = U(m_\alpha: \alpha \in \Lambda)$. Further let $l_\alpha = \max\{n_\alpha, m_\alpha\}$ and $W = U(l_\alpha: \alpha \in \Lambda)$. Then it is clear that $W \in \mathscr{U}$ and $W \subset U \cap V$.

(ii): For each $\alpha \in \Lambda$ and each $n \in N$, let $V(u_\alpha; n) = \{tu_\alpha: |t| < 1/n\}$. For any $U = U(n_\alpha: \alpha \in \Lambda) \in \mathscr{U}$, let $V = U(2n_\alpha: \alpha \in \Lambda)$. Then we shall prove that $V + V \subset U$. For any $x, y \in V$, let $x = a_1 v_{\alpha_1} + \cdots + a_n v_{\alpha_n}$, $y = b_1 w_{\beta_1} + \cdots + b_m w_{\beta_m}$, where $v_{\alpha_i} \in V(u_{\alpha_i}; 2n_{\alpha_i})$, $w_{\beta_i} \in V(u_{\beta_i}; 2n_{\beta_i})$, $a_1 + \cdots + a_n = 1$, $b_1 + \cdots + b_m = 1$ and $a_i > 0$, $b_j > 0$ for each $i = 1, \dots, n$, $j = 1, \dots, m$. Let $v'_{\alpha_i} = 2v_{\alpha_i}$ and $w'_{\beta_i} = 2w_{\beta_i}$. Then $v'_{\alpha_i} \in V(u_{\alpha_i}; n_{\alpha_i})$ and $w'_{\beta_i} \in V(u_{\beta_i}; n_{\beta_i})$. In case, $\alpha_i \neq \beta_j$ for any α_i and β_j,

$$x + y = a'_1 v'_{\alpha_1} + \cdots + a'_n v'_{\alpha_n} + b'_1 w'_{\alpha_1} + \cdots + b'_n w'_{\alpha_n}$$

where $a'_i = a_i/2$ and $b'_j = b_j/2$. Therefore, since $a'_1 + \cdots + a'_n + b'_1 + \cdots + b'_m = 1$, $x + y \in U$. In the other case $\alpha_i = \beta_j$ for some α_i and β_j, there is $x_{\alpha_i} \in V(u_{\alpha_i}; n_{\alpha_i})$ such that $a'_i v'_{\alpha_i} + b'_j w'_{\beta_j} = (a'_i + b'_j) x_{\alpha_i}$. Therefore by the same of the former case, $x + y \in U$.

(iii): Since $U \in \mathscr{U}$ is convex, $aU \subset U$ for each scalar a with $|a| \leqq 1$.

(iv): Let $x = a_1 u_{\alpha_1} + \cdots + a_k u_{\alpha_k}$ and $U = U(n_\alpha: \alpha \in \Lambda)$. Further let $c = \max\{|a_i|: i = 1, \dots, k\}$ and $d = \min\{1/n_{\alpha_i}: i = 1, \dots, k\}$. Then there is a positive number a with $c < ad$. This number a satisfies $x \in aU$.

Thus, the proof of Lemma 2.2 is completed.

PROOF OF LEMMA 2.3. First, it is obvious that each $U \in \mathscr{U}$ is open in the finite topology. Next, let $\mathscr{T}'$ be a locally convex topology contained in the finite topology. For any convex neighborhood U (in $\mathscr{T}'$) of 0, since U is open in the finite topology, $U \cap \langle u_\alpha \rangle$ is open in $\langle u_\alpha \rangle$ for each $\alpha \in \Lambda$. Therefore there is $n_\alpha \in N$ for each $\alpha \in \Lambda$ such that $\{tu_\alpha: |t| < 1/n_\alpha\} \subset U$. Then $U(n_\alpha: \alpha \in \Lambda) \in \mathscr{U}$ and since U is convex, $U(n_\alpha:$

$\alpha\in\Lambda)\subset U$. Thus $\mathscr{T}$ is the strongest locally convex topology contained in the finite topology. This completes the proof of Lemma 2.3.

§4. Proofs of Lemmas 2.6 and 2.7

PROOF OF LEMMA 2.6. Let W be a neighborhood of $x\ (=a_1u_{\alpha_1}+\cdots+a_ku_{\alpha_k})$. Then there is $U=U(n_\alpha: \alpha\in\Lambda)\in\mathscr{U}$ with $x+U\subset W$. Let m be a natural number satisfying $2k/m<\min\{|a_i|, 1/n_{\alpha_i}: i=1,\dots, k\}$. Let $E=\langle u_{\alpha_1},\dots, u_{\alpha_k}\rangle$. Then there is $x_j(E)$ such that the distance of $x_j(E)$ and x is smaller than $1/mk$. For these numbers m and j, it is easily verified that there is $V\in\mathscr{U}^j_m(E)$ with $x\in x_j(E)+V\subset x+U\subset W$. This completes the proof of Lemma 2.6.

PROOF OF LEMMA 2.7. First, we prove that $\mathscr{U}$ is closure preserving. Let $x\notin\overline{U}$ for each $U\in\mathscr{U}$ and $E=\langle u_{\alpha_1},\dots, u_{\alpha_n}\rangle$, where $x=a_1u_{\alpha_1}+\cdots+a_nu_{\alpha_n}$ ($a_i\neq 0$ for $i=1,\dots, n$). Then in E, $\mathscr{U}\cap E=\{U\cap E: U\in\mathscr{U}\}$ is locally finite by the construction of $\mathscr{U}$. Therefore there is a convex neighborhood V_n of x in E such that $V_n\cap(\cup(\mathscr{U}\cap E))=\emptyset$. We can construct a convex neighborhood V of x in L such that $V\cap E=V_n$ and $V\cap(\cup\mathscr{U})=\emptyset$. Thus $\mathscr{U}$ is closure preserving.

Next, we prove that $\mathscr{U}^m_n$ is closure preserving for each $m, n\in N$. Let $x\notin\overline{V}$ for each $V\in\mathscr{U}^m_n$ and $E=\langle u_{\alpha_1},\dots, u_{\alpha_k}\rangle$, where $x=a_1u_{\alpha_1}+\cdots+a_ku_{\alpha_k}$ ($a_i\neq 0$ for $i=1,\dots, k$). In case $k<n$, by the construction of $\mathscr{U}^m_n$, $E\cap\overline{V}=\emptyset$ for each $V\in\mathscr{U}^m_n$. Therefore there is a convex neighborhood W of x such that $W\cap(\cup\mathscr{U}^m_n)=\emptyset$. In the other case $k\geqq n$, by the construction of $\mathscr{U}^m_n$, $\mathscr{U}^m_n\cap E=\{V\cap E: V\in\mathscr{U}^m_n\}$ is locally finite in E. Therefore there is a convex neighborhood W_k of x in E such that $W_k\cap(\cup(\mathscr{U}^m_n\cap E))=\emptyset$. We can construct a convex neighborhood W of x in L such that $W\cap E=W_k$ and $W\cap(\cup\mathscr{U}^m_n)=\emptyset$. Thus, $\mathscr{U}^m_n$ is closure preserving. This completes the proof of Lemma 2.7.

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