Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
38 (2005), pp. 17–30

OSCILLATION AND NONOSCILLATION THEOREMS FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Communicated by Jitsuro Sugie

(Received: March 1, 2005)

Abstract. In this paper, we are concerned with the oscillation problem for the nonlinear differential equation $(\phi_p(a(t)x'))' + b(t)g(x) = 0$, where $\phi_p(y)$ is the one-dimensional p-Laplacian operator, and $g(x)$ satisfies $xg(x) > 0$ if $x \neq 0$.

1. Introduction

The purpose of this paper is to improve oscillation and nonoscillation theorems for the nonlinear differential equation

$$(\phi_p(a(t)x'))' + b(t)g(x) = 0, \tag{1.1}$$

where $\phi_p(y)$ is a real-valued function defined by $\phi_p(y) = |y|^{p-2}y$ with $p > 1$ a fixed real number, and $a(t)$ and $b(t)$ are positive and continuous on some half-line (α, ∞), and $g(x)$ is a continuous function on $\mathbb{R}$ satisfying the signum condition

$$xg(x) > 0 \quad \text{if} \;\; x \neq 0. \tag{1.2}$$

We assume throughout this paper that

$$\int_\alpha^\infty \frac{1}{a(t)} dt < \infty. \tag{1.3}$$

A nontrivial solution $x(t)$ of (1.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, it is said to be *nonoscillatory*.

When $p = 2$, equation (1.1) becomes the nonlinear self-adjoint differential equation

$$(a(t)x')' + b(t)g(x) = 0, \tag{1.4}$$

which has been devoted to the study of the oscillation problem by many authors (for example, see [3, 6, 7, 8, 9, 10]). Especially, using phase plain analysis of the Liénard system, Sugie et al. [8] discussed the oscillation problem for equation (1.4)

2000 *Mathematics Subject Classification.* 34C10, 34C15.

Key words and phrases. Oscillation, One-dimensional p-Laplacian, Riccati technique.

whether the integral of the function $1/a(t)$ is divergent or convergent. In case (1.3), they gave the following pair of an oscillation theorem and a nonoscillation theorem.

Theorem A ([8, Theorem 3.1]). *Let* (1.2) *and* (1.3) *hold. Suppose that* $a(t)$ *and* $b(t)$ *satisfy*

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(\tau)}d\tau\right)^2 \ge 1$$

for t *sufficiently large, and that there exists a* λ *with* $\lambda > 1/16$ *such that*

$$\frac{g(x)}{x} \ge \frac{1}{4} + \frac{\lambda}{(\log|x|)^2}$$

for $|x|$ *sufficiently small. Then all nontrivial solutions of* (1.4) *are oscillatory.*

Theorem B ([8, Theorem 3.2]). *Let* (1.2) *and* (1.3) *hold. Suppose that* $a(t)$ *and* $b(t)$ *satisfy*

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(\tau)}d\tau\right)^2 \le 1$$

for t *sufficiently large, and that* $g(x)$ *satisfies*

$$G(x) \stackrel{\text{def}}{=} \int_0^x g(\chi)d\chi \le \frac{1}{2}x^2 \quad \textit{for } x \in \mathbb{R}$$

and

$$\frac{g(x)}{x} \le \frac{1}{4} + \frac{1}{16(\log|x|)^2}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently small. Then all nontrivial solutions of* (1.4) *are nonoscillatory.*

Equation (1.1) also includes a half-linear differential equation. In the study of half-linear differential equations, its associated Riccati inequality plays an important role in the oscillation of solutions (see [1, 2, 4, 5]). Using Riccati technique, we can extend Theorems A and B to Theorems 1.1 and 1.2 below, respectively.

Theorem 1.1. *Let* (1.2) *and* (1.3) *hold. Suppose that* $a(t)$ *and* $b(t)$ *satisfy*

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(\tau)}d\tau\right)^p \ge 1 \tag{1.5}$$

for t *sufficiently large, and that there exists a* λ *with*

$$\lambda > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p+1} \tag{1.6}$$

such that

$$\frac{g(x)}{\phi_p(x)} \ge \left(\frac{p-1}{p}\right)^p + \frac{\lambda}{(\log|x|)^2} \tag{1.7}$$

for $|x|$ *sufficiently small. Then all nontrivial solutions of* (1.1) *are oscillatory.*

Theorem 1.2. *Let* (1.2) *and* (1.3) *hold. Suppose that* $a(t)$ *and* $b(t)$ *satisfy*

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(\tau)}d\tau\right)^p \le 1 \tag{1.8}$$

for t *sufficiently large, and that* $g(x)$ *satisfies*

$$G(x) \le \frac{p-1}{p}|x|^p \quad \textit{for } x \in \mathbb{R} \tag{1.9}$$

and there exists a λ *with*

$$0 < \lambda < \frac{1}{2}\left(\frac{p-1}{p}\right)^{p+1} \tag{1.10}$$

such that

$$\frac{g(x)}{\phi_p(x)} \le \left(\frac{p-1}{p}\right)^p + \frac{\lambda}{(\log|x|)^2} \tag{1.11}$$

for $x > 0$ *or* $x < 0$, $|x|$ *sufficiently small. Then all nontrivial solutions of* (1.1) *are nonoscillatory.*

Remark. Since Theorem 1.1 coincides with Theorem A when $p = 2$, Theorem 1.1 is a complete generalization of Theorem A. Also Theorem 1.2 includes Theorem B if $\lambda \ne ((p-1)/p)^{p+1}/2$. From Theorem B, we see that all nontrivial solutions of (1.1) are nonoscillatory in the case $\lambda = ((p-1)/p)^{p+1}/2$ with $p = 2$. For this reason, we may conjecture that even if $\lambda = ((p-1)/p)^{p+1}/2$ with $p \ne 2$, then all nontrivial solutions of (1.1) are nonoscillatory.

2. Transformation into a special case

In this section, we will find the canonical form of (1.1). For this purpose, we define

$$s = s(t) = \left(\int_t^\infty \frac{1}{a(\tau)}d\tau\right)^{-1}, \qquad u(s) = x(t(s)),$$

where $t(s)$ is the inverse function of $s(t)$. Then we have

$$x'(t) = \frac{ds}{dt}\dot{u}(s) = \frac{s^2}{a(t)}\dot{u}(s),$$

$$\begin{aligned}(\phi_p(a(t)x'(t)))' &= \frac{ds}{dt}(\phi_p(s^2\dot{u}(s)))^\cdot = \frac{s^2}{a(t)}(s^{2(p-1)}\phi_p(\dot{u}(s)))^\cdot \\ &= \frac{s^{2p}}{a(t)}\left((\phi_p(\dot{u}(s)))^\cdot + \frac{2(p-1)}{s}\phi_p(\dot{u}(s))\right),\end{aligned}$$

where $^\cdot = d/ds$, and therefore, equation (1.1) becomes the equation

$$(\phi_p(\dot{u}))^\cdot + \frac{2(p-1)}{s}\phi_p(\dot{u}) + \frac{a(t(s))b(t(s))}{s^{2p}}g(u) = 0. \tag{2.1}$$

Since $a(t)$ is positive for $t \ge \alpha$ and satisfies (1.3), the functions $s(t)$ and $t(s)$ are increasing and $s(t) \to \infty$ as $t \to \infty$. Hence, all nontrivial solutions of (1.1) are oscillatory (resp., nonoscillatory) if and only if all nontrivial solutions of (2.1) are oscillatory (resp., nonoscillatory).

Let $c(s) = a(t(s))\,b(t(s))/s^{2p}$. Then conditions (1.5) and (1.8) coincide with $s^p c(s) \geq 1$ and $s^p c(s) \leq 1$, respectively. Thus, Theorems 1.1 and 1.2 are rewritten as an oscillation theorem and a nonoscillation theorem for equation

$$(\phi_p(x'))' + \frac{2(p-1)}{t}\phi_p(x') + c(t)g(x) = 0, \tag{2.2}$$

respectively.

Proposition 2.1. *Assume* (1.2) *and suppose that* $c(t)$ *satisfies*

$$t^p c(t) \geq 1 \tag{2.3}$$

for t *sufficiently large, and that there exists a* λ *with* (1.6) *satisfying* (1.7) *for* $|x|$ *sufficiently small. Then all nontrivial solutions of* (2.2) *are oscillatory.*

Proposition 2.2. *Let* (1.2) *and* (1.9) *hold. Suppose that* $c(t)$ *satisfies*

$$t^p c(t) \leq 1 \tag{2.4}$$

for t *sufficiently large, and that there exists a* λ *with* (1.10) *satisfying* (1.11) *for* $x > 0$ *or* $x < 0$, $|x|$ *sufficiently small. Then all nontrivial solutions of* (2.2) *are nonoscillatory.*

3. Proof of Proposition 2.1

To prove Proposition 2.1, we prepare some lemmas. We first consider the following lemma concerning properties of nonoscillatory solutions of (2.2).

Lemma 3.1. *Assume* (1.2) *and suppose that* $c(t)$ *satisfies* (2.3) *for* t *sufficiently large, and that equation* (2.2) *has a nonoscillatory solution. Then the solution tends to zero as* $t \to \infty$. *Furthermore, if the solution is eventually positive, then its derivative is eventually negative.*

Proof. Let $x(t)$ be a nonoscillatory solution of (2.2). Then, without loss of generality, we may assume that there exists a $T > 0$ such that $x(t) > 0$ for $t \geq T$.

To begin with, we will show that there exists a $t_1 \geq T$ such that $x'(t_1) < 0$. By way of contradiction, we suppose that $x'(t) \geq 0$ for $t \geq T$. Then $x(t) \geq x(T) > 0$ for $t \geq T$. Hence, by (1.2) and (2.3), we have

$$(t^{2(p-1)}\phi_p(x'(t)))' = -t^{2(p-1)}c(t)g(x(t)) \leq -t^{p-2}g(x(t)) < 0 \quad \text{for } t \geq T.$$

Integrating both sides of this inequality from T to t, we get

$$t^{2(p-1)}\phi_p(x'(t)) \leq T^{2(p-1)}\phi_p(x'(T)) \quad \text{for } t \geq T,$$

and therefore, $x'(t) \leq T^2 x'(T)/t^2$ for $t \geq T$. Integrate this inequality to obtain

$$x(t) \leq T^2 x'(T)\left(\frac{1}{T} - \frac{1}{t}\right) + x(T) \leq T x'(T) + x(T) \quad \text{for } t \geq T.$$

Define $m_1 = \min\{g(x) \mid x(T) \leq x \leq T x'(T) + x(T)\}$. Then we have

$$(t^{2(p-1)}\phi_p(x'(t)))' \leq -t^{p-2}g(x(t)) \leq -t^{p-2}m_1 \quad \text{for } t \geq T.$$

Integrating both sides of this inequality from T to t, we get

$$t^{2(p-1)}\phi_p(x'(t)) \le -\frac{m_1}{p-1}(t^{p-1}-T^{p-1}) + T^{2(p-1)}\phi_p(x'(T)) \quad \text{for } t \ge T.$$

Hence, $x'(t)$ is negative for t sufficiently large. This is a contradiction to the assumption that $x'(t) \ge 0$ for $t \ge T$. Thus, $x'(t_1) < 0$ for some $t_1 > T$.

Next, we will show that $x'(t) < 0$ for $t \ge t_1$. Suppose that there exists a $t_2 > t_1$ such that

$$x'(t) < 0 \quad \text{for } t_1 \le t < t_2 \qquad \text{and} \qquad x'(t_2) = 0. \tag{3.1}$$

Since $x(t)$ is a solution of (2.2), we have

$$x''(t) = -\frac{2}{t}x'(t) - \frac{c(t)g(x(t))}{(p-1)|x'(t)|^{p-2}} = -x'(t)\left(\frac{2}{t} + \frac{c(t)g(x(t))}{(p-1)\phi_p(x'(t))}\right)$$

for $t_1 \le t < t_2$. From (3.1) we see that

$$\frac{2}{t} + \frac{c(t)g(x(t))}{(p-1)\phi_p(x'(t))} \to -\infty \quad \text{as } t \to t_2 - 0.$$

Hence, there exists a $\tau > 0$ such that $x''(t) < 0$ for $t_2 - \tau \le t < t_2$, and therefore, $x'(t)$ is decreasing for $t_2 - \tau \le t \le t_2$. Thus, we obtain $x'(t) > x'(t_2)$ for $t_2 - \tau \le t < t_2$, which is a contradiction to (3.1). We therefore conclude that $x(t)$ is decreasing for $t \ge t_1$.

Finally, we will show that $x(t)$ tends to zero as $t \to \infty$. Suppose that $x(t)$ does not tend to zero as $t \to \infty$. Since $x(t)$ is positive and decreasing for $t \ge t_1$, there exists a $\mu > 0$ such that $x(t) \to \mu$ as $t \to \infty$. Let $m_2 = \min\{g(x) \mid \mu \le x \le x(t_1)\}$. Then we have

$$(t^{2(p-1)}\phi_p(x'(t)))' \le -t^{p-2}g(x(t)) \le -t^{p-2}m_2 \quad \text{for } t \ge t_1.$$

Hence, integrating both sides of this inequality from t_1 to t, we get

$$t^{2(p-1)}\phi_p(x'(t)) \le -\frac{m_2}{p-1}(t^{p-1}-t_1^{p-1}) + t_1^{2(p-1)}\phi_p(x'(t_1)) \quad \text{for } t \ge t_1.$$

Let L be a positive number satisfying $L^{p-1} < m_2/(p-1)$. Then there exists a $t_3 > t_1$ such that

$$t^{2(p-1)}\phi_p(x'(t)) \le -L^{p-1}t^{p-1} \quad \text{for } t \ge t_3,$$

and therefore, $x'(t) \le -L/t$ for $t \ge t_3$. Thus, we see that

$$x(t) \le -L\log\frac{t}{t_3} + x(t_3) \quad \text{for } t \ge t_3.$$

This is a contradiction to the assumption that $x(t)$ is positive for $t \ge T$, thereby completing the proof. □

We next consider some differential inequalities of the first order. For simplicity, we denote

$$H(\xi) = (p-1)\left((-\xi)^{p/(p-1)} + \xi + \frac{(p-1)^{p-1}}{p^p}\right)$$

for $\xi < 0$ and

$$\gamma_p = \left(\frac{p-1}{p}\right)^{p-1}$$

with $p > 1$ a fixed real number.

Lemma 3.2. *Suppose that the differential inequality*

$$\dot{\xi} + H(\xi) \le 0 \tag{3.2}$$

has a negative solution on $[s_0, \infty)$ *with* $s_0 > 0$. *Then the solution tends to* $-\gamma_p$ *as* $s \to \infty$.

Proof. Since

$$H(-\gamma_p) = (p-1)\left\{\left(\frac{p-1}{p}\right)^p - \left(\frac{p-1}{p}\right)^{p-1} + \frac{(p-1)^{p-1}}{p^p}\right\} = 0$$

and

$$\frac{d}{d\xi}H(\xi) = -p(-\xi)^{1/(p-1)} + p - 1,$$

we see that $H(\xi) > 0$ if $\xi \neq -\gamma_p$. Let $\xi(s)$ be a negative solution of (3.2) on $[s_0, \infty)$. Then $\xi(s)$ satisfies

$$\dot{\xi}(s) = -H(\xi(s)) \le 0 \quad \text{for} \;\; s \ge s_0. \tag{3.3}$$

Let $u(s)$ be the positive function defined by

$$u(s) = \exp\left(-\int_{s_0}^{s} (-\xi(\sigma))^{1/(p-1)} d\sigma\right)$$

for $s \ge s_0$. Differentiate $u(s)$ to obtain

$$\dot{u}(s) = -u(s)(-\xi(s))^{1/(p-1)} < 0 \quad \text{for} \;\; s \ge s_0.$$

Hence, we get

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} \quad \text{for} \;\; s \ge s_0. \tag{3.4}$$

Differentiating both sides, we have

$$\dot{\xi}(s) = \frac{(\phi_p(\dot{u}(s)))^{\cdot}}{\phi_p(u(s))} - (p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^p \quad \text{for} \;\; s \ge s_0.$$

Hence, by (3.3) and (3.4), $u(s)$ satisfies

$$(\phi_p(\dot{u}(s)))^{\cdot} + (p-1)\phi_p(\dot{u}(s)) + \left(\frac{p-1}{p}\right)^p \phi_p(u(s)) \le 0 \quad \text{for} \;\; s \ge s_0. \tag{3.5}$$

Put $v(s) = \dot{u}(s) + u(s)$. Then, by (3.5) we have

$$\dot{v}(s) = \ddot{u}(s) + \dot{u}(s) \le -\frac{(p-1)^{p-1}\phi_p(u(s))}{p^p|\dot{u}(s)|^{p-2}} < 0 \quad \text{for} \;\; s \ge s_0.$$

Suppose that there exists an $s_1 > s_0$ such that $v(s_1) < 0$. Then we get $v(s) \le v(s_1)$ for $s \ge s_1$. Since $u(s) > 0$ for $s \ge s_1$, we obtain

$$\dot{u}(s) = v(s) - u(s) < v(s_1) \quad \text{for} \;\; s \ge s_1.$$

Therefore, we have

$$u(s) = v(s_1)(s - s_1) + u(s_1) \to -\infty \quad \text{as} \;\; s \to \infty.$$

This is a contradiction. Hence, we see that $\dot{u}(s) + u(s) = v(s) \ge 0$ for $s \ge s_0$. Thus, we get

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} \ge -1 \quad \text{for} \;\; s \ge s_0. \tag{3.6}$$

Using (3.3) and (3.6), we can find a μ such that $-1 \le \mu < 0$ and $\xi(s) \to \mu$ as $s \to \infty$. If $\mu \ne -\gamma_p$, there exists an $s_2 \ge s_0$ such that

$$\dot{\xi}(s) \le -H(\xi(s)) \le -H\left((\mu - \gamma_p)/2\right) < 0$$

for $s \ge s_2$. Then we obtain $\xi(s) \to -\infty$ as $s \to \infty$ which is a contradiction to (3.6). The proof is complete. □

Lemma 3.3. *Suppose that the differential inequality*

$$\dot{\eta} + \left(\eta + \frac{1}{2}\right)^2 + \frac{\delta}{s^2} \le 0 \tag{3.7}$$

has a solution on $[s_0, \infty)$ *with* $s_0 > 0$ *where* δ *is a positive parameter. Then* $\delta \le 1/4$.

Proof. Let $\eta(s)$ be a solution of (3.7) and define

$$h(s) = -\dot{\eta}(s) - \left(\eta(s) + \frac{1}{2}\right)^2 \quad \text{for} \;\; s \ge s_0. \tag{3.8}$$

Then we have

$$h(s) \ge \frac{\delta}{s^2} \quad \text{for} \;\; s \ge s_0. \tag{3.9}$$

Changing variable $t = e^s$ and let $u(s)$ be the positive function defined by

$$x(t) = \exp\left(\int_{s_0}^{\log t} \eta(\sigma) d\sigma\right) \quad \text{for} \;\; t \ge e^{s_0}.$$

Then, by (3.8) $x(t)$ is a nonoscillatory solution of the linear differential equation

$$x'' + \frac{2}{t}x' + \frac{1}{t^2}\left(\frac{1}{4} + h(\log t)\right)x = 0$$

It follows from (3.9) and Sturm's comparison theorem that all nontrivial solutions of

$$x'' + \frac{2}{t}x' + \frac{1}{t^2}\left(\frac{1}{4} + \frac{\delta}{(\log t)^2}\right)x = 0 \tag{3.10}$$

are nonoscillatory. It is known that all nontrivial solutions of (3.10) are nonoscillatory if and only if $\delta \le 1/4$ (for details, see [8]). The proof is now complete. □

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Let t_0 be a large number satisfying (2.3) for $t \geq t_0$ and let ε_0 be a small number satisfying (1.7) for $0 < |x| < \varepsilon_0$. Since ε_0 is sufficiently small, by (1.6) we see that

$$\frac{\gamma_p}{2}(1+\varepsilon_0)\left(\frac{p-1}{p}+\varepsilon_0\right)^2 < \lambda. \tag{3.11}$$

The proof is by contradiction. Suppose that equation (2.2) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. By Lemma 3.1 there exists a $t_1 > t_0$ such that $0 < x(t) < \varepsilon_0$ and $x'(t) < 0$ for $t \geq t_1$.

Making the change of variable $t = e^s$, we can transform equation (2.2) into the equation

$$(\phi_p(\dot{u}))^{\cdot} + (p-1)\phi_p(\dot{u}) + e^{ps}c(e^s)g(u) = 0. \tag{3.12}$$

Let $u(s)$ be the solution of (3.12) corresponding to $x(t)$. Then we have $0 < u(s) < \varepsilon_0$ and $\dot{u}(s) = tx'(t) < 0$ for $s \geq \log t_1$. Define

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))}$$

which is negative for $s \geq \log t_1$. Differentiating $\xi(s)$ and using (1.2), (1.7), (2.3) and (3.12), we have

$$\begin{aligned}
\dot{\xi}(s) &= \frac{(\phi_p(\dot{u}(s)))^{\cdot}}{\phi_p(u(s))} - (p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^p \\
&= \frac{-(p-1)\phi_p(\dot{u}(s)) - e^{ps}c(e^s)g(u(s))}{\phi_p(u(s))} - (p-1)\,|\xi(s)|^{p/(p-1)} \\
&\leq -(p-1)\xi(s) - \left(\frac{p-1}{p}\right)^p - \frac{\lambda}{(\log u(s))^2} - (p-1)|\xi(s)|^{p/(p-1)} \\
&= -H(\xi(s)) - \frac{\lambda}{(\log u(s))^2} \quad \text{for } s \geq \log t_1. \qquad (3.13)
\end{aligned}$$

Hence, by Lemma 3.2 we see that

$$\xi(s) \searrow -\gamma_p \quad \text{as } s \to \infty, \tag{3.14}$$

and therefore, we have

$$\frac{\dot{u}(s)}{u(s)} \geq -\frac{p-1}{p} \quad \text{for } s \geq \log t_1.$$

Integrating both sides of this inequality from $\log t_1$ to s, we obtain

$$u(s) \geq u(\log t_1)\exp\left\{-\frac{p-1}{p}(s-\log t_1)\right\} \quad \text{for } s \geq \log t_1.$$

Hence, there exists an $s_1 > \log t_1$ such that

$$u(s) \geq \exp\left\{-\left(\frac{p-1}{p}+\varepsilon_0\right)s\right\} \quad \text{for } s \geq s_1.$$

Thus, together with (3.13), we get

$$\dot{\xi}(s) \leq -H(\xi(s)) - \frac{\lambda}{((p-1)/p+\varepsilon_0)^2 s^2} \quad \text{for} \ \ s \geq s_1. \tag{3.15}$$

From Taylor's expansion theorem, there exists a $0 < K(\xi) < (p-1)/p$ such that

$$H(\xi) = \frac{1}{2\gamma_p}(\xi+\gamma_p)^2 + \frac{p(p-2)}{6(p-1)^2}K(\xi)^{3-2p}(\xi+\gamma_p)^3 \quad \text{for} \ \ -\gamma_p \leq \xi < 0. \tag{3.16}$$

Therefore, we can find an $\varepsilon_1 > 0$ such that

$$H(\xi) \geq \frac{1}{2\gamma_p(1+\varepsilon_0)}(\xi+\gamma_p)^2 \quad \text{for} \ \ -\gamma_p \leq \xi \leq -\gamma_p+\varepsilon_1. \tag{3.17}$$

By (3.14) there exists an $s_2 \geq s_1$ such that $-\gamma_p \leq \xi(s) \leq -\gamma_p + \varepsilon_1$ for $s \geq s_1$. Let

$$\eta(s) = \frac{\xi(s) - \varepsilon_0\gamma_p}{2\gamma_p(1+\varepsilon_0)}.$$

Then, from (3.17) we have

$$\begin{aligned} H(\xi(s)) &\geq \frac{1}{2\gamma_p(1+\varepsilon_0)}(\xi(s)+\gamma_p)^2 = \frac{1}{2\gamma_p(1+\varepsilon_0)}\{2\gamma_p(1+\varepsilon_0)\eta(s)+\gamma_p(1+\varepsilon_0)\}^2 \\ &= 2\gamma_p(1+\varepsilon_0)\left(\eta(s)+\frac{1}{2}\right)^2 \quad \text{for} \ \ s \geq s_2. \end{aligned}$$

Hence, by (3.15) we obtain

$$\begin{aligned} \dot{\eta}(s) &= \frac{\dot{\xi}(s)}{2\gamma_p(1+\varepsilon_0)} \leq \frac{1}{2\gamma_p(1+\varepsilon_0)}\left\{-H(\xi(s)) - \frac{\lambda}{((p-1)/p+\varepsilon_0)^2 s^2}\right\} \\ &\leq -\left(\eta(s)+\frac{1}{2}\right)^2 - \frac{\lambda}{2\gamma_p(1+\varepsilon_0)((p-1)/p+\varepsilon_0)^2 s^2} \quad \text{for} \ \ s \geq s_2. \end{aligned}$$

Thus, from Lemma 3.3 we have

$$\frac{\lambda}{2\gamma_p(1+\varepsilon_0)((p-1)/p+\varepsilon_0)^2} \leq \frac{1}{4},$$

which is a contradiction to (3.11). This completes the proof of Proposition 2.1. □

4. Proof of Proposition 2.2

Before proving Proposition 2.2, we will show that oscillatory solutions of (2.2) tend to zero.

Lemma 4.1. *Let* (1.2) *and* (1.9) *hold. Suppose that* $c(t)$ *satisfies* (2.4) *for* t *sufficiently large, and that equation* (2.2) *has a nontrivial oscillatory solution. Then the solution tends to zero as* $t \to \infty$.

Proof. Let $x(t)$ be a nontrivial oscillatory solution of (2.2). Changing variable $t = e^s$, we can transform equation (2.2) into the system

$$\begin{aligned} \dot{u} &= \phi_q(v), \\ \dot{v} &= -(p-1)v - e^{ps}c(e^s)g(u), \end{aligned} \tag{4.1}$$

where $q = p/(p-1)$. Let $(u(s), v(s))$ be the solution of (4.1) corresponding to $x(t)$. Then there exists a sequence $\{s_n\}$ such that $u(s_n) = 0$. Consider the function

$$U(u,v) = \begin{cases} \dfrac{1}{q}|v|^q + G(u) & \text{if } \ uv \le 0, \\ G(u + \phi_q(v)) & \text{if } \ uv > 0. \end{cases}$$

Using (1.2) and (2.4), we have

$$\begin{aligned} \dot{U}_{(4.1)}(u,v) &= -\phi_q(v)((p-1)v + e^{ps}c(e^s)g(u)) + g(u)\phi_q(v) \\ &= -(p-1)|v|^q + (1 - e^{ps}c(e^s))g(u)\phi_q(v) \\ &\le -(p-1)|v|^q \le 0 \quad \text{for } \ vu \le 0, \\ \dot{U}_{(4.1)}(u,v) &= g(u+\phi_q(v))\{\phi_q(v) - (q-1)|v|^{q-2}((p-1)v - e^{ps}c(e^s)g(u))\} \\ &= g(u+\phi_q(v))\{(1-(p-1)(q-1))\phi_q(v) - (q-1)e^{ps}c(e^s)|v|^{q-2}g(u)\} \\ &= -(q-1)e^{ps}c(e^s)|v|^{q-2}g(u)g(u+\phi_q(v)) < 0 \quad \text{for } \ vu > 0. \end{aligned}$$

Hence, by (1.9) we obtain

$$\begin{aligned} \lim_{s \to s_n + 0} U(u(s), v(s)) &= G(\phi_q(v(s_n))) \\ &\le \frac{p-1}{p}|v(s_n)|^{p(q-1)} = \frac{1}{q}|v(s_n)|^q = \lim_{s \to s_n - 0} U(u(s), v(s)). \end{aligned}$$

Put

$$V(s) = \begin{cases} U(u(s), v(s)) & \text{if } \ s \ne s_n, \\ \lim\limits_{s \to s_n - 0} U(u(s), v(s)) & \text{if } \ s = s_n. \end{cases}$$

Then we conclude that the function $V(s)$ is piecewise continuous and decreasing for $s \ge s_1$. Hence, $v(s)$ is bounded, namely, there exists a $B > 0$ such that $|v(s)| < B$ for $s \ge s_1$.

To complete the proof, it suffices to show that $V(s)$ tends to zero as $s \to \infty$. Suppose that there exists a $V_0 > 0$ such that

$$V(s) \searrow V_0 \qquad \text{as } \ s \to \infty.$$

Let $S_{V_0} = \{(u,v) \mid U(u,v) < V_0\}$. Then the solution $(u(s), v(s))$ does not enter S_{V_0} for $s \ge s_1$. The region S_{V_0} consists of two bounded and disjointed parts and encircles the origin.

We can find an ε_0 so small that

$$\{(u,v) \mid |u| < \varepsilon_0 \ \text{ and } \ |v| < \varepsilon_0\} \subset S_{V_0}$$

Since the positive orbit of (4.1) corresponding to $(u(s), v(s))$ rotates around the region S_{V_0} in a clockwise direction, there exists a sequence $\{\sigma_n\}$ such that $\sigma_n < s_n$, $|u(\sigma_n)| = \varepsilon_0$ and $|v(s)| > \varepsilon_0$ for $\sigma_n < s < s_n$. Hence, we have

$$\varepsilon_0 = |u(s_n) - u(\sigma_n)| = \left| \int_{\sigma_n}^{s_n} \dot{u}(s)ds \right| = \left| \int_{\sigma_n}^{s_n} \phi_q(v(s))ds \right| \le \phi_q(B)(s_n - \sigma_n),$$

and therefore,

$$\begin{aligned} V_0 - V(s_1) &\le \lim_{s\to\infty} (U(u(s),v(s)) - U(u(\tau_1),v(\tau_1))) = \int_{s_1}^{\infty} \frac{d}{ds} U(u(s),v(s))ds \\ &\le -(p-1)\sum_{n=1}^{\infty}\int_{\sigma_n}^{s_n} |v(s)|^q ds \le -(p-1)\varepsilon_0^q \sum_{n=1}^{\infty}(s_n-\sigma_n)ds \\ &< -(p-1)\varepsilon_0^q \sum_{n=1}^{\infty}\frac{\varepsilon_0}{\phi_q(B)} = -\infty, \end{aligned}$$

which is a contradiction. The lemma is proved. □

We are now able to prove Proposition 2.2

Proof of Proposition 2.2. We prove only the case that condition (1.11) is satisfied for $x > 0$ sufficiently small, because the other case is carried out in the same manner.

By (1.10) and (3.16), there exist an $\varepsilon_0 > 0$ and an $\varepsilon_1 > 0$ such that

$$H(\xi) \le \frac{1+\varepsilon_0}{2\gamma_p}(\xi+\gamma_p)^2 \quad \text{for} \quad -\gamma_p \le \xi \le -\left(\frac{p-1}{p} - \varepsilon_1\right)^{p-1} \tag{4.2}$$

and

$$(1+\varepsilon_0)\lambda < \frac{1}{2}\gamma_p\left(\frac{p-1}{p}-\varepsilon_1\right)^2.$$

Note that ε_1 depends on ε_0. Let

$$\delta = \frac{(1+\varepsilon_0)\lambda}{2\gamma_p\left((p-1)/p-\varepsilon_1\right)^2}. \tag{4.3}$$

Then $0 < \delta < 1/4$. Define $\eta(s) = -1/2 + z/s$, where $z = (1+\sqrt{1-4\delta})/2 > 0$. Then we see that $\eta(s)$ satisfies equation

$$\dot\eta = -\left(\eta + \frac{1}{2}\right)^2 - \frac{\delta}{s^2}. \tag{4.4}$$

We also see that there exists a $\tau > 0$ such that

$$\eta(\tau) = -\frac{1}{2} + \frac{(1+\varepsilon_0)}{2\gamma_p}\left(-\left(\frac{p-1}{p}-\varepsilon_1\right)^{p-1} + \gamma_p\right) \tag{4.5}$$

and

$$\eta(s) > -\frac{1}{2} \quad \text{for} \quad s \ge \tau. \tag{4.6}$$

Let ε_2 be a positive number satisfying

$$\log\varepsilon_2 < -\left(\frac{p-1}{p}-\varepsilon_1\right)\tau. \tag{4.7}$$

and (1.11) for $0 < x < \varepsilon_2$.

The proof is by contradiction. Suppose that (2.2) has an oscillatory solution $x(t)$. Then, from Lemma 4.1 we see that $x(t)$ tends to zero as $t \to \infty$. Let $u(s)$

be the solution of (3.12) corresponding to $x(t)$. Since $u(s)$ is also oscillatory and tends to zero as $s \to \infty$, there exist an s_1 and an s_2 such that

$$u(s_1) = u(s_2) = 0, \quad \dot{u}(s_1) > 0, \quad \dot{u}(s_2) < 0 \tag{4.8}$$

and

$$0 < u(s) < \varepsilon_2 \quad \text{for } \ s_1 < s < s_2. \tag{4.9}$$

Note that we may assume that

$$e^{ps} c(e^s) \le 1 \quad \text{for } \ s_1 < s < s_2 \tag{4.10}$$

by (2.4). Let

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} \quad \text{for } \ s_1 < s < s_2.$$

Then, by (4.8) we have

$$\lim_{s \to s_1 + 0} \xi(s) = \infty \quad \text{and} \quad \lim_{s \to s_2 - 0} \xi(s) = -\infty.$$

Since $\xi(s)$ is continuous on the bounded open interval (s_1, s_2), there exist an s_* and an s^* such that $s_1 < s_* < s^* < s_2$,

$$\xi(s_*) = -\left(\frac{p-1}{p} - \varepsilon_1\right)^{p-1}, \qquad \xi(s^*) = -\gamma_p \tag{4.11}$$

and

$$-\gamma_p \le \xi(s) \le -\left(\frac{p-1}{p} - \varepsilon_1\right)^{p-1} \quad \text{for } \ s_* \le s \le s^*. \tag{4.12}$$

Hence, we have

$$\frac{\dot{u}(s)}{u(s)} = \xi(s)^{1/(p-1)} \le -\left(\frac{p-1}{p} - \varepsilon_1\right) \quad \text{for } \ s_* \le s \le s^*.$$

Integrating both sides of this inequality from s_* to $s \le s^*$, and using (4.7) and (4.9), we obtain

$$\begin{aligned}
\log u(s) &\le -\left(\frac{p-1}{p} - \varepsilon_1\right)(s - s_*) + \log u(s_*) \\
&< -\left(\frac{p-1}{p} - \varepsilon_1\right)(s - s_*) + \log \varepsilon_2 \\
&< -\left(\frac{p-1}{p} - \varepsilon_1\right)(s - s_*) - \left(\frac{p-1}{p} - \varepsilon_1\right)\tau \\
&= -\left(\frac{p-1}{p} - \varepsilon_1\right)(s - s_* + \tau) \quad \text{for } \ s_* \le s \le s^*.
\end{aligned}$$

Hence, together with (1.11), (3.12), (4.2), (4.10) and (4.12), we have

$$\begin{aligned}
\dot{\xi}(s) &= \frac{(\phi_p(\dot{u}(s)))^{\cdot}}{\phi_p(u(s))} - (p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^p \\
&= \frac{-(p-1)\phi_p(\dot{u}(s)) - e^{ps}c(e^s)g(u(s))}{\phi_p(u(s))} - (p-1)\left|\frac{\dot{u}(s)}{u(s)}\right|^p \\
&\geq -(p-1)\left\{|\xi(s)|^{p/(p-1)} + \xi(s) + \frac{(p-1)^{p-1}}{p^p}\right\} - \frac{\lambda}{(\log u(s))^2} \\
&= -H(\xi(s)) - \frac{\lambda}{(\log u(s))^2} \\
&> -\frac{1+\varepsilon_0}{2\gamma_p}(\xi(s)+\gamma_p)^2 - \frac{\lambda}{((p-1)/p-\varepsilon_1)^2(s-s_*+\tau)^2} \quad \text{for } s_* \leq s \leq s^*.
\end{aligned}$$

Put

$$\zeta(s) = -\frac{1}{2} + \frac{1+\varepsilon_0}{2\gamma_p}(\xi(s+s_*-\tau)+\gamma_p) \quad \text{for } \tau \leq s \leq s^*-s_*+\tau. \tag{4.13}$$

Then, from (4.3) we have

$$\begin{aligned}
\dot{\zeta}(s) &= \frac{1+\varepsilon_0}{2\gamma_p}\dot{\xi}(s+s_*-\tau) \\
&> -\left(\frac{1+\varepsilon_0}{2\gamma_p}\right)^2(\xi(s+s_*-\tau)+\gamma_p)^2 - \frac{1+\varepsilon_0}{2\gamma_p}\frac{\lambda}{((p-1)/p-\varepsilon_1)^2 s^2} \\
&= -\left(\zeta(s)+\frac{1}{2}\right)^2 - \frac{\delta}{s^2} \quad \text{for } \tau \leq s \leq s^*-s_*+\tau. \qquad (4.14)
\end{aligned}$$

By (4.5) and (4.11), we have

$$\eta(\tau) = -\frac{1}{2} + \frac{1+\varepsilon_0}{2\gamma_p}(\xi(s_*)+\gamma_p) = \zeta(\tau).$$

Comparing equation (4.4) and inequality (4.14), we see that

$$\eta(s) \leq \zeta(s) \quad \text{for } \tau \leq s \leq s^*-s_*+\tau. \tag{4.15}$$

However, by (4.6), (4.11) and (4.13), we have

$$\eta(s^*-s_*+\tau) > -\frac{1}{2} \quad \text{and} \quad \zeta(s^*-s_*+\tau) = -\frac{1}{2}.$$

This is a contradiction to (4.15) with $s = s^*-s_*+\tau$. The proof is now complete. □

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