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UNIQUE MIDPOINT PROPERTY IN METRIZABLE SPACES: A SURVEY

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ABSTRACT. We shall give a brief survey of metric spaces with the unique midpoint property (abbreviated UMP).

1. Introduction

A metric space (X, ρ) is said to have the *unique midpoint property* (abbreviated *UMP*) if for every pair of distinct points x and y of X there is an exactly one point $p \in X$ such that $\rho(x,p) = \rho(y,p)$. We call the point p the *midpoint* of x and y. For example, the real line with the usual metric and its subspaces of the rational numbers and the closed interval $[0,1]$ have UMP, while the metric subspace of the irrationals, the usual Cantor set and the union $[-1,0] \cup [1,2]$ of exactly two of disjoint closed intervals do not have UMP. Several authors studied the relation between the unique midpoint property and the embeddability of spaces into the real line $\mathbb{R}$, cf., [1], [4], [6], [9] and [10]. In the present note, we shall present a brief survey in this field.

In sections 2 and 3, we shall describe some results about connected and non-connected metric spaces with UMP from [1] and [9] respectively. We shall present a proof of Theorem 2.2, because the original proof from [1] contains a gap and we correct it. In section 4, we shall consider UMP as a topological property, not metric one: A metrizable space X is said to have UMP if there is a compatible metric ρ on X such that (X, ρ) has UMP in original sense. As mentioned above, although the space of irrational numbers, the Cantor set and $[-1,0] \cup [1,2]$ do not have UMP as metric subspaces of $\mathbb{R}$, it seems to be natural that we ask whether if they have any compatible metrics with UMP. The main interest in the section is the following question: Which subspaces of the real line $\mathbb{R}$ have UMP?

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We refer to [2] and [5] for general terminology and basic facts on general topology and to [3] for other results on metric spaces having special properties which relates with UMP.

2. Connected metric spaces with UMP

We will use the following notation throughout this section: For a metric space (X, ρ), a point $x \in X$ and $\varepsilon > 0$,

$$B(x, \varepsilon) = \{q \in X : \rho(x, q) < \varepsilon\},$$
$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}.$$

A point c of a topological space X is said to be a *cut point* of X if and only if there exist disjoint open sets U and V of X such that $X - \{c\} = U \cup V$.

It is well known that (cf. [5]) if X is a metric contimuun (=comact connencted space) and X has just two non-cut points, then X is a homeomorphic to the closed unit interval $[0, 1]$ in the real line $\mathbb{R}$. In [1], A. D. Berard considered non-comapact metric space with UMP and obtained the following characterization of such spaces.

Theorem 2.1 ([1])**.** *If X is a connected metric space with* UMP*, then X is homeomorphic to an interval in $\mathbb{R}$. Furtheremore, we have the following:*

(i) *If X has two distinct non-cut points, then X is a closed interval.*
(ii) *If X has an exactly one non-cut point, then X is a half open interval.*
(iii) *If X has no non-cut points, then X is an open interval.*

We notice that every connected metric space with UMP has at most two distinct non-cut points ([1, Theorem 3]).

To prove the theorem above, A. D. Berard introduced a partial order in a connected metric space with a non-cut point as follows: Let (X, ρ) be a connected metric space with a non-cut point z. Then for $x, y \in X$ we say $x < y$ if $\rho(x, z) < \rho(y, z)$.

The following theorem is a key of the proof of Theorem 2.1.

Theorem 2.2 ([1])**.** *Let (X, ρ) be a connected metric space with the* UMP *and z a non-cut point of X. Then the order topology induced by the partial order defined above coincides the original (metric) topology.*

We shall describe the proof of the theorem, because the original proof due to A. D. Berard contains a gap (see Remark 1 below).

We need three lemmas ([1, Lemma 4], [1, Lemma 7] and [1, Lemma 8]) to prove the theorem.

Lemma 2.1 ([1])**.** *Let (X, ρ) be a connected metric space with* UMP*, and z_1 and z_2 two distinct non-cut points of X. Then $X \subseteq [B(z_1, \varepsilon) \cap B(z_2, \varepsilon)] \cup \{z_1, z_2\}$, where $\varepsilon = \rho(z_1, z_2)$*

Lemma 2.2 ([1])**.** *Let (X, ρ) be a connected metric space with* UMP *and z a non-cut point of X. Then for each $a, b \in X$ with $a < b$, $I(a, b)$ is a connected metric space with* UMP *and exactly two non-cut points a and b, where $I(a, b) = \{x \in X : a \le x \le b\}$.*

Lemma 2.3 ([1]). *Let (X, ρ) be a connected metric space with* UMP *and z a non-cut point of X. Then for each cut point $m \in X$ and each $\varepsilon > 0$ there exist $x, y \in B(m, \varepsilon)$ such that $x < m < y$.*

Now, we are ready for proving Theorem 2.2.

Proof of Theorem 2.2. It is not difficult to show that the order topology induced by the partial order is weaker than the original (metric) topology. Hence, it suffices to show that the order topology induced by the partial order is stronger than the original (metric) topology. To show it, it suffices to show that for any $m \in X$ and any $\varepsilon > 0$, there exist x and y such that $m \in (x, y) \subseteq B(m, \varepsilon)$, where $x \in X$ or x is the symbol $-\infty$ and $y \in X$ or y is the symbol $+\infty$.

Case 1. Let m be a cut point of X. Then by Lemma 2.3, it follows that there exist $x, y \in B(m, \varepsilon/4)$ with $x < m < y$. Then it follows from Lemma 2.2 that $I(x, y) \subseteq B(m, \varepsilon)$.

Case 2. If $m = z$, the result is obvious with $x = -\infty$ and y being the unique point which is a distance ε from z.

Case 3. Let m be a non-cut point of X with $m \neq z$. We may assume that $\varepsilon < \rho(z, m)$. Let x be the unique point with $\rho(m, x) = \varepsilon$. By Lemma 2.1, it follows that $X \subseteq B(z, \rho(z, m)) \cup \{m\}$, hence we have $\rho(z, x) < \rho(z, m)$. Hence $x < m$. By Lemma 2.2, $I(x, m)$ is a connected metric space with UMP and has two distinct non-cut points x and m. Therefore, $I(x, m) \subseteq [B(x, \varepsilon) \cap B(m, \varepsilon)] \cup \{x, m\}$ by Lemma 2.2, and $I(x, m) \subseteq B(m, \varepsilon) \cup \{x\}$, i.e., $(x, +\infty) \subseteq B(m, \varepsilon)$. This completes the proof. □

Remark 2.1. In the third case of the original proof of [1], A. D. Berard took the unique point $x \in X$ with $\rho(z, x) = \rho(z, m) - \varepsilon/4$ and $y = +\infty$. But, the next example shows that they do not satisfy $(x, +\infty) \subseteq B(m, \varepsilon)$. Consider

$$X = \{(x, y) : (x - 1)^2 + y^2 = 1, 0 \leq y \leq 1\} \subseteq \mathbb{R}^2.$$

Then, (X, d) is a connected metric space with UMP and has two distinct non-cut points $z = (0, 0)$ and $m = (2, 0)$, where d is the metric on X induced by the Euclidean metric on $\mathbb{R}^2$. Now, for each ε with $0 < \varepsilon \leq 1/4$, let x be the unique point with $d(x, z) = d(z, m) - \varepsilon/4 = 2 - \varepsilon/4$. Then,

$$d(x, m) = \sqrt{2^2 - \left(2 - \frac{\varepsilon}{4}\right)^2} = \frac{1}{4}\sqrt{\varepsilon(16 - \varepsilon)}.$$

This implies that

$$\begin{aligned} d(x, m) - \varepsilon &= \frac{1}{4}\sqrt{\varepsilon(16 - \varepsilon)} - \varepsilon \\ &= \frac{1}{4}\sqrt{\varepsilon}(\sqrt{16 - \varepsilon} - 4\sqrt{\varepsilon}) \\ &> \frac{1}{4}\sqrt{\varepsilon}(\sqrt{16} - \sqrt{\varepsilon} - 4\sqrt{\varepsilon}) \\ &= \frac{1}{4}\sqrt{\varepsilon}(4 - 5\sqrt{\varepsilon}) > 0. \end{aligned}$$

Hence, $(x, +\infty) \nsubseteq B(m, \varepsilon)$.

We describe the examples of metric spaces with UMP which have several properties. A metric space (X, ρ) is said to be *convex* if for each pair of distinct points x and y there is exactly one point $p \in X$ such that $\rho(x, y) = \rho(x, p) + \rho(y, p)$. The convex metric seems to have any relation with UMP. The following example is a complete connected metric space with UMP for which the metric is not convex.

Example 2.1 ([1])**.** There is a complete connected metric space (X, ρ) with UMP for which the metric ρ is not convex. Indeed, for each $x, y \in [-1, 1] = X$ let $\rho(x, y) = |x - y|/(1 + |x - y|)$. Then (X, ρ) is the desiered metric space.

The following examples show the connectedness of a metric space is not a necessary condition for which has UMP.

Example 2.2 ([9])**.** Let

$$X = \{(x, x^2) \in \mathbb{R}^2 : x \geq 0\} \cup \{(-1, 1)\}.$$

Then, (X, d) is a locally compact separable metric space with UMP which is homeomorphic to a subspace of the real line, where d is the metric induced by the Euclidean metric on $\mathbb{R}^2$. Furthermore, it is neither connected nor totally-disconnected.

Example 2.3 ([1])**.** There is a totally disconnected complete metric space (X, ρ) such that (X, ρ) has UMP.

Indeed, consider

$$Y = \left\{ x : x \in [0, 1],\ x = \sum_{i=1}^{\infty} a_i 4^{-i} (a_i = 0 \ or\ 3) \right\}.$$

Notice that Y is homeomorphic to the Cantor set. Let h and k be linear homeomorphisms defined by

$$h : Y \to [0, 1/4] \times \{0\},$$
$$k : Y \to [1, 5/4] \times \{0\}.$$

Let $q = (5/8, 1) \in \mathbb{R}^2$ and put $X = \{q\} \cup h(Y) \cup k(Y)$. Let $\rho : X \times X \to \mathbb{R}^+$ be defined by

$$\begin{cases} \rho(q, q) = 0, & \\ \rho(x, q) = \sigma(q, x) = 1 & \text{for } x \in h(Y) \cup k(Y), \\ \rho(x, y) = \sigma(y, x) = |x - y| & \text{for } x, y \in h(Y) \cup k(Y). \end{cases}$$

Then it is easy to see that (X, ρ) is a totally disconnected complete metric space with UMP.

3. Non-connected metric spaces with UMP

In this section, we shall consider certain non-connected metric spaces with UMP. As is shown at the end of the previous section (cf. Examples 2.2 and 2.3), we do not need the connectedness for being homeomorphic to subspaces of the real line $\mathbb{R}$. Follows from Theorem 2.1, one may expect that non-connected metric spaces with UMP are homeomorphic to subspaces of the real line $\mathbb{R}$. In this direction, S. D. Nadler, Jr. proved the following theorem.

Theorem 3.1 ([9])**.** *If (X, d) is a locally compact separable metric space with* UMP*, then X is homeomorphic to a subspace of the real line $\mathbb{R}$.*

A result from [7] is used in the proof of the theorem.

The next example shows that the separability is essential in Theorem 3.1.

Example 3.1 ([9])**.** Let (X, σ) be any uncountable metric space with UMP (e.g., the real line $\mathbb{R}$ with the usual metric). We define the new metric ρ as follows:

$$\begin{cases} \rho(x, y) = 1 + \sigma(x, y) & \text{if } x \neq y, \\ \rho(x, y) = 0 & \text{if } x = y. \end{cases}$$

Then, (X, ρ) is a discrete metric space and ρ has UMP. It is obvious that (X, ρ) is not separable and hence, (X, ρ) can not be topologically embedded in $\mathbb{R}$.

There are examples of totally disconnected separable metric spaces with UMP, e.g., the space of rationals. We also notice that the space of irrationals and the Cantor set have compatible metirc with UMP (see the next section). Although there are totally disconnected separable metric spaces of arbitrary dimension (cf. [8, Theorem 3.9.3]), we don't know any totally disconnected separable metric space with UMP which have the positive dimension. Hence we can ask the following.

Question 3.1 ([9]). Is every totally disconnected separable metric space with UMP 0-dimensional?

We don't know the local compactness can be dropped in Theorem 3.1, i.e., the following question seems to remain open.

Question 3.2. Is separable metric space with UMP homeomorphic to a subspace of the real line?

Concerning Question 3.2, we have the following.

Theorem 3.2 ([4])**.** *The following hold.*

(1) *A separable metric space (X, ρ) is homeomorphic to a subspace of the real line if (X, ρ) satisfies the following two conditions.*
   (a) *The cardinality of any midset is at most 1, i.e., $|\{z \in X : \rho(x, z) = \rho(y, z)\}| \leq 1$ for each distinct points $x, y \in X$.*
   (b) *The cardinality of any subset consisting of points which are equidistant from a point is at most two, i.e., for each $x \in X$ and each positive number $\varepsilon > 0$ $|\{y \in X : \rho(x, y) = \varepsilon\}| \leq 2$.*

(2) *If X is locally compact and satisfies (a), then X is homeomorphic to a subspace of the real line $\mathbb{R}$.*

We don't know whether if the second part of the theorem holds for every rim-compact space (= every point has a neighborhood base consisting of the open sets with compact boundaries) (see [4, Remark 1]).

4. Topological aspects of the unique midpoint property

In this section, we shall consider the unique midpoint property as a topological property. As is mentioned in Introduction, the real line $\mathbb{R}$ with the usual metric and its metric subspace of rational numbers $\mathbb{Q}$ have UMP, while the usual Cantor set and the space of irrational numbers $\mathbb{P}$ do not have UMP as the metric subspace of $\mathbb{R}$. Then the following question arises: Does there exist a compatible metric with UMP on $\mathbb{P}$ or the Cantor set? Now, we say that a metrizable space X has the *unique midpoint property* (abbreviated *UMP*) if there is a compatible metric d on X such that (X, d) has UMP. Connecting the question above, M. Ito, H. Ohta and J. Ono [6] proved the following.

Theorem 4.1 ([6])**.** *The discrete space D with cardinality $\mathfrak{n}$ has* UMP *if and only if $\mathfrak{n} \neq 2, 4$ and $\mathfrak{n} \leq \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum.*

Theorem 4.2 ([6])**.** *Let D be a discrete space with cardinality not greater than $\mathfrak{c}$. Then the countable power $D^{\mathbb{N}}$ of D has* UMP*. In particular, the Cantor set $2^{\mathbb{N}}$ and the space $\mathbb{N}^{\mathbb{N}}$ of irrational numbers have* UMP*.*

We notice that not all infinite subspaces of $(D_{\mathfrak{c}})^{\mathbb{N}}$ have UMP. Let $x_n \to x$ be a convergent sequence in $(D_{\mathfrak{c}})^{\mathbb{N}}$ such that $x_m \neq x_n$ for $m \neq n$. Y. Hattori and H. Ohta showed that the subspace $S = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ does not have UMP ([4, Remark 2]). On the other hand, since the countable power $S^{\mathbb{N}}$ is homeomorphic to the Cantor set, it follows from Theorem 4.2 that $S^{\mathbb{N}}$ has UMP.

We alo notice that the metric of the space of irrationals (or the Cantor set) with UMP described in the proof of Theorem 4.2 is not intutive and it may be difficult to how to metrize the spaces. Hence the following question seems to be interested.

Question 4.1 ([6]). Is there a subspace X of the Euclidean soace $\mathbb{R}^n$ or the Hilbert space $\mathbb{R}^\infty$ such that X is homeomorphic to $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ and the metric on X induced by the usual metric of $\mathbb{R}^n$ or $\mathbb{R}^\infty$ has UMP?

Remark 4.1 ([6]). We have an intuitive way of showing that the topological sum $\mathbb{Q} \oplus \mathbb{P}$ of the space $\mathbb{Q}$ of rational numbers and the space $\mathbb{P}$ of irrational numbers has UMP. Indeed, let $h : \mathbb{R} \to (-1, 1)$ be the homeomorphism defined by $h(x) = x/(1 + |x|)$ for $x \in \mathbb{R}$. Define

$$X = \{(x, \sqrt{1 - x^2}) : x \in h(\mathbb{Q})\} \cup \{(x, -\sqrt{1 - x^2}) : x \in h(\mathbb{P})\} \subseteq \mathbb{R}^2.$$

Then X is homeomorphic to $\mathbb{Q} \oplus \mathbb{P}$ and the metric on X induced by the Euclidean metric on $\mathbb{R}^2$ has UMP.

Theorems 4.1 and 4.2 are proved by an approach from the graph theory. We shall make a short description of this approach, because we may be intersted in itself.

The edge connecting vertices x and y is denoted by xy. By a *colouring* of a graph G we mean a map defined on the set of edges $E(G)$ of G. For a colouring $\varphi : E(G) \to A$, we call $\varphi(e)$ the colour of e for $e \in E(G)$. A colouring φ of G is said to have the the *unique midpoint property* (abbreviated *UMP*) if for every pair of distinct vertices x, y of G there is an exactly one vertex p of G such that xp

and yp are edges of G and $\varphi(xp) = \varphi(yp)$. Theorem 4.1 is a direct consequence of the following theorem which connects with the unique midpoint property of the discrete space and of the complete graph.

Theorem 4.3 ([6]). *Let $\mathfrak{n}$ be a cardinal and $K_{\mathfrak{n}}$ the complete graph (i.e., each vertex of $K_{\mathfrak{n}}$ is adjacent to every other vertices) with $\mathfrak{n}$ vertices. Then we have the following.*

(1) *The discrete space D with cardinality $\mathfrak{n}$ has* UMP *if and only if $\mathfrak{n} \le \mathfrak{c}$ and the complete graph $K_{\mathfrak{n}}$ has a colouring with* UMP.
(2) *The complete graph $K_{\mathfrak{n}}$ has a colouring with* UMP *if and only if $\mathfrak{n} \ne 2, 4$.*

We turn our attention to the graphs which have a colouring with UMP. Let G be a finite graph having a colouring with UMP. Then we denote the smallest number of colours we need to colour G with UMP by $\mathrm{ump}(G)$:

$$\mathrm{ump}(G) = \min\{|\varphi(E(G))| : \varphi \text{ is a colouring of } G \text{ with UMP}\}.$$

Then M. Ito, H. Ohta and J. Ono [6] proved the following.

Proposition 4.1 ([6]). *For each $k \ge 0$, $\mathrm{ump}(K_{2k+1}) = k$.*

Proposition 4.2 ([6]). *For each $k \ge 3$, $k \le \mathrm{ump}(K_{2k}) \le 2k - 1$.*

We notice that the following equalities are already known ([6]):

$$\mathrm{ump}(K_6) = 4, \ \mathrm{ump}(K_8) = 5 \text{ and } \mathrm{ump}(K_{10}) = 5.$$

Moreover, we also have the following inequalities ([6]):

$$\mathrm{ump}(K_{12}) \le 8 \text{ and } \mathrm{ump}(K_{14}) \le 10.$$

The following question still remains open.

Question 4.2 ([6]). Determine the values of $\mathrm{ump}(K_{2k})$ for each $k \ge 6$.

Now, we shall consider subspaces of the real line $\mathbb{R}$ which have UMP. For $n \in \mathbb{N}$, let I_n be the union of n-many disjoint closed intervals in $\mathbb{R}$. H. Ohta and J. Ono [10] proved the following theorems.

Theorem 4.4 ([10]). *Let X be a compact disconnected metrizable space. If either exactly one of the components is or exactly two of the components are nondegenerate, then X does not have* UMP. *In particular, I_2 does not have* UMP. *On the other hand, I_{2n-1} has* UMP *for each $n \in \mathbb{N}$.*

Theorem 4.5 ([10]). *Let X be a disconnected metrizable space and Y a subspace of X such that Y is homeomorphic to either I_1 or I_2. If $|X - Y| < \mathfrak{c}$, then X does not have* UMP.

Theorem 4.6 ([10]). *If X is a subspace of the real line $\mathbb{R}$ satisfying the following conditions, then X has* UMP*:*

(i) *Each nondegenerate component of X is an open set of X.*
(ii) *The union of one-point components of X is an open set of X.*
(iii) *At least one component is not compact.*

The following is a direct consequence of Theorems 4.4, 4.5 and 4.6.

Corollary 4.1 ([10]).

(1) *Let I and J be disjoint intervals in $\mathbb{R}$. Then $I \cup J$ has* UMP *if and only if either I or J is not compact.*
(2) *The union of an odd number of disjoint closed intervals in $\mathbb{R}$ has* UMP.
(3) *The subspaces $[0,1] \cup \mathbb{Z}$ and $[0,1] \cup \mathbb{Q}$ of $\mathbb{R}$ do not have* UMP, *where $\mathbb{Z}$ denotes the set of integers.*
(4) *Let X be a space which is the union of at most countably many subspaces $\{X_n : n \in A\}$ of $\mathbb{R}$. Assume that at least one of the spaces X_n is a non-compact interval and others are either intervals or totally disconnected. Then, X has* UMP.
(5) *Let X be a space with exactly two components. Then, X has* UMP *if and only if X is not compact and each component is either homeomorphic to an interval or a singleton.*

J. Ono showed that I_4 does not have UMP, but I_6, I_{10} and I_ω have UMP. This arises the following question.

Question 4.3 ([10]). For $n = 8$ or an even number $n \geq 12$, does I_n have UMP?

Connecting Corollary 4.1, we can also ask:

Question 4.4 ([10]). Does every subspace of the real line $\mathbb{R}$ containing a non-compact interval as a clopen set have UMP?

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