Mem. Fac. Sci. Shimane Univ.,
21, pp. 1–15 Dec. 20, 1987

Affine Homogeneous Structures on Analytic Loops

Dedicated to Professor Masahiro Sugawara on his 60th birthday

Michihiko KIKKAWA

Department of Mathematics, Shimane University, Matsue, Japan
(Received September 5, 1987)

In this paper, we introduce the concept of affine homogeneous structures which is a generalization of the concept of homogeneous structures on Riemannian manifolds ([15]). By considering affine homogeneous structures on an analytic loop with the canonical connection, we treat the problem of changing it for the other analytic loop which is homogeneous, without changing the system of geodesics.

Introduction

The concept of homogeneous structures on Riemannian manifolds is formulated by F. Tricerri-L. Vanhecke [16] from the earlier work of W. Ambrose-I. M. Singer [5]. It presents a criterion for a Riemannian manifold to be homogeneous. For instance, K. Abe [1] has classified all homogeneous Riemannian structures of 3-dimensional space forms by using homogeneous structures. In this paper, we try to find homogeneous loops ([8], [9], [10]) on a given analytic loop (G, μ) by changing its multiplication μ for homogeneous one, where we assume that the unit e of μ is unchanged and that any 1-parameter subgroup of μ is changed for a 1-parameter subgroup of the homogeneous loop obtained. To do this, we introduce first the concept of affine homogeneous structure of a linearly connected manifold (M, ∇) in §1. Then, in §2, we define the canonical connection ∇ for any analytic loop (G, μ), which is a generalization of the canonical connection of homogeneous loops ([8], [9]) or of the loops with the left inverse property ([11]). The loop (G, μ) is said to be geodesic at e if it is coincident with the geodesic local loop ([4], [7]) of the canonical connection centered at e. We show (Proposition 2.4) that, if (G, μ) is geodesic at e, any geodesic curve $c(t)$ through the unit $e=c(0)$ is a 1-parameter subgroup of the loop. Next, we show that a connected and simply connected geodesic loop (G, μ) can be changed for a homogeneous loop (G, μ') with the same unit, by changing every 1-parameter subgroup of μ for some 1-parameter subgroup of μ', if the canonical connection of (G, μ) admits an affine homogeneous structure (Theorem 3). By

This paper contains partly the results presented at the conference on Web Geometry, Szeged, August 1987.

using this result we treat the problem of changing a homogeneous loop for homogeneous one and we get a criterion for existence of such a change in terms of the tangent Lie triple algebras.

§ 1. Affine homogeneous structures

Let M be a differentiable manifold with a linear connection ∇. The torsion tensor field S and the curvature tensor field R of ∇ are given by

$$S(X, Y)=[X, Y]-\nabla_X Y+\nabla_Y X$$

$$R(X, Y)Z=\nabla_{[X,Y]}Z-\nabla_X\nabla_Y Z+\nabla_Y\nabla_X Z$$

for any (differentiable) vector fields X, Y, Z on M. If we consider the other linear connection $\tilde{\nabla}$ on M, then we have a (1, 2)-tensor field T on M given by

$$T(X, Y)=\nabla_X Y-\tilde{\nabla}_X Y, \tag{1.1}$$

and, conversely, any (1, 2)-tensor field T determines a linear connection $\tilde{\nabla}$ by the equation (1.1).

Now, we introduce the concept of affine homogeneous structures as follows:

DEFINITION. A tensor field T of type (1, 2) on M is called an *affine homogeneous structure* of (M, ∇), if

$$\tilde{\nabla}\tilde{S}=0, \tag{1.2}$$

$$\tilde{\nabla}\tilde{R}=0, \tag{1.3}$$

$$\tilde{\nabla}T=0 \tag{1.4}$$

are satisfied, where $\tilde{\nabla}$ is the linear connection determined by the equation (1.1) and $\tilde{S}$, $\tilde{R}$ are its torsion and curvature tensors, respectively.

REMARK. Let (M, g) be a Riemannian manifold with the Riemannian connection ∇. Then, a (1, 2)-tensor field T on M is a (Riemannian) homogeneous structure in the sense of [16] if and only if it is an affine homogeneous structure of (M, ∇) with the property

$$g(T(X, Y), Z)+g(Y, T(X, Z))=0$$

for any vector fields X, Y and Z on M.

We can easily show that the condition (1.4) in Definition above is equivalent to

$$(\nabla_X T)(Y, Z)=T(X, T(Y, Z))-T(Y, T(X, Z))-T(T(X, Y), Z). \tag{1.5}$$

PROPOSITION 1.1. *Let T be a tensor field of type* (1, 2) *on a linearly connected* *manifold* (M, ∇), *and* $\tilde{\nabla}=\nabla-T$ *the linear connection determined by* T. *Then, the torsion* $\tilde{S}$ *and the curvature* $\tilde{R}$ *of* $\tilde{\nabla}$ *are given by the following formulas*;

$$\tilde{S}(X, Y)=S(X, Y)+T(X, Y)-T(Y, X), \tag{1.6}$$

$$\tilde{R}(X, Y)Z=R(X, Y)Z-T(\tilde{S}(X, Y), Z)+T(X, T(Y, Z)) \\ -T(Y, T(X, Z))+(\tilde{\nabla}_X T)(Y, Z)-(\tilde{\nabla}_Y T)(X, Z). \tag{1.7}$$

PROOF. They are obtained from the definition of the torsion $\tilde{S}$ and the curvature $\tilde{R}$ under a straightforward calculation.

PROPOSITION 1.2. *A* (1, 2)-*tensor field* T *is an affine homogeneous structure of* (M, ∇) *if and only if the following equations are satisfied*:

$$(\nabla_X S)(Y, Z)=T(X, S(Y, Z))-S(T(X, Y), Z)-S(Y, T(X, Z)), \tag{1.8}$$

$$(\nabla_X R)(Y, Z)W=T(X, R(Y, Z)W)-R(T(X, Y), Z)W \\ -R(Y, T(X, Z))W-R(Y, Z)T(X, W), \tag{1.9}$$

and (1.5).

PROOF. Under the assumption that the equation (1.5) is valid, we can show

$$(\tilde{\nabla}_X\tilde{S})(Y, Z)=(\nabla_X S)(Y, Z)+S(T(X, Y), Z)+S(Y, T(X, Z)) \\ -T(X, S(Y, Z)), \tag{1.10}$$

$$(\tilde{\nabla}_X\tilde{R})(Y, Z)W=\tilde{\nabla}_X(\tilde{R}(Y, Z)W)-\tilde{R}(\tilde{\nabla}_X Y, Z)W-\tilde{R}(Y, \tilde{\nabla}_X Z)W \\ -\tilde{R}(Y, Z)\tilde{\nabla}_X W \\ =(\nabla_X R)(Y, Z)W-T(X, R(Y, Z)W)+R(T(X, Y), Z)W \\ +R(Y, T(X, Z))W+R(Y, Z)T(X, W). \tag{1.11}$$

Hence, we agree to the assertion of the proposition. q. e. d.

From Proposition 1.1 we can obtain;

PROPOSITION 1.3. *A* (1, 2)-*tensor field* T *is an affine homogeneous structure of* (M, ∇) *if and only if the tensor fields* S, R *and* T *are parallel with respect to the linear connection* $\tilde{\nabla}=\nabla-T$.

In the same way as [16] for Riemannian homogeneous structures of Riemannian manifolds, we obtain the following;

THEOREM 1. *Let* M *be a connected and simply connected analytic manifold* *with a complete analytic linear connection ∇. If (M, ∇) has a skew-symmetric analytic affine homogeneous structure T, then*

(1) *there exists a connected group A of affine transformations of $\tilde{\nabla}=\nabla-T$ such that $M=A/K$, K being the isotropy subgroup of A at any fixed point e, is a reductive homogeneous space* ([15]) *with the canonical connection $\tilde{\nabla}$;*

(2) *if the tensor field T is invariant by A, every element of A acts as an affine transformation of ∇.*

Proof. Since T is assumed to be skew-symmetric, any geodesic of $\tilde{\nabla}$ is a geodesic of ∇ too. Hence, $\tilde{\nabla}$ is a complete connection and the assertion (1) follows from Theorem 2.8 (2) in Ch. X of [12 Vol. II]. Moreover, if T satisfies $\phi T(X, Y)=T(\phi X, \phi Y)$ for each element ϕ of A, then we have

$$\phi\nabla_X Y=\phi\tilde{\nabla}_X Y+\phi T(X, Y)=\tilde{\nabla}_{\phi X}\phi Y+T(\phi X, \phi Y)=\tilde{\nabla}_{\phi X}\phi Y,$$

which verifies the second assertion. q.e.d.

§2. Canonical connections of analytic loops

An analytic loop (G, μ) is an analytic manifold G equipped with a binary operation $\mu: G\times G\to G$ denoted by $xy=\mu(x, y)$ such that (cf. [2], [14])

(i) μ is an analytic map,

(ii) for each x in G, the left translation $L_x: G\to G$, $L_x y=xy$, and the right translation $R_x: G\to G$, $R_x y=yx$, are analytic diffeomorphisms of G, and the maps

$$L^{-1}: G\times G \longrightarrow G; \quad L^{-1}(x, y)=L_x^{-1}y,$$

$$R^{-1}: G\times G \longrightarrow G; \quad R^{-1}(x, y)=R_x^{-1}y$$

are analytic,

(iii) μ has the two-sided unit e.

Let (G, μ) be an analytic loop with the unit e. By setting

$$\eta(x, y, z)=L_x\mu(L^{-1}(x, y), L^{-1}(x, z)) \quad \text{for} \quad x, y, z, \quad \text{in} \quad G \tag{2.1}$$

an analytic ternary system $\eta: G\times G\times G\to G$ is associated with (G, μ). For any x, y in G, the analytic diffeomorphism $\eta(x, y)$ of G given by $\eta(x, y)z=\eta(x, y, z)$ is called a *displacement* of (G, η) from x to y. The ternary system (G, η) satisfies;

$$\eta(x, x, y)=y \tag{2.2}$$

$$\eta(x, y, x)=y \tag{2.3}$$

$$\eta(e, x, \eta(e, y, z))=\eta(x, \eta(e, x, y), \eta(e, x, z)) \quad \text{for} \quad x, y, z \quad \text{in} \quad G. \tag{2.4}$$

Moreover, η satisfies the following equality (2.5) if and only if the loop (G, μ) has the *let inverse property*, i.e., $L_x^{-1}=L_{x^{-1}}$ for $x^{-1}=L_x^{-1}e$;

$$\eta(x, e, \eta(e, x, y))=\eta(e, x, \eta(x, e, y))=y. \tag{2.5}$$

The loop (G, μ) is said to be *homogeneous* if it has the left inverse property and if every left inner mapping $L_{x,y}=L_{xy}^{-1}L_xL_y$ is an automorphism of the loop μ. (G, μ) is homogeneous if and only if the equalities (2.5) and

$$\eta(e, x, \eta(u, v, w))=\eta(\eta(e, x, u), \eta(e, x, v), \eta(e, x, w)) \tag{2.6}$$

are valid for the ternary system η.

Let (G, η) be the ternary system associated with an analytic loop (G, μ). With the notation used in [11] we can describe various tangential formulas of η obtained from the equalities above. In particular, we get from (2.4) the following formula;

$$\eta(e, x, \eta(e, Y_y, Z_z))+\eta(e, x, \eta(e, Y_y, z)\cdot\eta(e, y, Z_z)) \tag{2.7}$$
$$=\eta(x, \eta(e, x, Y_y), \eta(e, x, Z_z)).$$

DEFINITION. The *canonical connection* ∇ of an analytic loop (G, μ) is an analytic linear connection on G defined by the following;

$$(\nabla_X Y)_x=X_xY-\eta(x, X_x, Y_x), \tag{2.8}$$

for each x in G and any vector fields X and Y on G.

PROPOSITION 2.1. *For the canonical connection ∇ of (G, μ) the torsion S and the curvature R are given by*

$$S_x(X_x, Y_x)=\eta(x, X_x, Y_x)-\eta(x, Y_x, X_x), \tag{2.9}$$

$$R_x(X_x, Y_x)Z_x=\eta(X_x, Y_x, Z_x)-\eta(Y_x, X_x, Z_x) \tag{2.10}$$
$$-\eta(x, X_x, \eta(x, Y_x, Z_x))+\eta(x, Y_x, \eta(x, X_x, Z_x))$$
$$-\eta(x, X_x, Y_x\cdot Z_x)+\eta(x, Y_x, X_x\cdot Z_x).$$

By Lemma in §3 of [11] we can show the following proposition for analytic loops in general (cf. Prop. 3 [11]);

PROPOSITION 2.2. $(\nabla S)_e=0$.

Let $\mathfrak{G}=T_e(G)$ denote the tangent space of G at the unit e. Since $\mu(e, e)=e$, the multiplication μ of the loop induces a bilinear operation on $\mathfrak{G}$ denoted by $d\mu: \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$. It is obtained by differentiating $\mu(x, y)$ in the directions X_x and Y_y at $x=y=e$. We have

$$d\mu(X_e, Y_e)=\eta(e, X_e, Y_e) \quad \text{for} \quad X_e, Y_e \quad \text{in} \quad \mathfrak{G}. \tag{2.11}$$

On the other hand, we can obtain endomorphisms $dL(X_e, Y_e)$ of $\mathfrak{G}$ by differentiating the linear automorphism $dL_{x,y}$ of $\mathfrak{G}$ in the directions X_x and Y_y at $x=y=e$, where $dL_{x,y}$ denotes the differential of the left inner mapping $L_{x,y}$ of (G, μ). We see that the endomorphism $dL(X_e, Y_e)$ is described in terms of the ternary system η associated with (G, μ) as follows;

$$dL(X_e, Y_e)Z_e=\eta(X_e, Y_e, Z_e)+\eta(e, Y_e, \eta(e, X_e, Z_e)) \\ +\eta(e, Y_e, X_e \cdot Z_e)+\eta(e, X_e \cdot Y_e, Z_e) \tag{2.12}$$

for X_e, Y_e, Z_e in $\mathfrak{G}$.

PROPOSITION 2.3.

$$S_e(X_e, Y_e)=d\mu(X_e, Y_e)-d\mu(Y_e, X_e), \tag{2.13}$$

$$R_e(X_e, Y_e)=dL(X_e, Y_e)-dL(Y_e, X_e). \tag{2.14}$$

PROOF. The equalities are shown directly by applying (2.11) and (2.12) to the formulas (2.9) and (2.10) in Proposition 2.2. q. e. d.

For the torsion S and the curvature R of a linear connection ∇, following formulas are well-known as Bianchi's identities (cf. [12]):

$$\mathfrak{S}\{R(X, Y)Z+S(S(X, Y), Z)-(\nabla_X S)(Y, Z)\}=0,$$

(*Bianchi's 1st identity*)

$$\mathfrak{S}\{(\nabla_X R)(Y, Z)-R(S(X, Y), Z)\}=0,$$

(*Bianchi's 2nd identity*)

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z.

If we set

$$[X, Y]=d\mu(X, Y)-d\mu(Y, X),$$

$$\langle X, Y, Z\rangle=dL(X, Y)Z$$

on the tangent space $\mathfrak{G}=T_e(G)$ of the analytic loop (G, μ), then we get the *tangent Akivis algebra* $\{\mathfrak{G}; [X, Y], \langle X, Y, Z\rangle\}$ of the loop ([3], [6]). In fact, by formulas in Prop. 2.3, we have

$$S_e(X, Y)=[X, Y], \tag{2.15}$$

$$R_e(X, Y)Z=\langle X, Y, Z\rangle-\langle Y, X, Z\rangle. \tag{2.16}$$

Then, Bianchi's 1st identity with Prop. 2.2 implies the axiom of Akivis algebra;

$$[X, Y]+[Y, X]=0,$$

$$\mathfrak{S}_{X,Y,Z}\{\langle X, Y, Z\rangle-\langle Y, X, Z\rangle+[[X, Y], Z]\}=0.$$

Moreover, if the canonical connection satisfies $(\nabla R)_e=0$, then, by Bianchi's 2nd identity, we see that the tangent Akivis algebra admits the following relation;

$$\mathfrak{S}_{X,Y,Z}\{\langle [X, Y], Z, W\rangle-\langle Z, [X, Y], W\rangle\}=0.$$

Especially, if the canonical connection is invariant by parallelism ([12–II]), i.e., $\nabla S=0$ and $\nabla R=0$ are satisfied on G, then, by setting

$$[X, Y, Z]=\langle X, Y, Z\rangle-\langle Y, X, Z\rangle,$$

the tangent Akivis algebra induces a Lie triple algebra $\{\mathfrak{G}; [X, Y], [X, Y, Z]\}$ ([8], [17]). In fact, it satisfies

$$[X, Y]+[Y, X]=0,$$

$$[X, Y, Z]+[Y, X, Z]=0,$$

$$\mathfrak{S}_{X,Y,Z}\{[X, Y, Z]+[[X, Y], Z]\}=0,$$

$$\mathfrak{S}_{X,Y,Z}\{[[X, Y], Z, W]\}=0,$$

and, from Ricci's identity (p. 87 [13]) applied for S and R, we obtain the following two relations;

$$[X, Y, [U, V]]=[[X, Y, U], V]+[U, [X, Y, V]],$$

$$[X, Y, [U, V, W]]=[[X, Y, U], V, W]+[U, [X, Y, V], W]$$
$$+[U, V, [X, Y, W]].$$

Thus, we have;

THEOREM 2. *Let (G, μ) be an analytic loop with the unit e, $\mathfrak{G}=T_e(G)$ the tangent space of G at e. For the tangent Akivis algebra $\{\mathfrak{G}; [X, Y], \langle X, Y, Z\rangle\}$ of (G, μ), the two operations are related with the canonical connection ∇ of the loop in the following equations;*

$$S_e(X, Y)=[X, Y],$$

$$R_e(X, Y)Z=\langle X, Y, Z\rangle-\langle Y, X, Z\rangle.$$

If the canonical connection is invariant by parallelism, then the tangent Akivis algebra induces a Lie triple algebra $\{\mathfrak{G}; [X, Y], [X, Y, Z]\}$ where $[X, Y, Z]=\langle X, Y, Z\rangle-\langle Y, X, Z\rangle$.

Let (G, μ) be a connected analytic loop with the canonical connection. For a neighborhood of the unit e, we can define the geodesic local loop $\tilde{\mu}$ centered at e by means of parallel displacements of geodesics through e along each geodesic curve through e (cf. [3], [7]). Moreover, with respect to the canonical connection on G, we can consider a geodesic local loop $\tilde{\mu}_x$ centered at any fixed point x. On the other hand, an analytic loop (G, μ_x) with the unit x is defined by

$$\mu_x(y, z)=\eta(x, y, z),$$

which is isomorphic to the original loop (G, μ) under the left translation L_x.

DEFINITION. An analytic loop (G, μ) will be called *geodesic at e* if $\tilde{\mu}(x, y)=\mu(x, y)$ holds as far as the multiplication $\tilde{\mu}(x, y)$ of the geodesic local loop centered at e is defined. If, for every $x \in G$,

$$\tilde{\mu}_x(y, z)=\mu_x(y, z)$$

holds whenever the geodesic local loop $\tilde{\mu}_x$ is defined, (G, μ) will be said to be *geodesic*.

REMARK. Suppose that the loop (G, μ) is homogeneous. Then, any displacement $\eta(x, y)$ is an affine transformation of the canonical connection. In [8] we have defined the homogeneous loop (G, μ) to be geodesic if, for any geodesic c, the displacement $\eta(c(t_0), c(t_1))$ induces the parallel displacement of the tangent space at any point on c along c. Thus, a homogeneous loop is geodesic in our sense if and only if it is geodesic in the sense of [8]. We can see that a homogeneous loop is geodesic if and only if it is geodesic at the unit e.

Let X_0 be a tangent vector of G at the unit e. We can construct an analytic vector field X on G by the left translations of (G, μ) as follows;

$$X_x=\eta(e, x, X_0), \qquad \text{for} \quad x \quad \text{in} \quad G.$$

Let $c(t)$ denote the maximal integral curve of the vector field X through the unit $e=c(0)$. We have the following;

PROPOSITION 2.4. *Assume that (G, μ) is geodesic at e. Then, the integral curve $c(t)$ above is a geodesic of the canonical connection of (G, μ), and it is a 1-parameter subgroup of the loop (G, μ).*

PROOF. Let $\gamma(t)$, $t \in I$, be a geodesic through $e=\gamma(0)$ which is tangent to the vector X_0. Since (G, μ) is assumed to be geodesic at e, for any fixed $t \in I$, the curve $\gamma_t(s)=\eta(e, \gamma(t), \gamma(s))$, $s \in I$, is a geodesic through $\gamma_t(0)=\gamma(t)$ and it is tangent to the vector

$$\frac{d\gamma}{dt}=\eta(e, \gamma(t), X_0) \tag{2.17}$$

at $\gamma(t)$. On the other hand, the curve $\tilde{\gamma}_t(s)=\gamma(t+s)$ is also a geodesic through $\tilde{\gamma}_t(0)=\gamma(t)$ and it is tangent to $d\gamma/dt$ at $\gamma(t)$. Hence, by the uniqueness of geodesics, we get $\gamma(t+s)=\eta(e, \gamma(t), \gamma(s))$ for $t, s\in I$. From this fact we see that the domain I of the geodesic γ can be extended to the whole real numbers $\boldsymbol{R}$ and that the set $\{\gamma(t), t\in \boldsymbol{R}\}$ forms an analytic associative subloop, i.e., a 1-parameter subgroup of (G, μ). The equation (2.17) shows that the curve $\gamma(t)$ is a maximal integral curve of X through $e=\gamma(0)$, that is to say $\gamma=c$. q.e.d.

The proposition above implies the following;

COROLLARY. *If a connected analytic loop (G, μ) is geodesic, then the canonical connection is complete.*

For geodesic analytic loops we have the following;

PROPOSITION 2.5. *Assume that an anlytic loop (G, μ) is geodesic. Then the relation*

$$L_{c(t),c(s)}=1, \qquad \textit{for} \quad t, s\in \boldsymbol{R},$$

holds for any 1-parameter subgroup $c(t)$ obtained in the proposition above.

PROOF. If (G, μ) is geodesic, then the displacement $\eta(c(t), c(t+s))$ induces the parallel displacement of tangent vectors from the point $c(t)$ to $c(t+s)$ along the curve c. Hence, we have

$$\eta(c(t), c(t+s))\eta(e, c(t))=\eta(e, c(t+s))$$

which shows the relation $L_{c(t),c(s)}=1$. q.e.d.

On the 1-parameter subgroup $c(t)$ above, the inverse of $c(t)$ is given by $c(-t)$ and this fact lead us to the following;

COROLLARY. *If (G, μ) is geodesic, then it has the left inverse property.*

§3. Geodesic preserving changes of analytic loops

Let (G, μ) be a connected analytic loop with the canonical connection ∇ and S, R the torsion and the curvature tensor fields of ∇. In what follows, we assume that the analytic loop (G, μ) is geodesic. Then, by the corollary to Prop. 2.5, (G, μ) has the left inverse property and its tangent Akivis algebra is reduced to $\{\mathfrak{G}; S_e, (1/2)R_e\}$ because the Chern's formula $dL(X, X)=0$ holds (cf. [11]).

Now, suppose that the multiplication μ on G is changed for a geodesic analytic loop (G, μ') with the same unit e, every geodesic of ∇ being a geodesic of the canonical connection ∇' of (G, μ') too. Such a change will be called a *geodesic preserving change* of loops. Let S' and R' be the torsion and the curvature of ∇'. If we set

$$T(X, Y)=\nabla_X Y-\nabla'_X Y$$

for any vector fields X, Y, then, we have a (1, 2)-tensor field T on G and it is skew-symmetric since the change of the loop is geodesic preserving. Let $\{\mathfrak{G}; [X, Y]', \langle X, Y, Z\rangle'\}$ denote the tangent Akivis algebra of the new loop (G, μ'). The formulas (1.6) and (1.7) in Prop. 1.1 show the relations between the Akivis algebras; i.e.,

$$[X, Y]=[X, Y]'-2T_e(X, Y), \tag{3.1}$$

$$\begin{aligned} 2\langle X, Y, Z\rangle=&2\langle X, Y, Z\rangle'+T_e([X, Y]', Z)-T_e(X, T_e(Y, Z)) \\ &+T_e(Y, T_e(X, Z))-(\nabla'_X T)(Y, Z)+(\nabla'_Y T)(X, Z). \end{aligned} \tag{3.2}$$

In the following, we consider the case where the geodesic loop (G, μ') obtained is homogeneous. Then the Akivis algebra of (G, μ') is reduced to the tangent Lie triple algebra of (G, μ'), that is;

THEOREM 3. *Let (G, μ) be an analytic geodesic loop on a connected and simply connected analytic manifold G. Assume that (G, μ) is changed for an analytic geodesic loop (G, μ') by a geodesic preserving change. Let ∇ (resp. ∇') denote the canonical connection of (G, μ) (resp. (G, μ')) and T the* (1, 2)*-tensor field on G given by $T=\nabla-\nabla'$.*

If T gives an affine homogeneous structure of ∇, then (G, μ') is a homogeneous loop with the tangent Lie triple algebra $\{\mathfrak{G}; [X, Y]', [X, Y, Z]'\}$ given by

$$[X, Y]'=[X, Y]+2T_e(X, Y), \tag{3.3}$$

$$\begin{aligned} [X, Y, Z]'=&2\langle X, Y, Z\rangle-T_e([X, Y], Z)-2T_e(T_e(X, Y), Z) \\ &+T_e(X, T_e(Y, Z))-T_e(Y, T_e(X, Z)), \end{aligned} \tag{3.4}$$

where $[X, Y]'$ and $[X, Y, Z]'=2\langle X, Y, Z\rangle'$ are the operations induced from the tangent Akivis algebra of (G, μ'). Moreover, the torsion S and the curvature R of ∇ are ∇'-parallel tensor fields on G.

Conversely, if (G, μ') is a homogeneous loop and if the torsion tensor S of ∇ is ∇'-parallel on G, then the tensor field $T=\nabla-\nabla'$ is an affine homogeneous structure of ∇. In this case, the tensor field T is completely determined by the equation (3.3) *above.*

PROOF. Suppose that the tensor field T is an affine homogeneous structure of ∇. By Corollary to Proposition 2.4, the connection ∇ is complete and Theorem 1 can be applicable to (G, ∇). We see that (G, ∇') is a reductive homogeneous space A/K with the canonical connection ∇'. If we choose the unit e as the origin of $G=A/K$, then the geodesic local loop of A/K is coincident with that of (G, μ') and the ternary system η' associated with (G, μ') satisfies (2.5) and (2.6) in a neighborhood of e.

Since η' is analytic and G is assumed to be connected, these equations are valid on the whole manifold G, that is, the geodesic loop (G, μ') is homogeneous. The equations $\nabla' T=0$ and (3.1), (3.2) imply the relations (3.3) and (3.4) of the tangent Akivis algebras of (G, μ) and (G, μ'). By Proposition 1.3 we see that the torsion S and the curvature R of ∇ is parallel with respect to ∇'.

Conversely, assume that the geodesic loop (G, μ') be homogeneous. In [8], we have shown that G is a reductive homogeneous space A/K with the canonical connection ∇', where the isotropy subgroup K is a Lie group generated by all left inner mappings of μ'. Hence, we have $\nabla' S'=0$ and $\nabla' R'=0$. If, moreover, $\nabla' S=0$ is satisfied, then from $S'=S+2T$ we get $\nabla' T=0$, that is, $T=\nabla-\nabla'$ is an affine homogeneous structure of (G, ∇). Since (G, μ') is geodesic, the tensor field T satisfies

$$T_x(X_x, Y_x)=\eta'(e, x, T_e(\eta'(x, e, X_x), \eta'(x, e, Y_x)))$$

in a normal neighbourhood of e, and this is valid for all x of G since η' is analytic on G. Hence, we see that the tensor field T is determined by its value at the unit e, which is given by (3.3). q. e. d.

An analytic homogeneous loop (G, μ) is said to be *symmetric* (cf. [8]) if the relation

$$(xy)^{-1}=x^{-1}y^{-1}$$

holds on G. In [8] we have shown that every connected symmetric homogeneous loop is geodesic, and that it can be regarded as an affine symmetric space A/K with the canonical connection. As for geodesic preserving changes of symmetric homogeneous loops, we have

THEOREM 4. *Let (G, μ) be an analytic symmetric homogeneous loop. If a geodesic preserving change (G, μ') of (G, μ) is symmetric too, then $\mu'=\mu$.*

PROOF. Suppose that there exists a geodesic preserving change (G, μ') of the symmetric homogeneous loop (G, μ), and let $T=\nabla-\nabla'$ be the associated (1, 2)-tensor field on G. Since (G, μ) is symmetric, the torsion S of the canonical connection ∇ of (G, μ) vanishes on G, and we have $S'=2T$ for the torsion of ∇'. If (G, μ') is symmetric too, then $S'=0$ and $\nabla'=\nabla$. Two loops must be coincident because they are geodesic at e. q. e. d.

The theorem above shows that there exists no geodesic preserving change except for the trivial change if we restrict ourselevs in the class of symmetric homogeneous loops. Now we consider geodesic preserving changes in the wider class consisting of geodesic homogeneous loops. In what follows we assume that the analytic manifold G is connected and simply connected.

DEFINITION. Two geodesic homogeneous loops (G, μ) and (G, μ') given on G are said to be *geodesically equivalent* if there exists a geodesic preserving change of (G, μ) for (G, μ') such that the associated (1, 2)-tensor field $T=\nabla-\nabla'$ is an affine homogeneous structure of the canonical connection ∇ of (G, μ) and $-T=\nabla'-\nabla$ is also an affine homogeneous structure of the canonical connection ∇' of (G, μ').

THEOREM 5. *Let (G, μ) and (G, μ') be geodesic homogeneous loops with the same unit e, where G is a connected and simply connected analytic manifold. Then, they are geodesically equivalent if and only if the followings are satisfied on the tangent space $\mathfrak{G}$ at e:*

$$T_e(X, X)=0 \tag{3.5}$$

$$T_e(X, S_e(Y, Z))=S_e(T_e(X, Y), Z)+S_e(Y, T_e(X, Z)) \tag{3.6}$$

$$T_e(X, R_e(Y, Z)W)=R_e(T_e(X, Y), Z)W+R_e(Y, T_e(X, Z))W + R_e(Y, Z)T_e(X, W) \tag{3.7}$$

$$T_e(X, T_e(Y, Z))=T_e(T_e(X, Y), Z)+T_e(Y, T_e(X, Z)) \tag{3.8}$$

$$R_e(X, Y)T_e(Z, W)=T_e(R_e(X, Y)Z, W)+T_e(Z, R_e(X, Y)W), \tag{3.9}$$

where $T=\nabla-\nabla'$ is the tensor field associated with the canonical connections ∇ and ∇', and where S and R denote the torsion and the curvature tensor fields of ∇, respectively.

PROOF. Suppose that two geodesic homogeneous loops (G, μ) and (G, μ') are geodesically equivalent. Then, there exists a geodesic preserving change of (G, μ) for (G, μ') such that T and $-T$ are respectively affine homogeneous structures of ∇ and ∇'. Since T is skew-symmetric, (3.5) is clear. The torsion S and the curvature R of ∇ are ∇-parallel because (G, μ) is homogeneous. Hence, the equalities (3.6) and (3.7) follow from (1.8) and (1.9) in Proposition 1.2. From the fact that T is an affine homogeneous structure of ∇ and $-T$ is that of ∇', we can see that T satisfies the equations $\nabla T=0$ and $\nabla' T=0$. Then, (1, 5) implies

$$T(T(X, Y), Z)=T(X, T(Y, Z)-T(Y, T(X, Z)) \tag{3.10}$$

for any vector fields X, Y and Z on G. Thus, (3.8) is obtained. Since $\nabla T=0$ holds on G, T_e is invariant by any left inner mapping $L_{x,y}$ and we get

$$dL(X, Y)T_e(Z, W)=T_e(dL(X, Y)Z, W)+T_e(Z, dL(X, Y)W).$$

We have seen in [10] and [11] that (2.14) in Proposition 2.3 is reduced to $R_e(X, Y)=2dL(X, Y)$ for homogeneous loops. Hence the equality (3.9) is proved.

Conversely, assume that the equations (3.5)–(3.9) are all valid on the tangent space $\mathfrak{G}$ at e. The equation (3.9) implies that the bilinear form T_e on $\mathfrak{G}$ is invariant by any left inner mapping of (G, μ). The ternary system (G, η) associated with (G, μ) is homogeneous in the sense of [9], and T_e is invariant by any composition of displacements of the form $\eta(y, e)\eta(x, y)\eta(e, x)$. Let $\tilde{T}$ be a (1, 2)-tensor field on G given by

$$\tilde{T}_x(X_x, Y_x)=\eta(e, x, T_e(\eta(x, e, X_x), \eta(x, e, Y_x)) \qquad \text{for} \quad x \in G.$$

Then we have

$$\eta(x, y, \tilde{T}_x(X_x, Y_x))=\tilde{T}_y(\eta(x, y, X_x), \eta(x, y, Y_x))$$

and, by Lemma of [11], we see that $\tilde{T}$ is ∇-parallel. Since the tensor fields S and R are ∇-parallel too, the equations (3.5)–(3.8) at e imply the following global equations;

(3.5)′ $\tilde{T}(X, X)=0$

(3.6)′ $\tilde{T}(X, S(Y, Z))=S(\tilde{T}(X, Y), Z)+S(Y, \tilde{T}(X, Z))$

(3.7)′ $\tilde{T}(X, R(Y, Z)W)=R(\tilde{T}(X, Y), Z)\,W+R(Y, \tilde{T}(X, Z))W$
$\qquad +R(Y, Z)\tilde{T}(X, W)$

(3.8)′ $\tilde{T}(X, \tilde{T}(Y, Z))=\tilde{T}(\tilde{T}(X, Y), Z)+\tilde{T}(Y, \tilde{T}(X, Z))$

for any vector fields X, Y, Z and W on G. From $\tilde{T}$, we can obtain a new linear connection $\tilde{\nabla}=\nabla-\tilde{T}$ on G whose torsion and the curvature will be denoted by $\tilde{S}$ and $\tilde{R}$, respectively. Taking the equation (1.5) into account, we can see that the equations (3.8)′ implies $\tilde{\nabla}\tilde{T}=0$. By the equations (1.8) and (1.9) in Proposition 1.2 applied for $\tilde{T}$, the equations $\tilde{\nabla}\tilde{S}=0$ and $\tilde{\nabla}\tilde{R}=0$ follow from (3.6)′ and (3.7)′. Hence, by Theorem 1, we conclude that $(G, \tilde{\nabla})$ is a reductive homogeneous space with the canonical connection $\tilde{\nabla}$. On the other hand, since (G, μ') is homogeneous too, G can be regarded as a reductive homogeneous space with the canonical connection ∇'. However, the two reductive homogeneous spaces must have the same tangent Lie triple algebra because, at the origin e,

$$S'_e(X, Y)=S_e(X, Y)+2T_e(X, Y)=\tilde{S}_e(X, Y),$$

$$R'_e(X, Y)Z=R_e(X, Y)Z-T_e(S_e(X, Y), Z)-T_e(T_e(X, Y), Z)$$
$$=\tilde{R}_e(X, Y)Z$$

for any tangent vectors X, Y and Z at e. Therefore, we see that $\tilde{\nabla}=\nabla'$ and $\tilde{T}=T$, that is, (G, μ) and (G, μ') are geodesically equivalent by the affine structure $\tilde{T}=T$.

q.e.d.

REMARK. In the theorem above, if we denote by T_X ($X \in \mathfrak{G}$) the endomorphism of the tangent Lie triple algebra $\{\mathfrak{G}; [X, Y], [X, Y, Z]\}$ of (G, μ) given by $T_X Y = T_e(X, Y)$, the equations (3.5)–(3.9) are replaced by;

$$T_X X = 0$$

$$T_X[Y, Z] = [T_X Y, Z] + [Y, T_X Z]$$

$$T_X[Y, Z, W] = [T_X Y, Z, W] + [Y, T_X Z, W] + [Y, Z, T_X W]$$

$$[T_X, T_Y] = T_{T_X Y}$$

$$[X, Y, T_Z W] = T_Z[X, Y, W] + T_{[X,Y,Z]} W.$$

EXAMPLE. Let (G, μ) be a connected and simply connected Lie group. Then, the canonical connection is reduced to the $(-)$-connection of E. Cartan and the tangent Lie triple algebra is reduced to the Lie algebra of (G, μ) because the curvature tensor R vanishes identically on G. By Theorem 5 above we can see that the other Lie group (G, μ') which is geodesically equivalent to (G, μ) is given by endomorphisms T_X $(X \in \mathfrak{G})$ satisfying the following;

$$T_X X = 0$$

$$T_X[Y, Z] = [T_X Y, Z] + [Y, T_X Z]$$

$$[T_X, T_Y] = T_{T_X Y} \quad \text{(cf. Remark above)},$$

and $$T_{[X,Y]} + T_{T_X Y} = 0 \text{ (cf. (3.4))}.$$

References

[1] K. Abe, The classification of homogeneous structures on 3-dimensional space forms, Math. J. Okayama Univ., **28** (1986), 173–189.

[2] M. A. Akivis, Local differentiable quasigroups and three webs of multidimensional surfaces (Russian), Studies in the theory of quasigroups and loops, Stiintsa, Kishiev (1973), 3–12.

[3] ———, Local algebras of a multidimensional web, Sib. Mat. Zh. **17** (1976), 5–11.

[4] ———, Geodesic loops and local triple systems in an affine connected spaces, Sib. Mat. Zh. **19** (1978), 243–253.

[5] W. Ambrose-I. M. Singer, On homogeneous Riemannian manifolds, Duke Math. J., **25** (1958), 647–669.

[6] K. H. Hofmann-K. Strambach, The Akivis algebra of a homogeneous loop, Preprint Nr. **908**, Fach. Mat., T. H. Darmstadt (1985).

[7] M. Kikkawa, On local loops in affine manifolds, J. Sci. Hiroshima Univ. A-I, **28** (1964), 199–207.

[8] ———, Geometry of homogeneous Lie loops, Hiroshima Math. J., **5** (1975), 141–179.

[9] ———, On homogeneous systems I-V, Mem. Fac. Lit. & Sci., Shimane Univ. **11** (1977), 9–17; **12** (1978), 5–13; Mem. Fac. Sci. Shimane Univ., **14** (1980), 41–46; **15** (1981), 1–7; **17** (1983), 9–13.

[10] ———, Canonical connections of homogeneous Lie loops and 3-webs, Mem. Fac. Sci. Shimane Univ., **19** (1985), 37–55.

[11] ———, Remarks on canonical connections of loops with the left inverse property, Mem. Fac. Sci. Shimane Univ., **20** (1986), 9–19.

[12] S. Kobayashi-K. Nomizu, Foundations of Differential Geometry Vols. I–II, Interscience Publ. 1963, 1969.

[13] A. Lichnerowicz, Theorie Globale des Connexions et des Groupes d'Holonomie, Ed. Cremonese, 1955.

[14] A. T. Mal'cev, Analytic loops (Russian), Mat. Sb. **36** (1955), 569–576.

[15] K. Nomizu, Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33–65.

[16] F. Tricerri-L. Vanhecke, Homogeneous Structures on Riemannian Manifolds, Cambridge Univ. Press, 1983.

[17] K. Yamaguti, On the Lie triple system and its generalization, J. Sci. Hiroshima Univ. **A-21** (1958), 155–160.