Mem. Fac. Sci. Shimane Univ.,
21, pp. 31–46 Dec. 20, 1987

The Equation $\Delta u = qu$ on an Infinite Network

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(Received September 5, 1987)

We shall discuss the structure of the spaces of some distinguished solutions of the partial difference equation $\Delta u = qu$ on an infinite network. The q-Green function of the network plays an important role in our study.

Introduction

We shall study the partial difference equation $\Delta u - qu = 0$ $(q \geq 0)$ on an infinite network. Our aim is to investigate the structure of the spaces of some distinguished solutions of this equation. As for the elliptic partial differential equation $\Delta u - qu = 0$ on a Riemann surface, the investigation of this direction has been established in [1], [3], [4], [5] and [6]. Most of our results have counterparts in these papers.

We say that a function u on the set X of nodes is q-harmonic (resp. q-superharmonic) at x if $\Delta_q u(x) = \Delta u(x) - q(x)u(x) = 0$ (resp. $\Delta_q u(x) \leq 0$), where Δ is the discrete Laplacian. Minimum principles for q-harmonic or q-superharmonic functions will be studied in §2. In this paper, the energy $E(u) = D(u) + \sum_{x \in X} q(x)u(x)^2$ of u plays the role of the discrete Dirichlet integral $D(u)$ in [8]. With the aid of the class of energy finite q-harmonic functions, we shall give in §3 a classification of infinite networks. The existence and some properties of q-Green function $\tilde{g}_a$ of the network with pole at a will be shown in §4. We shall prove the fundamental inequality: $\sum_{x \in X} q(x)\tilde{g}_a(x) \leq 1$. This result has a counterpart in [1] and [4]. We shall be concerned with the equality $\sum_{x \in X} q(x)\tilde{g}_a(x) = 1$ and its application in §5. A similar equality will be studied in §6. We shall list some fundamental results of the q-Green potentials in §7. The dependence of the q-Green function on q will be studied in §8 as in [1].

§ 1. Preliminaries

Let X be a countable set of nodes, Y be a countable set of arcs and K be the node-arc incidence function. We assume that the graph $G = \{X, Y, K\}$ is connected and locally finite and has no self-loop.

A sequence $\{G_n\}$ $(G_n = \{X_n, Y_n, K\} = \langle X_n, Y_n \rangle)$ of finite subgraphs of G is called an exhaustion of G if the following conditions are fulfilled:

(1.1) $$X_n \subset X_{n+1}, \quad Y_n \subset Y_{n+1}, \quad X = \cup_{n=1}^{\infty} X_n \quad \text{and} \quad Y = \cup_{n=1}^{\infty} Y_n.$$

(1.2) $$Y(x)=\{y\in Y;\ K(x, y)\neq 0\}\subset Y_{n+1} \quad \text{for all} \quad x\in X_n.$$

Here $Y(x)$ is the set of arcs which are incident to node x.

Let r be a strictly positive real function on Y and q be a non-negative real function on X. We call the trio $N(q)=\{G, r, q\}$ an infinite network in this paper. We have studied the network $N=N(0)$ in [7] and [8], i.e., the case where $q=0$. Hereafter we use the notation $N(q)$ only in the case $q\neq 0$.

For a subgraph $G'=\langle X', Y'\rangle$ of G, we can associate subnetworks $N'(q)=\{G', r', q'\}$ and $N'=N'(0)$, where r' is the restriction of r to Y' and q' is the restriction of q to X'.

For notation and terminology, we mainly follow [7] and [8].

Denote by $L(X)$ (resp. $L(Y)$) the set of all real functions on X (resp. Y). For $u\in L(X)$, the (discrete) derivative $du\in L(Y)$, the (discrete) Laplacian $\Delta u\in L(X)$ and the (discrete) Dirichlet integral $D(u)$ of u are defined by

$$du(y)=-r(y)^{-1}\sum_{x\in X}K(x, y)u(x),$$

$$\Delta u(x)=\sum_{y\in Y}K(x, y)[du(y)],$$

$$D(u)=\sum_{y\in Y}r(y)[du(y)]^2.$$

In this paper, we study the discrete analogues of the well-known properties of the solutions of the elliptic partial differential equation $\Delta u=qu$ on a Riemann surface. In order to emphasize the analogy to the continuous case, we shall omit the adjective "discrete" in what follows.

We introduce the q-Laplacian $\Delta_q u\in L(X)$ and the energy $E(u)$ of $u\in L(X)$ as follows:

$$\Delta_q u(x)=\Delta u(x)-q(x)u(x),$$

$$E(u)=D(u)+\sum_{x\in X}q(x)u(x)^2.$$

We say that $u\in L(X)$ is q-superharmonic (resp. q-harmonic) on a subset A of X if $\Delta_q u(x)\leq 0$ (resp. $\Delta_q u(x)=0$) on A.

§ 2. Minimum principles

We shall study minimum principles related to Δ_q. For our study, the following form of Δu is useful:

$$\Delta u(x)=-t(x)u(x)+\sum_{z\in X}t(z, x)u(z),$$

where $t(z, x)=\sum_{y\in Y}|K(z, y)K(x, y)|r(y)^{-1}$ for $z\neq x$, $t(x, x)=0$ and $t(x)=\sum_{y\in Y}|K(x, y)|r(y)^{-1}$. Note that $t(x, z)=t(z, x)$ and $t(x)=\sum_{z\in X}t(z, x)$.

REMARK 2.1. If $q \neq 0$, then a constant function u is q-harmonic on X if and only if $u=0$.

For $x \in X$, denote by $U(x)$ the set of all neighboring nodes of x, i.e., $U(x) = \cup \{e(y);\ y \in Y(x)\}$, where $e(y)$ is the set of end nodes of y. For a subset A of X, we put $U(A) = \cup \{U(x);\ x \in A\}$.

THEOREM 2.1. *Let X' be a finite subset of X and let $u \in L(X)$ be q-superharmonic on X'. If $u \geq 0$ on $X - X'$, then $u \geq 0$ on X'.*

PROOF. Suppose that $m = \min \{u(x);\ x \in X'\} < 0$. There exists $x_0 \in X'$ such that $u(x_0) = m$. Since $\Delta_q u(x_0) \leq 0$, we have

$$t(x_0)u(x_0) \leq \sum_{x \in X} t(x, x_0)u(x) \leq t(x_0)u(x_0) + q(x_0)u(x_0),$$

so that $q(x_0)u(x_0) \geq 0$. Thus $q(x_0) = 0$ and $u(x) = u(x_0)$ on the set $U(x_0)$. Similarly we have $u(x) = u(x_0)$ on $U(U(x_0))$. By repeating this argument a finite number of times, we see that $u(x) = u(x_0)$ on $U(X')$. Since $U(X') \cap (X - X') \neq \emptyset$, we arrive at a contradiction. Therefore $u \geq 0$ on X'.

COROLLARY. *Let X' be a finite subset of X and let u and v be q-harmonic on X'. If $u \geq v$ on $X - X'$, then $u \geq v$ on X'.*

REMARK 2.2. Note that Theorem 2.1 also holds in case $q=0$. This is the usual minimum principle for superharmonic functions.

Let $\{G_n\}$ $(G_n = \langle X_n, Y_n \rangle)$ be an exhaustion of G and Ω_n be the function on X defined by the conditions:

(2.1) $\quad \Omega_n$ is q-harmonic on X_n.

(2.2) $\quad \Omega_n = 1$ on $X - X_n$.

The uniqueness of Ω_n follows from the Corollary of Theorem 2.1. The existence of Ω_n can be shown by the aid of the optimal solution of the following extremum problem:

(2.3) $\quad$ Minimize $E(u)$ subject to $u = 1$ on $X_{n+1} - X_n$.

We see by Theorem 2.1 that $0 \leq \Omega_{n+1} \leq \Omega_n \leq 1$ on X, so that the limit function Ω of $\{\Omega_n\}$ exists. It is easily seen that the function Ω does not depend on the choice of an exhaustion of G. Note that Ω is q-harmonic on X and $0 \leq \Omega \leq 1$ on X. We call Ω the q-harmonic measure of the ideal boundary of $N(q)$.

By the same argument as in the proof of Theorem 2.1, we can prove

THEOREM 2.2. *Let u be q-superharmonic on X. If $u \geq 0$ on X and $u(x_0) = 0$ for some $x_0 \in X$, then $u(x) = 0$ on X.*

REMARK 2.3. We can not expect the following property in general unless $q=0$: If u is a non-constant q-superharmonic function on X, then u does not attain its minimum.

Next we prove a discrete analogue of Harnack's inequality.

THEOREM 2.3. *Let $a, b \in X$. There exists a positive constant α depend only on a and b such that $\alpha^{-1}u(b) \leq u(a) \leq \alpha u(b)$ for every non-negative q-superharmonic function u on X.*

PROOF. The condition $\Delta_q u(a) \leq 0$ implies that

$$t(x, a)u(x) \leq \sum_{x \in X} t(x, a)u(x) \leq [t(a)+q(a)]u(a),$$

so that $u(x) \leq v(x, a)u(a)$ with $v(x, a)=[t(a)+q(a)]/t(x, a)$ for every $x \in U(a)$, $x \neq a$. There exists a path P from a to b with $C_X(P)=\{x_k;\ k=0, 1,\dots, n\}$ ($x_0=a$ and $x_n=b$). Then $u(x_k) \leq v(x_k, x_{k-1})u(x_{k-1})$ for each k, so that $u(b) \leq c(b, a)u(a)$ with $c(b, a)=\prod_{k=1}^{n} v(x_k, x_{k-1})$. Taking $\alpha=\max[c(a, b), c(b, a)]$, we see that α satisfies our requirement.

We say that $u \in L(X)$ vanishes at the ideal boundary if for any $\varepsilon>0$, there exists a finite subset X' of X such that $|u(x)|<\varepsilon$ on $X-X'$.

THEOREM 2.4. *If u is q-harmonic on X and vanishes at the ideal boundary, then $u=0$ on X.*

PROOF. For any $\varepsilon>0$, there exists a finite subset X' of X such that $|u(x)|<\varepsilon$ on $X-X'$. Consider an exhaustion $\{G_n\}(G_n=\langle X_n, Y_n\rangle)$ of G. There exists n_0 such that $X' \subset X_n$ for all $n \geq n_0$. We have $|u(x)| \leq \varepsilon\Omega_n(x)$ on X for all $n \geq n_0$ by Theorem 2.1, so that $|u(x)| \leq \varepsilon\Omega(x) \leq \varepsilon$ on X. Thus $u=0$ on X.

THEOREM 2.5. *Let X' be a finite subset of X, $\varphi \in L(X)$ and let u and v satisfy the equations: $\Delta_q u=\Delta v=\varphi$ on X'. If $\varphi \leq 0$ on X' and if $v \geq u \geq 0$ on $X-X'$, then $v \geq u$ on X.*

PROOF. By Theorem 2.1, $u \geq 0$ on X. Put $f=v-u$. Then $\Delta f=-qu \leq 0$ on X' and $f \geq 0$ on $X-X'$. We have $f \geq 0$ on X by Theorem 2.1 with $q=0$.

§ 3. The spaces $E(N(q))$ and $E_0(N(q))$

Let us introduce some spaces of functions on X:

$$L_0(X)=\{u \in L(X);\ \{x \in X;\ u(x) \neq 0\} \text{ is a finite set}\},$$

$$E(N(q))=\{u \in L(X);\ E(u)<\infty\},$$

$$D(N)=\{u \in L(X);\ D(u)<\infty\}=E(N(0)),$$

$$\boldsymbol{H}(N(q)) = \{u \in \boldsymbol{L}(X);\ \Delta_q u = 0 \text{ on } X\}.$$

Needless to say, we have $L_0(X) \subset \boldsymbol{E}(N(q)) \subset \boldsymbol{D}(N)$.

In case $q \neq 0$, we see easily that $\boldsymbol{E}(N(q))$ is a Hilbert space with respect to the inner product:

$$E(u, v) = D(u, v) + \sum_{x \in X} q(x)u(x)v(x),$$

where $D(u, v) = \sum_{y \in Y} r(y)[du(y)][dv(y)]$ is the Dirichlet mutual integral of u and v. Denote by $\boldsymbol{E}_0(N(q))$ the closure of $L_0(X)$ in $\boldsymbol{E}(N(q))$ with respect to the norm $[E(u)]^{1/2}$ and by $\boldsymbol{D}_0(N)$ the closure of $L_0(X)$ with respect to the norm $[D(u) + u(b)^2]^{1/2}$, where b is a fixed node of X.

We shall write $N \in O_G$ if $\boldsymbol{D}(N) = \boldsymbol{D}_0(N)$, or equivalently $1 \in \boldsymbol{D}_0(N)$ (cf. [7], [8]).

THEOREM 3.1. $\boldsymbol{E}_0(N(q)) = \boldsymbol{D}_0(N) \cap \boldsymbol{E}(N(q))$.

PROOF. Since $\boldsymbol{E}_0(N(q)) \subset \boldsymbol{E}(N(q))$ and $\boldsymbol{E}_0(N(q)) \subset \boldsymbol{D}_0(N)$, we have $\boldsymbol{E}_0(N(q)) \subset \boldsymbol{D}_0(N) \cap \boldsymbol{E}(N(q))$. To prove the converse relation, let u be an element of $\boldsymbol{D}_0(N) \cap \boldsymbol{E}(N(q))$. Then there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $D(u - f_n) \to 0$ as $n \to \infty$, $f_n(x)$ converges pointwise to $u(x)$ and $|f_n(x)| \leq |u(x)|$ on X. Since $\sum_{x \in X} q(x)u(x)^2 < \infty$, we see that $\sum_{x \in X} q(x)[u(x) - f_n(x)]^2 \to 0$ as $n \to \infty$. Thus $E(u - f_n) \to 0$ as $n \to \infty$, i.e., $u \in \boldsymbol{E}_0(N(q))$. This completes the proof.

COROLLARY 1. *If $N \in O_G$, then $\boldsymbol{E}_0(N(q)) = \boldsymbol{E}(N(q))$.*

PROOF. If $N \in O_G$, then $\boldsymbol{D}_0(N) = \boldsymbol{D}(N)$, so that $\boldsymbol{E}_0(N(q)) = \boldsymbol{D}(N) \cap \boldsymbol{E}(N(q)) \supset \boldsymbol{E}(N(q))$ by Theorem 3.1. Hence $\boldsymbol{E}_0(N(q)) = \boldsymbol{E}(N(q))$.

The converse of this result does not hold in general. But we have

COROLLARY 2. *Assume that $\sum_{x \in X} q(x) < \infty$. Then $N \in O_G$ if and only if $\boldsymbol{E}_0(N(q)) = \boldsymbol{E}(N(q))$.*

PROOF. Assume that $\boldsymbol{E}_0(N(q)) = \boldsymbol{E}(N(q))$. Since $\sum_{x \in X} q(x) < \infty$, we have $1 \in \boldsymbol{E}(N(q)) = \boldsymbol{E}_0(N(q)) \subset \boldsymbol{D}_0(N)$. Thus $N \in O_G$.

In case $q \neq 0$, we introduce the following distinguished subspaces of $\boldsymbol{H}(N(q))$:

$$\boldsymbol{HD}(N(q)) = \boldsymbol{H}(N(q)) \cap \boldsymbol{D}(N), \quad \boldsymbol{HE}(N(q)) = \boldsymbol{H}(N(q)) \cap \boldsymbol{E}(N(q)).$$

$$\boldsymbol{HB}(N(q)) = \{u \in \boldsymbol{H}(N(q));\ u \text{ is bounded on } X\}.$$

It is easily seen that $\boldsymbol{HE}(N(q)) \subset \boldsymbol{HD}(N(q))$ and that $\boldsymbol{HE}(N(q))$ is a closed subspace of the Hilbert space $\boldsymbol{E}(N(q))$.

LEMMA 3.1. *$\boldsymbol{HE}(N(q))$ is the orthogonal complement of $\boldsymbol{E}_0(N(q))$ in $\boldsymbol{E}(N(q))$.*

PROOF. For $f \in L_0(X)$ and $h \in \boldsymbol{HE}(N(q))$, we have

$$E(f, h) = D(f, h) + \sum_{x \in X} q(x)f(x)h(x)$$
$$= -\sum_{x \in X} [\Delta h(x)]f(x) + \sum_{x \in X} q(x)f(x)h(x)$$
$$= -\sum_{x \in X} [\Delta_q h(x)]f(x) = 0,$$

so that $E(v, h) = 0$ for every $v \in \boldsymbol{E}_0(N(q))$ and $h \in \boldsymbol{HE}(N(q))$. Conversely, suppose that $h \in \boldsymbol{E}(N(q))$ satisfies $E(v, h) = 0$ for all $v \in \boldsymbol{E}_0(N(q))$. Let ε_x be the characteristic function of the set $\{x\}$. Since $\varepsilon_x \in L_0(X)$ and $E(\varepsilon_x, h) = -\Delta_q h(x)$, it follows that $h \in \boldsymbol{HE}(N(q))$.

By a standard argument, we obtain

THEOREM 3.2. *Every $u \in \boldsymbol{E}(N(q))$ is uniquely decomposed into $u = v + h$ with $v \in \boldsymbol{E}_0(N(q))$ and $h \in \boldsymbol{HE}(N(q))$.*

COROLLARY. *$\boldsymbol{HE}(N(q)) = \{0\}$ if and only if $\boldsymbol{E}_0(N(q)) = \boldsymbol{E}(N(q))$.*

THEOREM 3.3. *If $\inf\{q(x); x \in X\} > 0$, then $\boldsymbol{HE}(N(q)) = \{0\}$.*

PROOF. Let $h \in \boldsymbol{HE}(N(q))$ and put $c = \inf\{q(x); x \in X\}$. Then

$$c \sum_{x \in X} h(x)^2 \leq \sum_{x \in X} q(x)h(x)^2 \leq E(h) < \infty,$$

so that h vanishes at the ideal boundary. Thus $h = 0$ by Theorem 2.4.

The following result is due to Maeda [2]:

THEOREM 3.4. *If $N \in O_G$, then $\boldsymbol{HD}(N(q)) = \{0\}$.*

PROOF. Let $u \in \boldsymbol{HD}(N(q))$. Since $N \in O_G$, there exists a sequence $\{f_n\}$ such that $f_n \in L_0(X)$, $0 \leq f_n \leq 1$ on X, $D(f_n) \to 0$ as $n \to \infty$ and f_n converges pointwise to 1. For a positive integer m, we put $u_m(x) = \max\{-m, \min[u(x), m]\}$. Since $u_m f_n \in L_0(X)$, we have by Lemma 3.1

$$0 = E(u_m f_n, u) = D(u_m f_n, u) + \sum_{x \in X} q(x)u_m(x)f_n(x)u(x). \tag{3.1}$$

For $y \in Y$, put $e(y) = \{a(y), b(y)\}$ (the end of y). Then

$$(du_m f_n)(y) = f_n(b(y))[du_m(y)] + u_m(a(y))[df_n(y)],$$

$$D(u_m f_n, u) = \sum_{y \in Y} r(y)f_n(b(y))[du_m(y)][du(y)] + \sum_{y \in Y} r(y)u_m(a(y))[df_n(y)][du(y)]. \tag{3.2}$$

We have

$$|\sum_{y \in Y} r(y)u_m(a(y))[df_n(y)][du(y)]| \leq mD(f_n)^{1/2}D(u)^{1/2} \longrightarrow 0$$

as $n \to \infty$. Note that $[du_m(y)][du(y)] \geq 0$ for all $y \in Y$ and that $\sum_{x \in X} q(x)u_m(x)f_n(x)u(x) \geq 0$. We deduce from (3.1) and (3.2) that

$$\sum_{y \in Y} r(y)\,[du_m(y)]\,[du(y)] = 0,$$

and hence $[du_m(y)][du(y)] = 0$ for every $y \in Y$. Since this is true for any $m > 0$, we conclude that $du(y) = 0$ on Y, or u is a constant function. Thus $u = 0$ by Remark 2.1.

§4. q-Green function

In case $q \neq 0$, we say that a function $u \in L(X)$ is the q-Green function of $N(q)$ with pole at $a \in X$ if it satisfies the condition

(4.1) $\qquad u \in E_0(N(q))$ and $\Delta_q u(x) = -\varepsilon_a(x)$ on X,

where $\varepsilon_a(x) = 0$ if $x \neq a$ and $\varepsilon_a(a) = 1$. The uniqueness of the q-Green function follows from Lemma 3.1. Hereafter we denote by $\tilde{g}_a$ the q-Green function of $N(q)$ with pole at a. The existence of $\tilde{g}_a$ is assured by

THEOREM 4.1. *There exists a unique q-Green function $\tilde{g}_a$ of $N(q)$ with pole at a.*

PROOF. There exists $b \in X$ such that $q(b) > 0$. For any $x \in X$, we can find a constant M_x such that $|u(x)| \leq M_x[D(u) + |u(b)|^2]^{1/2}$ for all $u \in D(N)$ (cf. [7]). Let $M'_x = M_x[1 + q(b)^{-1}]^{1/2}$. Then we have $|u(x)| \leq M'_x[E(u)]^{1/2}$ for all $u \in E(N(q))$. Therefore $u(a)$ is a continuous linear functional on both $E_0(N(q))$ and $E(N(q))$ for every $a \in X$. By F. Riesz's theorem, there exists a reproducing kernel φ_a of $E_0(N(q))$, i.e., $\varphi_a \in E_0(N(q))$ and

(4.2) $\qquad E(\varphi_a, u) = u(a)$ for every $u \in E_0(N(q))$.

Since $\varepsilon_x \in E_0(N(q))$, we have by (4.2)

$$\begin{aligned}\varepsilon_x(a) &= D(\varepsilon_x, \varphi_a) + \sum_{z \in X} q(z)\varepsilon_x(z)\varphi_a(z) \\ &= -\Delta\varphi_a(x) + q(x)\varphi_a(x) = -\Delta_q\varphi_a(x).\end{aligned}$$

Namely φ_a is the q-Green function of $N(q)$ with pole at a.

COROLLARY. $\tilde{g}_a(a) = E(\tilde{g}_a) > 0$.

REMARK 4.1. In case $q = 0$, the harmonic Green function g_a of N with pole at a is defined by the condition:

(4.3) $\qquad g_a \in D_0(N)$ and $\Delta g_a(x) = -\varepsilon_a(x)$ on X.

The harmonic Green function g_a exists if and only if $N \notin O_G$ (cf. [8]).

By (4.2), we obtain the following fundamental properties of $\tilde{g}_a$:

THEOREM 4.2. *$\tilde{g}_a(x)=\tilde{g}_x(a)$ for every $a, x \in X$,*

LEMMA 4.1. *The function $\tilde{u}=\tilde{g}_a/\tilde{g}_a(a)$ is the unique optimal solution of the extremum problem:*

(4.4) *Minimize $E(u)$ subject to $u \in E_0(N(q))$ and $u(a)=1$.*

Let T be a normal contraction of the real line R, i.e., $|Ts_1-Ts_2| \leq |s_1-s_2|$ for every $s_1, s_2 \in R$. For every $u \in L(X)$, we define $Tu \in L(X)$ by $(Tu)(x)=Tu(x)$ for $x \in X$. Since $D(Tu) \leq D(u)$ and $\sum_{x \in X} q(x)(Tu(x))^2 \leq \sum_{x \in X} q(x)u(x)^2$, we have $E(Tu) \leq E(u)$ for every $u \in E(N(q))$. In our study, we often use the following normal contractions: (1) $Ts=|s|$, (2) $Ts=\max(s, 0)$, (3) $Ts=\min(s, c)$ $(c>0)$.

LEMMA 4.2. *Let T be a normal contraction of the real line. If $u \in E_0(N(q))$, then $Tu \in E_0(N(q))$.*

PROOF. Let $u \in E_0(N(q))$. Then $u \in D_0(N)$ and we proved in [9] that $Tu \in D_0(N)$. Since $Tu \in E(N(q))$ by the above observation, we see by Theorem 3.1 that $Tu \in E_0(N(q))$.

THEOREM 4.3. *$0<\tilde{g}_a(x) \leq \tilde{g}_a(a)$ on X.*

PROOF. Let $\tilde{u}$ be the function defined in Lemma 4.1. Since $E(\max(\tilde{u}, 0)) \leq E(\tilde{u})$ and $E(\min(\tilde{u}, 1)) \leq E(\tilde{u})$, we have $\max(\tilde{u}, 0)=\tilde{u}$ and $\min(\tilde{u}, 1)=\tilde{u}$ by Lemmas 4.1 and 4.2, and hence $0 \leq \tilde{u} \leq 1$ on X. We see by Theorem 2.2 that $\tilde{u}>0$ on X.

Let $G'=\langle X', Y' \rangle$ be a finite subgraph of G and $a \in X'$. The q-Green function $\tilde{g}'_a$ of $N'(q)$ with pole at a is defined by the condition:

(4.5) $\Delta_q \tilde{g}'_a(x)=-\varepsilon_a(x)$ on X' and $\tilde{g}'_a(x)=0$ on $X-X'$.

The existence and uniqueness of the q-Green function of $N'(q)$ can be shown by the standard argument as above. Note that $\tilde{g}'_a$ is characterized by the relation:

(4.6) $E(u, \tilde{g}'_a)=u(a)$ for all $u \in L(X)$ such that $u=0$ on $X-X'$.

Furthermore we see that $\tilde{u}'=\tilde{g}'_a/\tilde{g}'_a(a)$ is the optimal solution of the extremum problem:

(4.7) Minimize $E(u)$ subject to $u(a)=1$ and $u=0$ on $X-X'$.

By the same reasoning as above, we obtain the following properties of $\tilde{g}'_a$:

(4.8) $\tilde{g}'_a(x)=\tilde{g}'_x(a)$ for every $a, x \in X'$,

(4.9) $0<\tilde{g}'_a(x) \leq \tilde{g}'_a(a)$ on X'.

REMARK 4.2. The harmonic Green function g'_a of N' with pole at $a \in X'$ is defined by the condition:

(4.10) $$\Delta g'_a(x)=-\varepsilon_a(x) \quad \text{on} \quad X' \quad \text{and} \quad g'_a(x)=0 \quad \text{on} \quad X-X'.$$

We have $0\leq \tilde{g}'_a(x)\leq g'_a(x)$ on X by Theorem 2.5.

THEOREM 4.4. *Let $\{G_n\}$ $(G_n=\langle X_n, Y_n\rangle)$ be an exhaustion of G and $a\in X_1$. Denote by $\tilde{g}_a^{(n)}$ the q-Green function of $N_n(q)$ with pole at a. Then $0\leq \tilde{g}_a^{(n)}(x)\leq \tilde{g}_a^{(n+1)}(x)\leq \tilde{g}_a(x)$ on X and $\{\tilde{g}_a^{(n)}\}$ converges pointwise to $\tilde{g}_a$.*

PROOF. Put $u=\tilde{g}_a^{(n+1)}-\tilde{g}_a^{(n)}$ and $v=\tilde{g}_a-\tilde{g}_a^{(n)}$. Then u and v are q-harmonic on X_n and non-negative on $X-X_n$. Thus u and v are non-negative on X by Theorem 2.1. Therefore $0\leq \tilde{g}_a^{(n)}(x)\leq \tilde{g}_a^{(n+1)}(x)\leq \tilde{g}_a(x)$ on X. For $m>n$, we have

$$E(\tilde{g}_a^{(n)}, \tilde{g}_a^{(m)})=\tilde{g}_a^{(n)}(a)=E(\tilde{g}_a^{(n)}, \tilde{g}_a^{(n)})\leq \tilde{g}_a(a),$$

$$\begin{aligned} E(\tilde{g}_a^{(m)}-\tilde{g}_a^{(n)}) &= E(\tilde{g}_a^{(m)})-2E(\tilde{g}_a^{(m)}, \tilde{g}_a^{(n)})+E(\tilde{g}_a^{(n)}) \\ &= E(\tilde{g}_a^{(m)})-E(\tilde{g}_a^{(n)}). \end{aligned}$$

It follows that $\{\tilde{g}_a^{(n)}\}$ is a Cauchy sequence in the Hilbert space $E_0(N(q))$. There exists $f\in E_0(N(q))$ such that $E(\tilde{g}_a^{(n)}-f)\to 0$ as $n\to\infty$. Since $\{\tilde{g}_a^{(n)}\}$ converges pointwise to f, we see that $\Delta_q f(x)=-\varepsilon_a(x)$ on X, so that $f=\tilde{g}_a$.

COROLLARY. $E(\tilde{g}_a^{(n)}-\tilde{g}_a)\to 0$ as $n\to\infty$.

THEOREM 4.5. $\sum_{x\in X} q(x)\tilde{g}_a(x)\leq 1$.

PROOF. Let f_n be the characteristic function of X_n. Since $\Delta\tilde{g}_a^{(n)}(x)=q(x)\tilde{g}_a^{(n)}(x)-\varepsilon_a(x)$ on X_n, we have

$$\sum_{x\in X} q(x)\tilde{g}_a^{(n)}(x)=1+\sum_{x\in X} f_n(x)\,[\Delta\tilde{g}_a^{(n)}(x)]=1-D(f_n, \tilde{g}_a^{(n)}).$$

Denote by Z_n the set of all $y\in Y$ which connects X_n and $X-X_n$, i.e., $y\in Z_n$ if and only if $e(y)=\{x, x'\}$ with $x\in X_n$ and $x'\in X-X_n$. For $y\in Z_n$, let $x(y)$ be the node such that $x(y)\in e(y)$ and $x(y)\in X_n$. We have for $y\in Z_n$

$$df_n(y)=-r(y)^{-1}K(x(y), y),$$

$$d\tilde{g}_a^{(n)}(y)=-r(y)^{-1}K(x(y), y)\tilde{g}_a^{(n)}(x(y)),$$

so that

$$\begin{aligned} D(f_n, \tilde{g}_a^{(n)}) &= \sum_{y\in Y} r(y)\,[df_n(y)]\,[d\tilde{g}_a^{(n)}(y)] \\ &= \sum_{y\in Y} r(y)^{-1}K(x(y), y)^2\tilde{g}_a^{(n)}(x(y))\geq 0. \end{aligned}$$

Thus $\sum_{x\in X} q(x)\tilde{g}_a^{(n)}(x)\leq 1$. Our assertion follows from Theorem 4.4.

§5. The equality $\sum_{x\in X} q(x)\tilde{g}_a(x)=1$

We are concerned with the inequality in Theorem 4.5.

LEMMA 5.1. *Let* $\{G_n\}$ $(G_n=\langle X_n, Y_n\rangle)$ *be an exhaustion of* G *and* Ω_n *be the function defined in* §2, *i.e.,* Ω_n *is* q-*harmonic on* X_n *and* $\Omega_n=1$ *on* $X-X_n$. *Then* $\Omega_n(x)=1-\sum_{z\in X_n} q(z)\tilde{g}_z^{(n)}(x)$.

PROOF. Put $u(x)=1-\sum_{z\in X_n} q(z)\tilde{g}_z^{(n)}(x)$. For $x\in X_n$, we have

$$\begin{aligned}\Delta_q u(x)&=\Delta_q 1(x)-\sum_{z\in X_n} q(z)\Delta_q\tilde{g}_z^{(n)}(x)\\&=-q(x)-\sum_{z\in X_n} q(z)[-\varepsilon_z(x)]=0.\end{aligned}$$

Since $\tilde{g}_z^{(n)}(x)=0$ for every $x\in X-X_n$ and $z\in X_n$, we see that $u=1$ on $X-X_n$. Thus $u=\Omega_n$.

By Theorem 4.4 and Lemma 4.1, we obtain

THEOREM 5.1. *Let* Ω *be the* q-*harmonic measure of the ideal boundary of* $N(q)$. *Then* $\Omega(x)=1-\sum_{z\in X} q(z)\tilde{g}_z(x)$.

THEOREM 5.2. $\Omega=0$ *if and only if* $\boldsymbol{HB}(N(q))=\{0\}$.

PROOF. Let $u\in \boldsymbol{HB}(N(q))$ and $|u(x)|\leq c$ on X. By the Corollary of Theorem 2.1, we have $|u(x)|\leq c\Omega_n(x)$ on X, and hence $|u(x)|\leq c\Omega(x)$ on X. Thus $\Omega=0$ implies $\boldsymbol{HB}(N(q))=\{0\}$. Since $\Omega\in \boldsymbol{HB}(N(q))$, the converse is clear.

By Theorems 5.1 and 5.2, we have

THEOREM 5.3. $\sum_{z\in X} q(z)\tilde{g}_z(x)=1$ *for all* $x\in X$ *if and only if* $\boldsymbol{HB}(N(q))=\{0\}$.

REMARK 5.1. If $\sum_{z\in X} q(z)\tilde{g}_z(a)=1$ for some $a\in X$, then we see by Theorem 2.2 that $\sum_{z\in X} q(z)\tilde{g}_z(x)=1$ for all $x\in X$.

THEOREM 5.4. *If* $N\in O_G$, *then* $\sum_{z\in X} q(z)\tilde{g}_z(a)=1$ *for every* $a\in X$.

PROOF. Since $N\in O_G$, there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $0\leq f_n\leq 1$, $D(f_n)\to 0$ as $n\to\infty$ and $\{f_n\}$ converges pointwise to 1 (cf. [7]). We have

$$\begin{aligned}\sum_{z\in X} f_n(z)q(z)\tilde{g}_a(z)&=\sum_{z\in X} f_n(z)[\Delta\tilde{g}_a(z)+\varepsilon_a(z)]\\&=-D(f_n, \tilde{g}_a)+f_n(a).\end{aligned}$$

Since $0\leq f_n(z)\leq 1$ on X and $\sum_{z\in X} q(z)\tilde{g}_a(z)\leq 1$, we obtain

$$\lim_{n\to\infty}\sum_{z\in X}f_n(z)q(z)\tilde{g}_a(z)=\sum_{z\in X}q(z)\tilde{g}_a(z).$$

On the other hand, we have

$$|D(f_n,\tilde{g}_a)|\leq D(f_n)^{1/2}D(\tilde{g}_a)^{1/2}\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty.$$

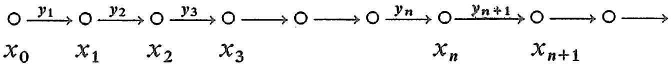
Thus we obtain the desired equality.

By Theorems 5.3 and 5.4, we obtain

THEOREM 5.5. *If $N\in O_G$, then $HB(N(q))=\{0\}$.*

This result has a counter part in [2] and [4].
We give an example which shows $HB(N(q))\neq\{0\}$:

EXAMPLE 5.1. Let us consider the infinite graph $G=\{X, Y, K\}$ shown as in the following figure, where $X=\{x_n;\ n=0, 1, 2,\dots\}$ and $Y=\{y_n;\ n=1, 2,\dots\}$:



Here $K(x_n, y_n)=1$ and $K(x_{n-1}, y_n)=-1$ for every positive integer n and $K(x, y)=0$ for any other pair. Let $q\in L^+(X)$, $q\neq 0$ and let $r\in L(Y)$ be strictly positive. Then $N(q)=\{G, r, q\}$ is an infinite network. For simplicity, we put $q_n=q(x_n)$, $r_n=r(y_n)$, $u_n=u(x_n)$ and $w_n=-du(y_n)=r_n^{-1}(u_n-u_{n-1})$.
Note that $\Delta u(x_0)=w_1$ and $\Delta u(x_n)=w_{n+1}-w_n$ for $n\geq 1$. Thus $u\in H(N(q))$ implies that $w_1=q_0u_0$ and $w_{n+1}-w_n=q_nu_n$. It follows that

$$u_{n+1}=u_n+r_{n+1}\sum_{k=0}^{n}q_ku_k$$

for $n\geq 0$. In case $u_0\geq 0$, we have $u_{n+1}\geq u_n\geq u_0$, so that

$$u_n+\alpha_nu_0\leq u_{n+1}\leq(1+\alpha_n)u_n \quad \text{with} \quad \alpha_n=r_{n+1}\sum_{k=0}^{n}q_k.$$

Therefore $(1+\sum_{k=0}^{n}\alpha_k)u_0\leq u_{n+1}\leq[\prod_{k=0}^{n}(1+\alpha_k)]u_0$. It is well-known that $\{\prod_{k=0}^{n}(1+\alpha_k)\}$ converges if and only if $\sum_{k=0}^{\infty}\alpha_k<\infty$. We see that $\{u_n\}$ is bounded if and only if $\sum_{k=0}^{\infty}\alpha_k<\infty$. In case $u_0<0$, we may consider $\{-u_n\}$ and obtain the same result. Since $u_0>0$ implies $u_n>0$ for all n, we conclude that $HB(N(q))\neq\{0\}$ if and only if $\sum_{k=0}^{\infty}\alpha_k<\infty$. As special choices of $\{r_n\}$ and $\{q_n\}$, we give three examples:
(1) $r_n=1$ and $q_n=1$. In this case $\alpha_n=n+1$ and $\sum_{k=0}^{\infty}\alpha_k=\infty$.
(2) $r_n=2^{-n}$ and $q_n=1$. In this case $\alpha_n=(n+1)2^{-(n+1)}$ and $\sum_{k=0}^{\infty}\alpha_k<\infty$.
(3) $r_n=2^{-n}$ and $q_n=2^n$. In this case $\alpha_n=1-2^{-(n+1)}$ and $\sum_{k=0}^{\infty}\alpha_k=\infty$.
Note that $N\in O_G$ in (1) and that $N\notin O_G$ in (2) and (3).

§6. The case where $HE(N(q))=\{0\}$

We introduce the reproducing kernel of $E(N(q))$. Let $a\in X$. Then $u(a)$ is a continuous linear functional on $\boldsymbol{E}(N(q))$ (cf. the proof of Theorem 4.1). By F. Riesz's theorem, we can find a unique $\tilde{k}_a \in \boldsymbol{E}(N(q))$ such that

(6.1) $$E(\tilde{k}_a, u) = u(a) \quad \text{for every} \quad u \in \boldsymbol{E}(N(q)).$$

The following properties of $\tilde{k}_a$ can be shown by the same reasoning as in §4:

(6.2) $$\Delta_q \tilde{k}_a(x) = -\varepsilon_a(x) \quad \text{on} \quad X.$$

(6.3) $$\tilde{k}_a(b) = \tilde{k}_b(a) \quad \text{for every} \quad a, b \in X.$$

(6.4) $$0 \le \tilde{k}_a(x) \le \tilde{k}_a(a) \quad \text{on} \quad X.$$

THEOREM 6.1. *$0 < \tilde{g}_a(x) \le \tilde{k}_a(x)$ on X.*

PROOF. Let $\{G_n\}$ be an exhaustion of G and let $\tilde{g}_a^{(n)}$ be the q-Green function of $N_n(q)$ with pole at a. We have $\tilde{g}_a^{(n)}(x) \le \tilde{k}_a(x)$ on X by Theorem 2.1, and hence $\tilde{g}_a(x) \le \tilde{k}_a(x)$ on X.

In order to obtain more fine properties of $\tilde{k}_a$, we consider an exhaustion $\{G_n\}$ ($G_n = \langle X_n, Y_n \rangle$) of G and the quantities $D_n(u, v)$, $E_n(u, v)$ and $E_n(u)$ defined by

$$D_n(u, v) = \sum_{y \in Y_n} r(y)\,[du(y)]\,[dv(y)],$$

$$E_n(u, v) = D_n(u, v) + \sum_{x \in X_n} q(x)u(x)v(x), \quad E_n(u) = E_n(u, u).$$

Let $a \in X_n$ and $q \ne 0$ on X_n. Then $L(X_n)$ is a Hilbert space with respect to the inner product $E_n(u, v)$. Thus there exists a unique $\tilde{k}_a^{(n)} \in L(X_n)$ such that

(6.5) $$E_n(\tilde{k}_a^{(n)}, u) = u(a) \quad \text{for every} \quad u \in L(X_n).$$

We can show the following properties similarly:

(6.6) $$\Delta_q^{(n)} \tilde{k}_a^{(n)}(x) = -\varepsilon_a(x) \quad \text{on} \quad X_n,$$

where $\Delta_q^{(n)} u(x) = \sum_{y \in Y_n} K(x, y)[du(y)] - q(x)u(x)$ and this is the q-Laplacian of $u \in L(X_n)$ on the network $N_n(q)$.

(6.7) $$\tilde{k}_a^{(n)}(b) = \tilde{k}_b^{(n)}(a) \quad \text{for every} \quad a, b \in X_n.$$

(6.8) $$0 \le \tilde{k}_a^{(n)}(x) \le \tilde{k}_a^{(n)}(a) \quad \text{on} \quad X_n.$$

LEMMA 6.1. *$\sum_{x \in X_n} q(x)\tilde{k}_a^{(n)}(x) = 1$.*

PROOF. Since $1 \in L(X_n)$ and $d1(y) = 0$ on Y_n, we have by (6.5)

$$1 = E_n(\tilde{k}_a^{(n)}, 1) = D_n(\tilde{k}_a^{(n)}, 1) + \sum_{x \in X_n} q(x)\tilde{k}_a^{(n)}(x)$$
$$= \sum_{x \in X_n} q(x)\tilde{k}_a^{(n)}(x).$$

LEMMA 6.2. *$\tilde{k}_a^{(n)}(x) \to \tilde{k}_a(x)$ as $n \to \infty$ for each $x \in X$.*

PROOF. For $m \geq n \geq j$, we have $E_n(\tilde{k}_a^{(n)}, \tilde{k}_a^{(m)}) = \tilde{k}_a^{(m)}(a)$ and

$$\begin{aligned}(6.9) \qquad E_j(\tilde{k}_a^{(n)} - \tilde{k}_a^{(m)}) &\leq E_n(\tilde{k}_a^{(n)} - \tilde{k}_a^{(m)}) \\ &= E_n(\tilde{k}_a^{(n)}) - 2E_n(\tilde{k}_a^{(n)}, \tilde{k}_a^{(m)}) + E_n(\tilde{k}_a^{(m)}) \\ &\leq \tilde{k}_a^{(n)}(a) - 2\tilde{k}_a^{(m)}(a) + E_m(\tilde{k}_a^{(m)}) \\ &= \tilde{k}_a^{(n)}(a) - \tilde{k}_a^{(m)}(a).\end{aligned}$$

It follows that $\{\tilde{k}_a^{(n)}(a)\}$ is a decreasing sequence, so that the restriction of $\tilde{k}_a^{(m)}$ to X_j converges pointwise to a function $\tilde{u}$ on X_j. Since j is arbitrary, we obtain a function $\tilde{u} \in L(X)$ such that $\Delta_q \tilde{u}(x) = -\varepsilon_a(x)$ on X and

$$(6.10) \qquad E_n(\tilde{u} - \tilde{k}_a^{(n)}) = \lim_{m \to \infty} E_n(\tilde{k}_a^{(m)} - \tilde{k}_a^{(n)}) \leq \tilde{k}_a^{(n)}(a) - \tilde{u}(a).$$

Since $E_n(\tilde{u} - \tilde{k}_a^{(n)}) = E_n(\tilde{u}) - 2\tilde{u}(a) + \tilde{k}_a^{(n)}(a)$, we have $E_n(\tilde{u}) \leq \tilde{u}(a)$ by (6.10), so that $E(\tilde{u}) = \lim_{n \to \infty} E_n(\tilde{u}) \leq \tilde{u}(a) < \infty$, i.e., $\tilde{u} \in \boldsymbol{E}(N(q))$. We shall show that $E(\tilde{u}, v) = v(a)$ for every $v \in \boldsymbol{E}(N(q))$. For any $\varepsilon > 0$, there exists n_0 such that $E_n(\tilde{u} - \tilde{k}_a^{(n)}) < \varepsilon^2$ for all $n \geq n_0$ by (6.10). Since $E_n(\tilde{u}, v) \to E(\tilde{u}, v)$ as $n \to \infty$, there exists n_1 such that $|E_n(\tilde{u}, v) - E(\tilde{u}, v)| < \varepsilon$ for all $n \geq n_1$. For $n \geq \max\{n_0, n_1\}$, we have

$$\begin{aligned}|E(\tilde{u}, v) - v(a)| &= |E(\tilde{u}, v) - E_n(\tilde{k}_a^{(n)}, v)| \\ &\leq |E_n(\tilde{u}, v) - E(\tilde{u}, v)| + |E_n(\tilde{u} - \tilde{k}_a^{(n)}, v)| \\ &\leq \varepsilon + [E_n(\tilde{u} - \tilde{k}_a^{(n)})]^{1/2}[E_n(v)]^{1/2} \\ &\leq (1 + [E(v)]^{1/2})\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have $E(\tilde{u}, v) = v(a)$, and hence $\tilde{u} = \tilde{k}_a$.

By Lemmas 6.1 and 6.2, we obtain

THEOREM 6.2. *$\sum_{x \in X} q(x)\tilde{k}_a(x) \leq 1$ for every $a \in X$.*

THEOREM 6.3. *If $\boldsymbol{HB}(N(q)) = \{0\}$, then $\boldsymbol{HE}(N(q)) = \{0\}$.*

PROOF. Suppose that $\boldsymbol{HB}(N(q)) = \{0\}$. Then $q \neq 0$ and $\sum_{x \in X} q(x)\tilde{g}_a(x) = 1$ by Theorem 5.3, so that $q(x)[\tilde{k}_a(x) - \tilde{g}_a(x)] = 0$ by Theorems 6.1 and 6.2. It follows from Theorem 2.2 that $\tilde{k}_a(x) = \tilde{g}_a(x)$ on X for every $a \in X$. Let $h \in \boldsymbol{HE}(N(q))$. For any $a \in X$, we have by Lemma 3.1

$$h(a) = E(\tilde{k}_a, h) = E(\tilde{g}_a, h) = 0.$$

Namely $\boldsymbol{HE}(N(q)) = \{0\}$.

THEOREM 6.4. *Assume that* $\sum_{x\in X} q(x)<\infty$. *Then the following conditions are equivalent*:

(a) $N \in O_G$.

(b) $\boldsymbol{HB}(N(q))=\{0\}$.

(c) $\boldsymbol{HE}(N(q))=\{0\}$.

PROOF. By Theorems 5.5 and 6.3, we see that (a) implies (b) and that (b) implies (c) without the assumption $\sum_{x\in X} q(x)<\infty$. Assume that $\boldsymbol{HE}(N(q))=\{0\}$. Then $\boldsymbol{E}_0(N(q))=\boldsymbol{E}(N(q))$ by the Corollary of Theorem 3.2, and hence $N \in O_G$ by the Corollary 2 of Theorem 3.1.

REMARK 6.1. In case $\sum_{x\in X} q(x)=\infty$, $\boldsymbol{HE}(N(q))=\{0\}$ does not imply $\boldsymbol{HB}(N(q))=\{0\}$ in general. In fact, consider the network defined in Example 5.1 and let u be q-harmonic on X with $u(x_0)>0$. Then we have $u(x_{n+1})\geq(1+\alpha_n)\geq u(x_0)$, so that $E(u)\geq \sum_{x\in X} q(x)u(x)^2 \geq \sum_{x\in X} q(x)u(x_0)^2=\infty$. Hence $\boldsymbol{HE}(N(q))=\{0\}$. On the other hand, we can choose r and q in such a way that $\boldsymbol{HB}(N(q))\neq\{0\}$.

§7. q-Green potentials

We define the q-Green potential $\tilde{G}\mu$ of $\mu \in L^+(X)$ and the mutual q-Green potential energy $\tilde{G}(\mu, \nu)$ of $\mu, \nu \in L^+(X)$ by

$$\tilde{G}\mu(x)=\sum_{a\in X} \tilde{g}_a(x)\mu(x),$$

$$\tilde{G}(\mu, \nu)=\sum_{x\in X} [\tilde{G}\mu(x)]\nu(x).$$

We call $\tilde{G}(\mu, \mu)$ the q-Green potential energy of μ. Let us put

$$M(\tilde{G})=\{\mu \in L^+(X);\ \tilde{G}\mu \in L(X)\},$$

$$E(\tilde{G})=\{\mu \in L^+(X);\ \tilde{G}(\mu, \mu)<\infty\}.$$

Denote by $SH^+(N(q))$ the set of all non-negative q-superharmonic functions on X.

We list the following results that can be proved by the same reasoning as in the case where $q=0$ (cf. [8]):

LEMMA 7.1. $\Delta_q\tilde{G}\mu=-\mu$ *for every* $\mu \in M(\tilde{G})$.

THEOREM 7.1 (Riesz's decomposition). *Every* $u \in SH^+(N(q))$ *can be decomposed uniquely in the form*: $u=\tilde{G}\mu+h$, *where* $\mu \in M(\tilde{G})$ *and* h *is non-negative and* q*-harmonic on* X. *In this decomposition*, $\mu=-\Delta_q u$ *and* h *is the greatest* q*-harmonic minorant of* u.

LEMMA 7.2. *If* $\mu \in E(\tilde{G})$, *then* $\tilde{G}\mu \in \boldsymbol{E}_0(N(q))$ *and* $E(\tilde{G}\mu)=\tilde{G}(\mu, \mu)$.

Denote by $\boldsymbol{P}(N(q))$ the subset of $\boldsymbol{E}_0(N(q))$ which consists of q-superharmonic functions on X, i.e.,

$$\boldsymbol{P}(N(q)) = \{u \in \boldsymbol{E}_0(N(q));\ \Delta_q u(x) \leq 0 \text{ on } X\}.$$

THEOREM 7.2. $\boldsymbol{P}(N(q)) = \{\tilde{G}\mu;\ \mu \in E(\tilde{G})\}$.

As an application of the above theory, we obtain another proof which shows that (b) implies (a) in Theorem 6.3.

THEOREM 7.3. *If* $\sum_{x \in X} q(x) < \infty$, *then* $\boldsymbol{HB}(N(q)) = \{0\}$ *implies* $N \in O_G$.

PROOF. Assume that $\boldsymbol{HB}(N(q)) = \{0\}$. Then $\sum_{z \in X} q(z)\tilde{g}_z(x) = 1$ for every $x \in X$ by Theorem 5.3. Let us put $v = \sum_{z \in X} q(z)\tilde{g}_z(x)$. Then $v = \tilde{G}\mu$ with $\mu = -\Delta_q v = q \in L^+(X)$ and $\tilde{G}(\mu, \mu) = \sum_{x \in X} q(x) < \infty$, i.e., $\mu \in E(\tilde{G})$. Thus $v \in \boldsymbol{E}_0(N(q)) \subset \boldsymbol{D}_0(N)$ by Lemma 7.2 and Theorem 3.1. Therefore $1 \in \boldsymbol{D}_0(N)$ and hence $N \in O_G$.

§ 8. Dependence of the q-Green function on q

In order to study the dependence of the q-Green function $\tilde{g}_a$ of the network $N(q)$ on q, denote it by $\tilde{g}_a^{(q)}$ in this section.

THEOREM 8.1. *Let* $q_1, q_2 \in L^+(X)$. *If* $q_1 \leq q_2$ *on* X, *then*

$$\tilde{g}_a^{(q_1)}(x) - \tilde{g}_a^{(q_2)}(x) = \sum_{z \in X} [q_2(z) - q_1(z)]\tilde{g}_a^{(q_1)}(z)\tilde{g}_x^{(q_2)}(z). \tag{8.1}$$

PROOF. Let $\{G_n\}$ ($G_n = \langle X_n, Y_n \rangle$) be an exhaustion of G and let $a, x \in X_n$. Denote by u_n the q_1-Green function of $N_n(q_1)$ with pole at a and by v_n the q_2-Green function on $N_n(q_2)$ with pole at x. Then we have by (4.6)

$$u_n(x) = D(u_n, v_n) + \sum_{z \in X} q_2(x)u_n(z)v_n(z),$$

$$v_n(a) = D(u_n, v_n) + \sum_{z \in X} q_1(z)u_n v_n(z),$$

so that

$$u_n(x) - v_n(a) = \sum_{z \in X} [q_2(z) - q_1(z)]u_n(z)v_n(z). \tag{8.2}$$

Since $q_1 \leq q_2$ on X and $\{u_n\}$ and $\{v_n\}$ increase to $\tilde{g}_a^{(q_1)}$ and $\tilde{g}_x^{(q_2)}$ by Theorem 4.4, we obtain (8.1) by letting $n \to \infty$ in (8.2).

COROLLARY 1. *If* $q_2 \geq q_1 \geq 0$, *then* $\tilde{g}_a^{(q_1)}(x) \geq \tilde{g}_a^{(q_2)}(x)$ *on* X.

COROLLARY 2. *If* $N \notin O_G$, *then*

$$g_a(x) - \tilde{g}_a^{(q)}(x) = \sum_{z \in X} q(x)g_a(z)\tilde{g}_x^{(q)}(z). \tag{8.3}$$

THEOREM 8.2. *If* $\{q_n\}$ *increases to* q, *then* $\{\tilde{g}_a^{(q_n)}\}$ *decreases to* $\tilde{g}_a^{(q)}$.

PROOF. Put $u_n=\tilde{g}_a^{(q_n)}$, $v=\tilde{g}_x^{(q)}$ and $u=\tilde{g}_a^{(q)}$. We have by Theorem 8.1

$$0\leq u_n(x)-u(x)=\sum_{z\in X}[q(z)-q_n(z)]u_n(z)v(z).$$

Since $\{[q(z)-q_n(z)]u_n(z)\}$ decreases to 0 for all z, we have the assertion.

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