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FLUX IN AN INFINITE NETWORK

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Abstract. In an infinite network, we introduce the notion of flux which is an important concept in the study of many physical problems. The basic tool needed, namely the discrete version of the Green's formula, is proved here in a generalized form.

1. Introduction

Flux is a very useful concept in the study of many physical problems. In the context of classical potential theory, an equation like $\int_\omega \Delta u\,(x)\,dx = \int_{\partial\omega} \frac{\partial u}{\partial n} ds$ relates the flux to the mass associated with a potential function u. In this note, we study the discrete analogue of this equation in the framework of an infinite network.

2. Preliminaries

We start with the description of what we call here an infinite network, which differs very little from the definition of Yamasaki's network [7]. Let X be a countable set of nodes; a countable set of arcs joining some pairs of nodes is given; the resulting graph is assumed to be connected, locally finite and without any self-loops. Two nodes x and z are said to be neighbours, denoted by $x \sim z$, if there is an arc joining x and z. Assume that for each pair of distinct nodes x and a in X is associated a number $t\,(x,a) \geq 0$ such that $t\,(x,a) = t\,(a,x)$; $t\,(x,a) > 0$ if and only if $x \sim a$; we take $t\,(a,a) = 0$ for any node a. (see Yamasaki [7,p.34] for the construction of such a function $t\,(x,a)$). For any $x \in X$, write $t\,(x) = \sum_{z\in X} t\,(x,z)$. Note $t\,(x) > 0$ for any x in X.

Let u be a real-valued function on X. Then the Laplacian of u at a node x is

$$\begin{aligned} \Delta u\,(x) &= \sum_{z\in X} t\,(x,z)\,[u\,(z) - u\,(x)] \\ &= -t\,(x)\,u\,(x) + \sum_{z\in X} t\,(x,z)\,u\,(z)\,. \end{aligned}$$

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For any x, the union of x and all its neighbours is denoted by $X(x)$. Given a set of nodes E, the interior $\overset{\circ}{E} = \{x \in E : X(x) \subset E\}$, and the boundary $\partial E = E \backslash \overset{\circ}{E}$; let $E^* = \bigcup_{x \in E} X(x) = X(E)$.

Definition 2.1. *Let $u(x) > -\infty$ be a function defined on a subset E of X. u is said to superharmoinc on E if and only if $\Delta u(x) \leq 0$ at each $x \in \overset{\circ}{E}$; u is said to be harmoinc on E if and only if u and $-u$ are superharmoinc on E.*

2.1. **A remark on the Dirichlet problem in an infinite network.** In a reversible Markov chain, the hitting time $H(x, y)$ is defined as the expected number of steps to reach state y starting from state x. An expression for $H(x, y)$ is given in Tetali [6] in terms of effective resistance whose definition in a finite network N involves solving the Dirichlet problem in $N \backslash \{x, y\}$ with boundary values 1 at x and 0 at y; also Ponzio [5] gives an expression of the Green function of N in terms of effective resistances. The above mentioned Dirichlet problem in the context of infinite networks can be solved as follows.

Theorem 2.2. *Let E and F be two disjoint sets (finite or not) in an infinite network X. Let f be a bounded function defined on $E \cup F$. Then there exists a function h on X, such that $h = f$ on $E \cup F$ and h is harmonic on $X \backslash (E \cup F)$.*

Proof. Let $\alpha \leq f \leq \beta$ on $E \cup F$.

Let
$$u = \begin{cases} f & \text{on} \quad E \cup F \\ \beta & \text{on} \quad X \backslash (E \cup F), \end{cases}$$
and
$$v = \begin{cases} f & \text{on} \quad E \cup F \\ \alpha & \text{on} \quad X \backslash (E \cup F). \end{cases}$$
Then u and v are defined on X and $v \leq u$. Let z be an interior node of $X \backslash (E \cup F)$. If z has a neighbour on $E \cup F$, then u is superharmoinc at z; otherwise, u is harmonic at z. Hence, on $X \backslash (E \cup F)$, u is superharmoinc. Similarly, v is subharmoinc on $X \backslash (E \cup F)$.

Let J be the family of functions g on X, such that $v \leq g \leq u$ on X, and g is subharmoinc on $X \backslash (E \cup F)$. Take a function $g \in J$. Suppose z is an interior point of $X \backslash (E \cup F)$. Note $t(z) g(z) \leq \sum_{y \in X} t(z, y) g(y)$.

Define
$$g_1(x) = \begin{cases} g(x) & \text{if} \quad z \neq x \\ \\ \frac{1}{t(z)} \sum t(z, y) g(y) & \text{if} \quad z = x. \end{cases}$$
Then g_1 is harmonic at $x = z$, $g_1 \geq g$ and $g_1 \in J$. This procedure can be repeated for each $g \in J$ to obtain a corresponding $g_1 \in J$ as above. Consequently, $h = \sup_J g = \sup_J g_1$ is harmonic at the node z. Since z is an arbitrary interior point of $X \backslash (E \cup F)$, $h(x)$ is harmonic on $X \backslash (E \cup F)$; further, $h = f$ on $E \cup F$.

This completes the proof of the theorem □

The above method can be used to solve the Dirichlet problem in an infinte network.

Theorem 2.3. *Let D be a set of nodes (finite or not), such that ∂D is a finite set. Let f be a real-valued function on ∂D. Then there exists a bounded harmoinc function h on D, such that $h = f$ on ∂D.*

Proof. Extend f arbitrarily on $X\backslash D$. Then in the above Theorem 2.2, take $F = X\backslash \overset{\circ}{D}$ and $E = \phi$. The function h is the Dirichlet solution in D, with boundary values f. □

Remark

(1) To prove the uniqueness of the solution h, we may require some restriction on E, such as E is finite.
(2) Suppose ∂E is an infinite set in the above theorem. Then also the above proof is valid, provided f is bounded on ∂E.

Definition 2.4. *Let E be a subset of nodes in X. E is said to be admissible if and only if there are only a finite number of nodes in ∂E.*

3. Green's formula

Consider the discrete case of lattice points in $\mathbb{R}^3$, of the form (a, b, c) where a, b and c take on the values $0, \pm1, \pm2, \cdots$. For a real function u on these lattice points, define the Laplacian operator D:

$$\begin{aligned} Du\,(a,b,c) = u\,(a+1,b,c) + u\,(a-1,b,c) + u\,(a,b+1,c) + u\,(a,b-1,c) \\ + u\,(a,b,c+1) + u\,(a,b,c-1) - 6u\,(a,b,c)\,. \end{aligned}$$

Using the operator D, Duffin [3,Lamma 1] has obtained a form of the classical Green's formula

$$\iint\limits_{\omega} (f\Delta g - g\Delta f)\, d\sigma = \int\limits_{\partial\omega} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) ds$$

in this discrete situatuation.

A similar formula is given in Bendito et al. [1, Proposition 3.1] in the framework of finite network. First we shall give a generalized version of this, in the context of an infinite network.

Definition 3.1. *Let E be a set of nodes. Let $u\,(x)$ be a real function defined on E^*. Let $s \in \partial E$. Then the outer normal derivative of u at s with respect to E is*

$$\frac{\partial u}{\partial n^+}\,(s) = \sum_{x \in E^*\backslash E} t\,(s,x)\,[u\,(x) - u\,(s)]\,.$$

Remark. The normal derivative defined in Bendito et al. [1, section 3] is at the nodes $E^*\backslash E$. It is differnt from the above definition of outer normal derivative defined at the nodes in $E\backslash \overset{\circ}{E}$.

Definition 3.2. *Let u be a real function defined on E^*. The outward flux of u with respect to E is $\sum_{s\in\partial E} \frac{\partial u}{\partial n^+}\,(s)$, if the series converges absolutely. If $A = \{x\}$, the outward flux of u with respect to A is $\Delta u\,(x)$.*

Since for the outer normal derivative, u has to be defined at some nodes outside E also, it is preferable to work with inner normal derivatives defined below.

Definition 3.3. *Let u be a real function defined on E .The inner normal derivative of u with respect to E at a node $s \in \partial E$ is*

$$\frac{\partial u}{\partial n^-}(s) = \sum_{x \in E} t(s,x)\left[u(x) - u(s)\right].$$

Thus, if u is defined on E^* and if $s \in \partial E$, then

$$\frac{\partial u}{\partial n^+}(s) + \frac{\partial u}{\partial n^-}(s) = \Delta u(s).$$

Notation.

1) Let u be a real function defined on E .Suppose $\sum_{s \in \partial E} \frac{\partial u}{\partial n^-}(s)$ is absolutely convergent; write

$$Flux(u; E) = -\sum_{s \in \partial E} \frac{\partial u}{\partial n^-}(s).$$

2) Let u and v be two real functions defined on E .Suppose

$$\sum_{x \in E} \sum_{y \in E} t(x,y)\, u(x)\left[v(y) - v(x)\right]$$

is absolutely convergent. Then write

$$(u,v)_E = -\sum_{x \in E} \sum_{y \in E} t(x,y)\, u(x)\left[v(y) - v(x)\right].$$

Suppose $(u,v)_E$ is defined. Let x and y be two neighbours in E. Then, there are two terms, namely

$$t(x,y)\, u(x)\left[v(y) - v(x)\right] \text{ and } t(y,x)\, u(y)\left[v(x) - v(y)\right]$$

in the expression $(u,v)_E$; and the sum of these two terms is

$$-t(x,y)\left[u(y) - u(x)\right]\left[v(y) - v(x)\right].$$

Hence, the assumption on the absolute convergence of the series implies that if $(u,v)_E$ exists,

$$(u,v)_E = \sum_{x,y \in E} t(x,y)\left[u(y) - u(x)\right]\left[v(y) - v(x)\right].$$

Theorem 3.4. *Let E be an admissible set (that is, ∂E is finite). Let u and v be two real functions on E, such that $(u,v)_E$ is defined. Then,*

$$\sum_{x \in \overset{\circ}{E}} u(x)\, \Delta v(x) + (u,v)_E = -\sum_{s \in \partial E} u(s) \frac{\partial v}{\partial n^-}(s).$$

Proof. Extend u and v arbitrarily on E^*. Then, for $x \in E$,

$$\Delta v(x) = \sum_{y \in X} t(x,y)\left[v(y) - v(x)\right],$$

so that

$$\begin{aligned}\sum_{x\in E} u(x)\,\Delta v(x) &= \sum_{x\in E}\sum_{y\in X} t(x,y)\,u(x)\left[v(y)-v(x)\right]\\ &= \sum_{x\in E}\sum_{y\in E} t(x,y)\,u(x)\left[v(y)-v(x)\right]\\ &\quad+\sum_{x\in E}\sum_{y\notin E} t(x,y)\,u(x)\left[v(y)-v(x)\right].\end{aligned}$$

Note in the second sum on the right side, there are only a finite number of terms. For in that sum, $t(x,y) > 0$ if and only if $x \in \partial E$, which is a finite set. Hence,

$$\begin{aligned}\sum_{x\in E} u(x)\,\Delta v(x) &= -(u,v)_E + \sum_{s\in\partial E} u(s)\,\frac{\partial v}{\partial n^+}(s)\\ &= -(u,v)_E + \sum_{s\in\partial E} u(s)\left[\Delta v(s) - \frac{\partial v}{\partial n^-}(s)\right].\end{aligned}$$

Consequently,

$$\sum_{x\in \overset{\circ}{E}} u(x)\,\Delta v(x) = -(u,v)_E - \sum_{s\in\partial E} u(s)\,\frac{\partial v}{\partial n^-}(s).$$

□

Corollary 3.5. *Let E be an admissible set. Let v be a real function on E, such that $(1,v)_E$ is defined. Then,*

$$Flux(v;E) = \sum_{x\in \overset{\circ}{E}} \Delta v(x).$$

Proof. Since $(1,v)_e$ exists, using the remark preceding Theorem 3.4, we have $(1,v)_E = 0$. Now, take $u \equiv 1$ in the above theorem to obtain

$$Flux(v;E) = -\sum_{s\in\partial E}\frac{\partial v}{\partial n^-}(s) = \sum_{x\in \overset{\circ}{E}} \Delta v(x).$$

□

Proposition 3.6. *Let E be an admissible set. Let v be a superharmonic function on E, such that $(1,v)_E$ is defined. Then v is harmonic on E if and only if $Flux(v;E) = 0$.*

Proof. If v is harmonic, the assertion follows from the abovr corollary. Assume now $Flux(v;E) = 0$. Then by the Theorem 3.4, $\sum_{x\in \overset{\circ}{E}} \Delta v(x) = 0$. But v being superharmonic, $\Delta v(x) \le 0$ on $\overset{\circ}{E}$, so that $\Delta v \equiv 0$ on $\overset{\circ}{E}$. □

Proposition 3.7. *Let $f(x)$ be a real function on x, such that $f = 0$ outside a finite set. Then $\sum_X \Delta f(x) = 0$.*

Proof. Let $f(x) = 0$ outside A. Choose a finite set E, such that $A^* \subset \overset{\circ}{E}$. Since E is admissible, $(1, f)_E = 0$; and if $s \in \partial E$, $\frac{\partial f}{\partial n^-}(s) = 0$. Hence, if we take $u \equiv 1$ and $v = f$ in Theorem 3.4, we have $\sum_{x \in \overset{\circ}{E}} \Delta f(x) = 0$. This implies that $\sum_X \Delta f(x) = 0$. □

3.1. **Representation of harmonic functions.** Let e be a fixed node in X and let $d(e, x) = |x|$ denote the geodesic distance between e and x, that is the number of arcs in the shortest path from e to x. Let $E_m = \{x \in X, |x| \leq m\}$, m integer.

Proposition 3.8. *Let u be superharmonic on X. Then,*

$$\begin{aligned} Flux\,(u; E_m) &\geq Flux\,(u; E_{m+1})\,, \text{ and} \\ \lim_{m\to\infty} Flux\,(u; E_m) &= \sum_X \Delta u(x)\,. \end{aligned}$$

Proof. The assertion is clear by Corollary 3.5 and the fact that $\Delta u(x) \leq 0$ on X.

Let now $a \in \overset{\circ}{E}_m$. Then there exists (see Yamasaki [7,pp.36-37]) the Green function g_a^m of E_m with pole at a; that is, $\Delta g_a^m(x) = -\delta_a(x)$ on E_m, $g_a^m(x) = 0$ on ∂E_m and $g_a^m(x) > 0$ on $\overset{\circ}{E}_m$. In Kellogg [4] and Brelot [2] is given a representation theorem for harmonic functions in $\mathbb{R}^n$, $n \geq 2$. An analogue of this in the context of networks is the following. □

Proposition 3.9. *Let h be harmonic on E_m and let $a \in \overset{\circ}{E}_m$. Then,*

$$h(a) = \sum_{|s|=m} h(s) \frac{\partial g_a^m}{\partial n^-}(s)\,.$$

Proof. In Theorem 3.4, take $E = E_m$ which is an admissible set, $u = h$ and $v = g_a^m$ on E. Then $(u, v)_E$ and $(v, u)_E$ are well-defined and $(u, v)_E = (v, u)_E$. Hence, we have

$$\sum_{|x|<m} h(x)\,\Delta g_a^m(x) + (h, g_a^m)_E = -\sum_{|s|=m} h(s) \frac{\partial g_a^m}{\partial n^-}(s)\,,$$

and interchanging u and v,

$$\sum_{|x|<m} g_a^m(x)\,\Delta h(x) + (g_a^m, h)_E = -\sum_{|s|=m} g_a^m(s) \frac{\partial h}{\partial n^-}(s)\,.$$

Subtracting,

$$-h(a) = -\sum_{|s|=m} h(s) \frac{\partial g_a^m}{\partial n^-}(s)\,.$$

□

Corollary 3.10. *Let $f(x)$ be a finite function on ∂E_m. Then the Dirichlet solution in E_m with boundary values f is given by*

$$h(x) = \sum_{|s|=m} f(s) \frac{\partial g_x^m}{\partial n^-}(s).$$

3.2. **Flux at infinity.** Let u be a superharmonic function on X. By Proposition 3.8, we know that $Flux(u; E_m)$ is a decreasing sequence of numbers. Let us write $Flux_\infty u = \lim_{m\to\infty} Flux(u; E_m)$ and call it the flux at infinity of u. Suppose h is a harmonic function outside a finite set. If h is harmonic on $E = \{x : i \le |x| < j\}$, then by Theorem 3.4, $\sum_{\partial E} \frac{\partial h}{\partial n^-}(s) = 0$. This means that

$$A + \sum_{|s|=j} \frac{\partial h}{\partial n^-}(s) = 0,$$

where

$$A = \sum_{|x|=i} \sum_{|y|\ge i} t(x,y)\left[u(y) - u(x)\right].$$

Thus, if $h(x)$ is harmonic for $|x| \ge i$, we have for any $j > i$,

$$-\sum_{|s|=j} \frac{\partial h}{\partial n^-}(s) = A.$$

Let us denote

$$A = Flux_\infty h.$$

Thus, for any harmonic function h defined outside a finite set, $Flux_\infty h$ is always finite $= -\sum_{|s|=j} \frac{\partial h}{\partial n^-}(s)$ for any large j.

Since $Flux_\infty u = \sum_X \Delta u(x)$, it is clear that for a superharmonic function u which is harmonic outside a finite set in X, $Flux_\infty u$ is finite. More generally we have the following Theorem 3.12.

Lemma 3.11. *Let $\{u_i\}$ be a sequence of superharmonic functions on X, such that $u(x) = \lim_{i\to\infty} u_i(x)$ exists for any $x \in X$. Then, u is superharmonic on X and*

$$Flux_\infty u = \lim_{i\to\infty} Flux_\infty u_i.$$

Proof. Since, at any node x, $\Delta u(x) = \lim_{i\to\infty} \Delta u_i(x)$ (see Yamasaki [7, Lamma 2.3]) and since $\Delta u_i(x) \le 0$, we have $\Delta u(x) \le 0$; that is, u is superharmonic on X. Now

$$\begin{aligned} Flux_\infty u_i &= \lim_{m\to\infty} Flux(u_i; E_m) \\ &= \lim_{m\to\infty} \left[\sum_{x\in \overset{\circ}{E}_m} \Delta u_i(x) \right] \\ &= \lim_{m\to\infty} \alpha_{im}, \end{aligned}$$

where $\alpha_{im} = \sum_{x\in \overset{\circ}{E}} \Delta u_i(x)$ is negative and decreasing in m.

Hence,

$$\begin{aligned}\lim_{i\to\infty} Flux_{\infty}u_i &= \lim_{i\to\infty}\lim_{m\to\infty}\alpha_{im}\\ &= \lim_{m\to\infty}\lim_{i\to\infty}\alpha_{im}\\ &= \lim_{m\to\infty}\left(\sum_{x\in\overset{\circ}{E}_m}\Delta u\left(x\right)\right)\\ &= Flux_{\infty}u.\end{aligned}$$

□

Theorem 3.12. *Let u be a superharmonic function on X. Then $Flux_{\infty}u$ is finite if and only if u has a harmonic minorant outside a finite set.*

Proof. **i)** Let $h\left(x\right)$ be a harmonic minorant of $u\left(x\right)$ in $|x|\geq m$. Let h_i be Dirichlet solution in $m\leq|x|\leq m+i$ with boundary values u.

Define

$$u_i\left(x\right)=\begin{cases}u\left(x\right) & \text{if } \ |x|<m \text{ or } \ |x|>m+i\\ h_i\left(x\right) & \text{if } \ m\leq|x|\leq m+i.\end{cases}$$

Then, $u_i\left(x\right)$ is superharmonic on X and since $u_i=u$ outside a finite set, $Flux_{\infty}u_i=Flux_{\infty}u$.

Let $H\left(x\right)$ be the greatest harmonic minorant of $u\left(x\right)$ in $|x|\geq m$. (That is, $H\left(x\right)=\sup_{v\in J}v\left(x\right)$, where J is the family of functions defined on $|x|\geq m$, majorized by u and subharmonic on $|x|>m$.) Note that $\lim_{i\to\infty}u_i\left(x\right)=H\left(x\right)$ when $|x|>m$, and $u_i\left(x\right)=u\left(x\right)$ if $|x|\leq m$.

Let $v\left(x\right)=\lim u_i\left(x\right)$. Since each u_i superharmonic on X, v is superharmonic on X and $v\left(x\right)=H\left(x\right)$ when $|x|>m$. Since v is a superharmonic function that is harmonic outside a finite set, $Flux_{\infty}v$ is finite and is equal to $Flux_{\infty}H$. Thus,

$$Flux_{\infty}H=Flux_{\infty}v=\lim_i Flux_{\infty}u_i=Flux_{\infty}u.$$

ii) Conversely. Suppose $Flux_{\infty}u$ is finite. Let h_m be the Dirichlet solution with boundary values u on $|x|=m+1$ and at e. Clearly, $\frac{\partial u}{\partial n^-}\geq\frac{\partial h_m}{\partial n^-}$ on $|x|=m+1$. Hence,

$$\begin{aligned}-\infty<Flux_{\infty}u &= -\lim_{m\to\infty}\sum_{|s|=m+1}\frac{\partial u}{\partial n^-}\left(s\right)\\ &\leq -\lim_{m\to\infty}\sum_{|s|=m+1}\frac{\partial h_m}{\partial n^-}\left(s\right)\\ &= -\lim_{m\to\infty}\left[-\Delta h_m\left(e\right)\right]\\ &= \lim_{m\to\infty}\left[\Delta h_m\left(e\right)\right] \qquad (1)\end{aligned}$$

Since h_m is a decreasing sequence, let $v=\lim h_m$. Then (h_m being superharmonic on $|x|<m+1$), $v\equiv-\infty$ or v is harmonic outside e and superharmonic on X.

Suppose $v \equiv -\infty$. Then, given $R > 0$, there exists m such that $h_m(x_0) < -R$ for some $x_0 \sim e$. However, note $h_m(e) = u(e)$ for all m. Hence,

$$\begin{aligned}
\Delta h_m(e) &= \sum_X t(e,x)[h_m(x) - h_m(e)] \\
&= \sum_{x \neq x_0} t(e,x)[h_m(x) - h_m(e)] + t(e,x_0)[h_m(x_0) - h_m(e)] \\
&< \sum_{x \neq x_0} t(e,x)[h_1(x) - u(e)] + t(e,x_0)[-R - u(e)] \\
&= -R\, t(e,x_0) + (\text{a constant independent of } m).
\end{aligned}$$

This means that $\lim_{m\to\infty}[\Delta h_m(e)] = -\infty$, contradicting (1). Hence v is superharmonic on X. Moreover, outside e, v is harmonic and $v \leq u$. Hence u has a harmonic minorant outside e. □

3.3. **Balayage.** In the context of finite networks, Bendito et al. [1] find expression for Green functions associated with various boundary value problems by means of equilibrium measures, which are solutions of suitable equilibrium problems. In the classical potential theory, the connection between equilibrium measures, capacity distributions and balayage with respect to compact sets is well-described in Brelot [2,pp. 118-119]. In this paragraph, we obtain the properties of balayage in the context of an infinite network.

Theorem 3.13. *Let u be a superharmonic function on X, such that $Flux_\infty u$ is finite. Let E be any proper subset of X. Then there exists a superharmonic function B_u^E on X, such that*

i) $B_u^E \leq u$ *on* X,

ii) $B_u^E = u$ *on* $X \backslash \overset{\circ}{E}$,

iii) B_u^E *is harmonic on* E, *and*

iv) $Flux_\infty B_u^E = Flux_\infty u$.

Proof. Fix a node e in $X \backslash E$. Since $Flux_\infty u$ is finite, we can construct h the greatest harmonic minorant of u outside e (as in the proof of the converse part in the above Theorem 3.12). Let J_1 be the family of subharmonic functions t on E, such that $t \leq u$ on E. Let $H = \sup_{J_1} t$. Since $h \in J_1$, H is harmonic on $\overset{\circ}{E}$; thus H is the greatest harmonic minorant of u on E. Hence $h \leq H \leq u$ on E.

i) and **ii)** Let J be the family of superharmonic functions v on X, such that $v \geq u$ on $X \backslash \overset{\circ}{E}$ and $v \geq H$ on $\overset{\circ}{E}$. Define $B_u^E = \inf_{v \in J} v$. Then B_u^E is superharmonic on X, $B_u^E \leq u$, and $B_u^E = u$ on $X \backslash \overset{\circ}{E}$.

iii) Suppose $v \in J$ and $a \in \overset{\circ}{E}$. Then,

$$\begin{aligned}
t(a)\,v(a) &\geq \sum t(a,y)\,v(y) \\
&\geq \sum t(a,y)\,H(y) \\
&= t(a)\,H(a).
\end{aligned}$$

Let

$$v_1(x) = \begin{cases} v(x) & \text{if} \quad x \neq a \\ \frac{1}{t(a)} \sum t(a,y)\, v(y) & \text{if} \quad x = a. \end{cases}$$

Then $v_1 \leq v$ on X, v_1 is superharmonic on X and v_1 is harmonic at $x = a$. Note also $v_1(a) \geq H(a)$; and if $x \in \overset{\circ}{E}$, $x \neq a$, then $v_1(x) = v(x) \geq H(x)$; moreover on $X \backslash \overset{\circ}{E}$, $v_1 = v \geq u$. Hence $v_1 \in J$.

This argument with respect to each $v \in J$ leads to the equality $B_u^E = \inf v = \inf v_1$, so that B_u^E is harmonic at $a \in \overset{\circ}{E}$. Similar arguments with respect to each node in $\overset{\circ}{E}$ shows that B_u^E is harmonic on $\overset{\circ}{E}$.

iv) Since $B_u^E \leq u$ on X and B_u^E is harmonic on $\overset{\circ}{E}$, $B_u^E \leq H$ on E (since H is the greatest harmonic minorant of u on E). But by the construction of B_u^E, we have $B_u^E \geq H$ on $\overset{\circ}{E}$, and on ∂E, $B_u^E = u \geq H$ and hence $B_u^E \geq H$ on E. Consequently $B_u^E = H$ on E.

Recall that h is the greatest harmonic minorant of u in $X \backslash e$, and $h \leq H \leq u$ on E, so that $h \leq B_u^E$ on $X \backslash e$. Hence, if h_1 is the greatest harmonic minorant of B_u^E outside e, then $h \leq h_1 \leq B_u^E \leq u$ on $X \backslash e$. This means that $h = h_1$. (Now recall that in the proof (i) of Theorem 3.12, we have proved that if u is superharmonic on X and if H is its greatest harmonic minorant outside a finite set, then $Flux_\infty H = Flux_\infty u$.) consequently,

$$Flux_\infty B_u^E = Flux_\infty h_1 = Flux_\infty h = Flux_\infty u.$$

This completes the proof of the theorem. □

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