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Matrices of Clifford Semigroups, and a Generalization of Rees's Theorem[1)]

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Let S be a completely regular semigroup, and $E(S)$ the partial subgroupoid of idempotents of S. Let γ be a relation on $E(S)$. If γ is a congruence on $E(S)$, that is, if γ is an equivalence relation on $E(S)$ and if $x\gamma y$ and $u\gamma v$ satisfy $xu\gamma yv$ (if both xu and yv are defined in $E(S)$), then S is called a CS-matrix. Firstly, several characterizations of a CS-matrix are given. Secondly, split CS-matrices are investigated. In particular, matrix representations of these semigroups are discussed.

§1. Preliminary

Let P be a partial groupoid, and γ a relation on P as follows:

(1.1) $x\gamma y$ if and only if both xy and yx are defined in P, and $xy=yx$.

If γ is a congruence on P, that is,

(1.2) (1) $x\gamma x$ for all $x\in P$,

(2) $x\gamma y$ implies $y\gamma x$,

(3) $x\gamma y$, $y\gamma z$ imply $x\gamma z$,

(4) if $x\gamma y$, $u\gamma v$ and if both xu and yv are defined in P, then $xu\,\gamma\, yv$,

then P is called *γ-compatible*. In a completely simple semigroup C, it is obvious that the partial groupoid $E(C)$ of idempotents of C (with respect to the multiplication in C) is γ-compatible. If the partial groupoid $E(S)$ of idempotents of a regular semigroup S is γ-compatible, then S is called γ-compatible. If a semigroup A is a rectangular band Δ of subsemigroups $\{A_\delta: \delta\in\Delta\}$ of type $\mathscr{T}$, then we shall say that *A is a matrix Δ of semigroups $\{A_\delta: \delta\in\Delta\}$ of type $\mathscr{T}$*. If A is a matrix of semigroups of type $\mathscr{T}$, then A is said to be a $\mathscr{T}$-semigroup matrix. For example, if A is a rectangular band of subgroups then A is called *a group matrix*. If A is a rectangular band of Clifford subsemigroups (that is, semilattices of groups), then A is called *a Clifford semigroup matrix* (abbrev.,

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1) An abstract of this paper was announced in the Proceedings of 9th Symposium on Semigroups and Related Topics, Naruto University of Teacher Education, 1985.

a CS-matrix).[2] Let B be a γ-compatible band, and ρ the least semilattice congruence on B. It is well-known that each ρ-class is a rectangular subband of B (therefore, B is a semilattice of rectangular bands). Now, it is also easy to see that $\rho \cap \gamma = \iota_S$ (the identity congruence on B). Hence, B is isomorphic to a subdirect product of B/ρ and B/γ. Let $e, f \in B$. Since $efee = eefe$, $efe\,\gamma\, e$. Hence, B/γ is a rectangular band. Since B/ρ is a semilattice, B is isomorphic to a subdirect product of a semilattice and a rectangular band.

Further, it is easy to see that the converse also holds; that is, a band B is γ-compatible if and only if it is isomorphic to a subdirect product of a semilattice and a rectangular band. Accordingly, in this case B is a rectangular band of semilattices, that is, *a semilattice matrix* (abbrev., *an SL-matrix*). Of course, an SL-matrix is a normal band. In this paper, we shall investigate the structure of CS-matrices and that of split CS-matrices. If S is a completely regular semigroup, it is well-known that S is uniquely decomposed into a semilattice Λ of completely simple semigroups $\{S_\lambda: \lambda \in \Lambda\}$ (see [1]). This decomposition is called *the structure decomposition of* S, and denoted by $S \sim \Sigma\{S_\lambda: \lambda \in \Lambda\}$.

Hereafter the terminology "a completely regular semigroup $S \equiv \Sigma\{S_\lambda: \lambda \in \Lambda\}$" means "$S$ is a completely regular semigroup and has $S \sim \Sigma\{S_\lambda: \lambda \in \Lambda\}$ as its structure decomposition". Further, it is also well-known that the least semilattice congruence ρ on a completely regular semigroup S induces the structure decomposition of S. Throughout the whole paper, if S is a completely regular semigroup, ρ_S and γ_S denote the least semilattice congruence on S and the γ-congruence on $E(S)$ respectively. Every terminology and notation should be referred to [1], unless otherwise stated.

§ 2. CS-matrices

Let S be a γ-compatible completely regular semigroup, and $E(S)$ the set of all idempotents of S. Since γ is an equivalence relation on $E(S)$, $E(S)$ is decomposed into γ-equivalence classes $\{E_\lambda: \lambda \in \Lambda\}$ (where each E_λ is a γ-class). Now, put $S_\lambda = \{x \in S: xx^*, x^*x \in E_\lambda$ for some $x^* \in V(x)\}$, where $V(x)$ is the set of inverses of x.

LEMMA 2.1. (1) *Each* S_λ *is a maximal Clifford subsemigroup of* S.

(2) $\mathscr{S} = \{S_\lambda: \lambda \in \Lambda\}$ *is the set of all maximal Clifford subsemigroups, and* $S = \Sigma\{S_\lambda: \lambda \in \Lambda\}$ (*where* Σ *denotes disjoint sum*).

PROOF. It is obvious that each E_λ is a subsemilattice of S. Let $x \in S_\lambda$. Then, $xx^*, x^*x \in E_\lambda$ for some $x^* \in V(x)$. For any $e \in E_\lambda$, xex^* is an idempotent and $xx^*xex^* = xex^* = xex^*xx^*$. Hence, $xx^*\,\gamma\, xex^*$, and accordingly $xex^* \in E_\lambda$. Therefore, $xE_\lambda x^* \subset$

2) Recently, the structure of CS-matrices has been also studied by Pastijn and Petrich [3], and the paper [3] has appeared after the author announced the abstract of this paper in the above-mentioned proceedings of the symposium. Some parts of this paper overlap with results of [3], though proofs and approach are quite different.

E_λ. Similarly, we have $x^*E_\lambda x \subset E_\lambda$. Hence, $S_\lambda = \{x \in S: xx^*, x^*x \in E_\lambda, xE_\lambda x^* \subset E_\lambda, x^*E_\lambda x \subset E_\lambda$ for some $x^* \in V(x)\}$. Therefore, S_λ is a maximal regular subsemigroup having E_λ as the set of idempotents. Since E_λ is a semilattice, S_λ is an inverse semigroup. Now, let $x \in S_\lambda$. Then, xx^*, $x^*x \in E_\lambda$ for some $x^* \in V(x)$. Since S is completely regular, there exists the group inverse x^{-1} of x. Now, $xx^* \gamma x^*x$ implies $xx^*xx^{-1} \gamma x^*xxx^{-1} = x^*xx^{-1}x$, and accordingly $xx^{-1} \gamma x^*x$. Therefore, $xx^{-1} \in E_\lambda$. Thus, $x^{-1} \in S_\lambda$. That is, x has the group inverse x^{-1} in S_λ. Consequently, S_λ is a union of groups, and accordingly S_λ is a Clifford semigroup.

(2) Let T be a maximal Clifford subsemigroup of S. For any $e, f \in E(T)$ (the set of idempotents of T), $ef = fe$. Hence, $e \gamma f$, and accordingly $e, f \in E_\lambda$ for some $\lambda \in \Lambda$. Therefore, $E(T) \subset E_\lambda$, and $T \subset S_\lambda$. Since T is a maximal Clifford subsemigroup, $T = S_\lambda$. Thus, $\mathscr{S}$ is the set of all maximal Clifford subsemigroups of S. It is obvious that $S = \cup \{S_\lambda: \lambda \in \Lambda\}$. Assume that $x \in S_\lambda \cap S_\delta$. Then, there exist $x^*, x^\# \in V(x)$ such that xx^*, $x^*x \in E_\lambda$ and $xx^\#$, $x^\#x \in E_\delta$. Since $xx^* \gamma x^*x$ and $xx^\# \gamma x^\#x$, we have $xx^\# = xx^*xx^\# \gamma x^*xx^\#x = x^*x$. Hence, $xx^\#$, $x^*x \in E_\lambda \cap E_\delta$, and $\lambda = \delta$. Therefore, $S_\lambda \cap S_\delta = \square$ for $\lambda \neq \delta$.

As characterizations of a CS-matrix, we have the following[3]:

THEOREM 2.2. *For a completely regular semigroup S, the following conditions* (1)–(6) *are equivalent*:

(1) *S is the disjoint sum of maximal Clifford subsemigroups of S.*

(2) *S is γ-compatible.*

(3) *S is a matrix of Clifford semigroups, that is, S is a CS-matrix.*

(4) *For the least matrix congruence (that is, the least rectangular band congruence) σ_S on S, each σ_S-class is a Clifford subsemigroup.*

(5) *S is an SL-matrix cryptogroup (that is, an SL-matrix of groups).*

(6) *The relation τ on S defined by*

$$x \tau y \text{ if and only if } [x][y] = [y][x] \tag{2.1}$$

is a matrix congruence, and $[x][y] = [y][x]$ if and only if $[xy][yx] = [yx][xy]$, where $[u]$ denotes the identity of the maximal subgroup H_u containing u.

PROOF. (2)⇒(1) follows from Lemma 2.1, and (1)⇔(3) has been shown in Pastijn [2]. Further, it is easy to see that (3)⇒(2). (3)⇒(4): Since S is a CS-matrix, there exists a matrix congruence η_S on S such that each η_S-class is a Clifford subsemigroup. Let e, f be idempotents of a η_S-class. Then, $ef = fe$. Hence, $e \sigma_S efe = fef \sigma_S f$. Therefore, $e \sigma_S f$. Let $x\eta_S$ be the η_S-class containing $x \in S$. For any $y \in x\eta_S$, there exists a unique inverse y' of y in $x\eta_S$. For $a \in S$, let $\bar{a}$ be the σ_S-class containing a. Now, $xx' \sigma_S x'x$ implies $\overline{xx'} = \overline{x'x}$, and hence $\bar{x} = \bar{x}'$, that is, $x \sigma_S x'$. Hence, $xx' \sigma_S x$.

3) Several other characterizations of a CS-matrix have been also given by [3].

Let $y \in x\eta_S$. Then, $x\eta_S = y\eta_S$, and $yy'\sigma_S y$. Since $yy'\eta_S xx'$, we have $yy'\sigma_S xx'$, and $x\sigma_S y$. Therefore, $y \in x\sigma_S$. Thus, $x\eta_S \subset x\sigma_S$. Since σ_S is the least matrix congruence, $\eta_S = \sigma_S$. (4)⇒(3): Obvious. (4)⇒(5): It is easy to see that $\sigma_S \cap \rho_S = \mathscr{H}_S$, where $\mathscr{H}_S$ is the Rees H-relation on S. Put $S/\sigma_S = \Lambda$ and $S/\rho_S = Y$. Then, S is H-compatible,[4] and $S/\mathscr{H}_S$ is isomorphic to a subdirect product $\Lambda \divideontimes Y$ of Λ and Y (where $\divideontimes$ denotes a subdirect product). Since $\Lambda \divideontimes Y$ is an SL-matrix and since each $\mathscr{H}_S$-class is a subgroup of S, S is an SL-matrix cryptogroup. (5)⇒(3): Obvious. (6)⇒(4): It is obvious that each σ_S-class is a union of groups. Let $e\sigma_S f$ for $e, f \in E(S)$. Hence, $ef\sigma_S fe$. Since σ_S is the least matrix congruence, $ef\tau fe$. Therefore, $[ef][fe] = [fe][ef]$ and $[e][f] = [f][e]$, that is, $ef = fe$. Thus, every σ_S-class is an inverse semigroup, and hence it is a Clifford subsemigroup. (5)⇒(6): Let S be an SL-matrix $\Lambda \divideontimes Y$ of groups $\{H_i^\alpha : (\alpha, i) \in \Lambda \divideontimes Y\}$, where Λ, Y are a rectangular band and a semilattice respectively, and $\Lambda \divideontimes Y$ is a subdirect product of Λ and Y. It is obvious that each H_i^α is an H-class of S. Suppose that $[x][y] = [y][x]$ for $x, y \in S$. Then, $(xy)\mathscr{H}_S = (yx)\mathscr{H}_S$, where $\mathscr{H}_S$ is the H-relation on S. Hence, $[xy][yx] = [yx][xy]$. Conversely, suppose that $[xy][yx] = [yx][xy]$. There exist H_i^λ, H_j^δ such that $x \in H_i^\lambda$ and $y \in H_j^\delta$. Then, $xy \in H_{ij}^{\lambda\delta}$ and $yx \in H_{ji}^{\delta\lambda}$. Hence, $[xy][yx] \in H_{iji}^{\lambda\delta\lambda}$ and $[yx][xy] \in H_{jij}^{\delta\lambda\delta}$, and hence $\lambda = \delta$. For any $\xi \in \Lambda$, $S_\xi = \cup \{H_k^\xi : (\xi, k) \in \Lambda \divideontimes Y, k \in Y\}$ is a Clifford subsemigroup of S. Since $[x] \in H_i^\lambda$ and $[y] \in H_j^\lambda$ and since S_λ is a Clifford semigroup, $[x][y] = [y][x]$. Next, suppose that $x\tau y$. Then, $[x][y] = [y][x]$, and hence $x, y \in S_\lambda$ for some $\lambda \in \Lambda$. Let τ^* be the congruence on S which gives the decomposition of S into the Clifford subsemigroups $\{S_\xi : \xi \in \Lambda\}$. Then, τ^* is a matrix congruence and satisfies $\tau \subset \tau^*$. Conversely, it is obvious that $\tau^* \subset \tau$. Accordingly, $\tau = \tau^*$. Thus, τ is a matrix congruence.

From the theorem above, it is easy to see that a matrix decomposition (that is, a rectangular band decomposition) of a CS-matrix S into Clifford subsemigroups $\{C_\alpha : \alpha \in \Gamma\}$ is unique, and it is given by the least matrix congruence σ_S on S. In this case, each C_α is a maximal Clifford subsemigroup of S. Further, it is also obvious that $\sigma_S | E(S)$ (the restriction of σ_S to $E(S)) = \gamma_S$.

LEMMA 2.3. *Let S be a CS-matrix, and $\{E_\lambda : \lambda \in \Lambda\}$ the γ-classes of $E(S)$. Then, $E_\lambda S E_\lambda$ is a Clifford subsemigroup of S, and $S = \cup \{E_\lambda S E_\lambda : \lambda \in \Lambda\}$.*

PROOF. For each $\lambda \in \Lambda$, let $S_\lambda = \{x \in S : xx^*, x^*x \in E_\lambda$ for some $x^* \in V(x)\}$. As was shown above, S_λ is a maximal Clifford subsemigroup. Hence, it is a σ_S-class. Therefore, we can consider Λ as a rectangular band and S as a matrix Λ of the maximal Clifford subsemigroups $\{S_\lambda : \lambda \in \Lambda\}$. Since Λ is a rectangular band, $E_\lambda S E_\lambda \subset S_\lambda$ for $\lambda \in \Lambda$. Conversely, let $x \in S_\lambda$. Then, there exists a group inverse x^{-1} of x in S_λ. Hence, $xx^{-1} = x^{-1}x \in E_\lambda$, and hence $x = xx^{-1}xx^{-1}x \in E_\lambda S E_\lambda$. Hence, $E_\lambda S E_\lambda = S_\lambda$.

To consider a description of all possible CS-matrices, we need only to construct all

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4) A semigroup S is said to be H-compatible if Green's H-relation is a congruence on S.

possible SL-matrices Ω of groups $\{N_\omega: \omega \in \Omega\}$ for a given SL-matrix Ω and given groups $\{H_\omega: \omega \in \Omega\}$. This can be obtained as a special case of Schein's theorem of [4] which has given a construction of bands of unipotent monoids (see also [5]). However, we omit to state it here again.

§ 3. Split CS-matrices

Let $M \equiv \Sigma\{M_\lambda: \lambda \in \Lambda\}$ be a CS-matrix, and ρ_M the congruence which gives the structure decomposition $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$. We can consider $M/\rho_M = \Lambda$ by identifying each ρ_M-class M_λ with $\lambda \in \Lambda$. Let f be the natural homomorphism of M onto $M/\rho_M = \Lambda$; that is, the homomorphism f such that $xf = \lambda$ if $x \in M_\lambda$. If there exists a homomorphism g of $\Lambda = M/\rho_M$ into M such that $gf = \varepsilon_\Lambda$ (the identity mapping on Λ), then M is called *split*. Let $M \equiv \{M_\lambda: \lambda \in \Lambda\}$ be a CS-matrix. Then, there exists an SL-matrix $(L \times R) \divideontimes \Lambda$, where L, R are a left zero semigroup, a right zero semigroup, $L \times R$ the direct product of L and R and $(L \times R) \divideontimes \Lambda$ a subdirect product of $L \times R$ and Λ, such that M is an SL-matrix $(L \times R) \divideontimes \Lambda$ of groups $\{H^\alpha_{(i,j)}: ((i, j), \alpha) \in (L \times R) \divideontimes \Lambda\}$ (where each $H^\alpha_{(i,j)}$ is an H-class of M). In this case, it is easy to see that $M_\lambda = \Sigma\{H^\lambda_{(i,j)}: (i, j) \in I_\lambda\}$, where $I_\lambda = \{(i, j): ((i, j), \lambda) \in (L \times R) \divideontimes \Lambda\}$, for each $\lambda \in \Lambda$.

LEMMA 3.1. *M splits if and only if*

(3.1) *there exists $(i, j) \in L \times R$ such that $H^\alpha_{(i,j)}$ exists in M for all $\alpha \in \Lambda$.*

PROOF. Suppose that M satisfies (3.1). There exists $(i, j) \in L \times R$ such that $H^\alpha_{(i,j)}$ exists in M for all $\alpha \in \Lambda$. Let $C_{(i,j)} = \cup \{H^\alpha_{(i,j)}: \alpha \in \Lambda\}$. Then, $C_{(i,j)}$ is a Clifford subsemigroup. Let $e^\alpha_{(i,j)}$ be the idempotent of $H^\alpha_{(i,j)}$ for all $\alpha \in \Lambda$. Now, define f: $\Lambda \to M$ by $\alpha f = e^\alpha_{(i,j)}$. Since $C_{(i,j)}$ is a Clifford semigroup, f is a homomorphism. On the other hand, the mapping h: $M \to \Lambda$ defined by $M_\alpha h = \{\alpha\}$ is a homomorphism of M onto Λ. The congruence induced by h is ρ_M. Since $fh = \varepsilon_\Lambda$ (the identity mapping on Λ), M splits. Conversely, suppose that M splits. Then, there exists a surjective homomorphism f: $M/\rho_M \to M$ such that $fh = \varepsilon_{M/\rho_M}$, where h is the natural homomorphism of M onto M/ρ_M. If we identify an element M_λ of M/ρ_M with λ, then we can consider h and f as a surjective homomorphism of M onto Λ and a homomorphism of Λ into M such that $fh = \varepsilon_\Lambda$. For every $\alpha \in \Lambda$, let $\alpha f = e_\alpha$. Then, $e_\alpha \in M_\alpha$. Hence, there exists $(u, v) \in L \times R$ such that $e_\alpha = e^\alpha_{(u,v)}$ and $((u, v), \alpha) \in (L \times R) \divideontimes \Lambda$. For $\beta \in \Lambda$, similarly there exists $(s, k) \in L \times R$ such that $((s, k), \beta) \in (L \times R) \divideontimes \Lambda$ and $e_\beta = e^\beta_{(s,k)}$. Now, $e_\alpha e_\beta = (\alpha f)(\beta f) = (\alpha\beta)f = (\beta\alpha)f = (\beta f)(\alpha f) = e_\beta e_\alpha$. Hence, $e^\alpha_{(u,v)} e^\beta_{(s,k)} = e^\beta_{(s,k)} e^\alpha_{(u,v)}$. Since $e^\alpha_{(u,v)} e^\beta_{(s,k)}$, $e^\alpha_{(s,k)} e^\alpha_{(u,v)}$ are idempotents contained in $H^{\alpha\beta}_{(u,k)}$, $H^{\alpha\beta}_{(s,v)}$ respectively, $u = s$ and $k = v$. Thus, $(u, v) = (s, k)$. Consequently, $H^\lambda_{(u,v)}$ exists for every $\lambda \in \Lambda$.

Now, let $M \sim \Sigma\{M_\lambda: \lambda \in \Lambda\}$ be the above-mentioned split CS-matrix.
Then, M is an SL-matrix $(L \times R) \divideontimes \Lambda$ of H-classes $\{H^\alpha_{(i,j)}: ((i, j), \alpha) \in (L \times R) \divideontimes \Lambda\}$.

Further, there exists $(i, j)\in L\times R$ such that $H^{\alpha}_{(i,j)}$ exists for all $\alpha\in\Lambda$. Denote (i, j) by (1, 1), and put $C_{(1,1)}=\cup\{H^{\alpha}_{(1,1)}: \alpha\in\Lambda\}$. It is obvious that $C_{(1,1)}$ is a Clifford subsemigroup of M. Let $I_{\alpha}=\{(i,j)\in L\times R: ((i,j),\alpha)\in(L\times R)\divideontimes\Lambda\}$ for every $\alpha\in\Lambda$. Then, $M_{\alpha}=\cup\{H^{\alpha}_{(s,k)}: (s,k)\in I_{\alpha}\}$. For any $x\in H^{\alpha}_{(s,k)}$, x is uniquely written in the form $x=e^{\alpha}_{s1}ue^{\alpha}_{1k}$, $u\in H^{\alpha}_{(1,1)}$, where e^{α}_{sk} is the identity of $H^{\alpha}_{(s,k)}$ for every $((s,k),\alpha)\in(L\times R)\divideontimes\Lambda$ (see [1]). For $x=e^{\delta}_{i1}ue^{\delta}_{1j}\in H^{\delta}_{(i,j)}$, $y=e^{\eta}_{s1}ve^{\eta}_{1k}\in H^{\eta}_{(s,k)}$, $u\in H^{\delta}_{(1,1)}$ and $v\in H^{\eta}_{(1,1)}$,

(3.2) $\quad xy=e^{\delta}_{i1}ue^{\delta}_{1j}e^{\eta}_{s1}ve^{\eta}_{1k}=e^{\delta}_{i1}ue^{\delta\eta}_{11}p^{(\delta,\eta)}_{js}e^{\delta\eta}_{11}ve^{\eta}_{1k}$, where $e^{\delta}_{1j}e^{\eta}_{s1}=p^{(\delta,\eta)}_{js}\in H^{\delta\eta}_{(1,1)}$.

Now,

(3.2) $=e^{\delta}_{i1}e^{\delta\eta}_{i1}ue^{\delta\eta}_{11}p^{(\delta,\eta)}_{js}e^{\delta\eta}_{11}ve^{\delta\eta}_{1k}e^{\eta}_{1k}$.

Next, $(e^{\delta}_{i1}e^{\delta\eta}_{i1})^2=e^{\delta}_{i1}e^{\delta}_{1i}e^{\delta\eta}_{i1}=e^{\delta}_{i1}e^{\delta\eta}_{i1}$. Hence, $e^{\delta}_{i1}e^{\delta\eta}_{i1}$ is an idempotent of $H^{\delta\eta}_{(i,1)}$, and hence $e^{\delta}_{i1}e^{\delta\eta}_{i1}=e^{\delta\eta}_{i1}$. Thus, (3.2)$=e^{\delta}_{i1}(ue^{\delta\eta}_{11}p^{(\delta,\eta)}_{js}e^{\delta\eta}_{11}v)e^{\delta\eta}_{1k}$. Put $e^{\delta\eta}_{11}p^{(\delta,\eta)}_{js}e^{\delta\eta}_{11}=q^{(\delta,\eta)}_{js}$. Then, $q^{(\delta,\eta)}_{js}\in H^{\delta\eta}_{(1,1)}$ and $q^{(\alpha,\beta)}_{11}=e^{\alpha\beta}_{11}$ for all $\alpha, \beta\in\Lambda$. Further, (3.2)$=e^{\delta\eta}_{i1}(uq^{(\delta,\eta)}_{js}v)e^{\delta\eta}_{1k}$. It is easy to see that $uq^{(\delta,\eta)}_{js}v\in C_{(1,1)}$ and the product of u, $q^{(\delta,\eta)}_{js}$ and v can be obtained in the semigroup $C_{(1,1)}$. Hence, if we rewrite x, y in the form $x=[u]^{\delta}_{ij}$, $y=[v]^{\eta}_{sk}$, then

$$xy=[u]^{\delta}_{ij}[v]^{\eta}_{sk}=[uq^{(\delta,\eta)}_{js}v]^{\delta\eta}_{ik}$$

and $M=\{[u]^{\delta}_{ij}: ((i,j),\delta)\in(L\times R)\divideontimes\Lambda,\ u\in H^{\delta}_{(1,1)}\}$.
Since $([u^{\delta}_{ij}][v]^{\eta}_{sk})[t]^{\xi}_{mn}=[u]^{\delta}_{ij}([v]^{\eta}_{sk}[t]^{\xi}_{mn})$. we have

$$[uq^{(\delta,\eta)}_{js}vq^{(\delta\eta,\xi)}_{km}t]^{\delta\eta\xi}_{in}=[uq^{(\delta,\eta\xi)}_{js}vq^{(\eta,\xi)}_{km}t]^{\delta\eta\xi}_{in}.$$

Hence,

(3.3) $\quad q^{(\delta\eta)}_{js}vq^{(\delta\eta,\xi)}_{km}=q^{(\delta,\eta\xi)}_{js}vq^{(\eta,\xi)}_{km}$ for all $v\in H^{\eta}_{11}$.

Conversely, let L, R and Λ be a left zero semigroup, a right zero semigroup and a semilattice respectively. Let $(L\times R)\divideontimes\Lambda$ be an SL-matrix, where $\divideontimes$ denotes a subdirect product, and assume that there exists $(s, k)\in L\times R$ such that $((s,k),\alpha)$ $(L\times R)\divideontimes\Lambda$ for all $\alpha\in\Lambda$.

Denote (s, k) by (1, 1). Let $C(\Lambda)$ be a semilattice Λ of groups $\{H^{\alpha}_{(1,1)}: \alpha\in\Lambda\}$. Of course, $C(\Lambda)$ is a Clifford semigroup. Put $M=\{[u]^{\delta}_{ij}: u\in H^{\delta}_{(1,1)}, ((i,j),\delta)\in(L\times R)\divideontimes\Lambda\}$. For δ, η, j, s such that $((1,j),\delta)$, $((s,1),\eta)\in(L\times R)\divideontimes\Lambda$, let $q^{(\delta,\eta)}_{js}$ be an element of $H^{\delta\eta}_{(1,1)}$. Put $Q=\{q^{(\delta,\eta)}_{js}: ((1,j),\delta), ((s,1),\eta)\in(L\times R)\divideontimes\Lambda\}$. Assume that Q satisfies (3.3) for $((1,j),\delta)$, $((s,1),\eta)$, $((1,k),\eta)$ and $((m,1),\xi)$ of $(L\times R)\divideontimes\Lambda$ and the following (3.4):

(3.4) $\quad q^{(\alpha,\beta)}_{11}=e^{\alpha\beta}_{11}$ for all $\alpha, \beta\in\Lambda$, where e^{τ}_{11} is the identity of $H^{\tau}_{(1,1)}$.

In this case, if multiplication is defined in M by

$$[u]^{\delta}_{ij}[v]^{\eta}_{sk}=[uq^{(\delta,\eta)}_{js}v]^{\delta\eta}_{ik},$$

then M becomes a split CS-matrix. The set Q above is called *the sandwich matrix* of M over the Clifford semigroup $C(\Lambda)$, and the split CS-matrix M above is denoted by $\mathscr{M}((L\times R)\divideontimes\Lambda;\ C(\Lambda);\ Q)$.

Now, it follows from the results above that:

THEOREM 3.2. *$\mathscr{M}((L\times R)\divideontimes\Lambda;\ C(\Lambda);\ Q)$ is a split CS-matrix. Conversely, every split CS-matrix can be obtained in this way.*

REMARK. In Theorem 3.2, consider the case where Λ consists of a single element α and $C(\Lambda)$ is a group $H^{\alpha}_{(1,1)}$. Then, $Q=\{q^{(\alpha,\alpha)}_{js}: (j, s)\in R\times L\}$. Denote $q^{(\alpha,\alpha)}_{js}$ simply by q_{js}, and $[u]^{\alpha}_{ij}$ simply by $[u]_{ij}$. Then, $\mathscr{M}((L\times R)\divideontimes\{\alpha\};\ H^{\alpha}_{(1,1)};\ Q)=\{[u]_{ij}: (i, j)\in L\times R\}$ and

$$[u]_{ij}[v]_{ks}=[uq_{jk}v]_{is}.$$

That is, it is the regular Rees $L\times R$-matrix semigroup with sandwich matrix Q over the group $H^{\alpha}_{(1,1)}$. Hence, $\mathscr{M}((L\times R)\divideontimes\Lambda;\ C(\Lambda);\ Q)$ in Theorem 3.2 is a generalization of the concept of a regular Rees matrix semigroup.

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