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REAL HYPERSURFACES OF COMPLEX SPACE FORMS WITH SYMMETRIC RICCI $*$-TENSOR

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Abstract. Real hypersurfaces M's in non-flat complex space forms such that the symmetric part of the Ricci $*$-tensor of M is a constant multiple of the metric are classified.

1. Introduction

This note is a continuation of our previous paper [1].

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold with Ricci tensor S. The Ricci $*$-tensor S^* is defined by

$$S^*(X,Y) = \frac{1}{2}\mathrm{trace}(Z \mapsto R(X, \phi Y)\phi Z), \;\; X, Y \in TM.$$

An almost contact metric manifold is said to be $*$-Einstein if S^* is a constant multiple of the metric g on the holomorphic distribution $T^\circ M$.

It should be remarked that Ricci $*$-tensor is *not* symmetric, in general. Thus the condition "$*$-Einstein" automatically requires a symmetric property of the Ricci $*$-tensor.

On real hypersurfaces in almost Hermitian manifolds, almost contact structures are naturally induced from the almost Hermitian structure of the ambient space. In our previous paper [1], the first named author investigated real hypersurfaces in non-flat complex space forms in terms of Ricci $*$-tensor. In particular, he classified $*$-Einstein real hypersurfaces in non-flat complex space forms whose structure vector fields are principal.

The purpose of present note is to generalize the classification result of [1]. We shall weaken the assumption "$*$-Einstein" to "the symmetric part of S^* is a constant multiple of g on $T^\circ M$". More precisely, we shall prove the following two results.

Theorem 1.1. *Let M be a connected real hypersurface of $P_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c > 0$. Assume that the symmetric part* $\mathrm{Sym}S^*$ *of*

Key words and phrases. real hypersurface, complex space form, Ricci $*$-tensor, $*$-Einstein.

Ricci $*$-tensor of M is a constant multiple of the induced metric over the holomorphic distribution and the structure vector field ξ is a principal curvature vector. Then M is an open subset of one of the following:

(i) *a geodesic hypersphere of radius r $(0 < r < \pi/(2\sqrt{c}))$,*
(ii) *a tube over a totally geodesic complex projective space $P_k(\mathbf{C})$ of radius $\pi/(4\sqrt{c})$, where $0 < k < n-1$,*
(iii) *a tube over a complex quadric Q_{n-1} of radius r $(0 < r < \pi/(4\sqrt{c}))$.*

Theorem 1.2. *Let M be a connected real hypersurface of $H_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c < 0$. Assume that the symmetric part $\mathrm{Sym}S^*$ of Ricci $*$-tensor of M is a constant multiple of the induced metric over the holomorphic distribution and the structure vector field ξ is a principal curvature vector. Then M is an open subset of one of the following:*

(i) *a geodesic hypersphere of radius r $(0 < r < \infty)$,*
(ii) *a tube over a totally geodesic complex hyperbolic hyperplane of radius r $(0 < r < \infty)$,*
(iii) *a tube over a totally real hyperbolic space $H^n(\mathbf{R})$ of radius r $(0 < r < \infty)$,*
(iv) *a horosphere.*

2. Preliminaries

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature $4c$ is called a complex space form, which is denoted by $\widetilde{M}_n(4c)$. A complete and simply connected complex space form is a *complex projective space* $P_n(\mathbf{C})$, a *complex Euclidean space* $\mathbf{C}^n$ or a *complex hyperbolic space* $H_n(\mathbf{C})$, according as $c > 0, c = 0$ or $c < 0$. Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(4c)$.

Take a local unit normal vector filed N of M in $\widetilde{M}_n(4c)$. Then the Riemannain connections $\tilde{\nabla}$ of $\widetilde{M}_n(4c)$ and ∇ of M are related by the following *Gauss formula* and *Weingarten formula*:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad X, Y \in \mathfrak{X}(M),$$
$$\widetilde{\nabla}_X N = -AX, \ X \in \mathfrak{X}(M).$$

Here g is the Riemannian metric of M induced by the Kähler metric G of the ambient space $\widetilde{M}_n(4c)$. The $(1,1)$-tensor field A is called the *shape operator* of M derived from N.

An eigenvector X of the shape operator A is called a *principal curvature vector*. The corresponding eigenvalue λ of A is called a *principal curvature*. As is well known, the Kähler structure (J, G) of the ambient space induces an almost contact metric structure (ϕ, ξ, η, g) on M. In fact, the *structure vector field* ξ and its dual 1-form η are defined by

$$\eta(X) = g(\xi, X) = G(JX, N), \quad X \in TM.$$

The $(1,1)$-tensor field ϕ is defined by

$$g(\phi X, Y) = G(JX, Y), \quad X, Y \in TM.$$

One can easily check that this structure (ϕ, ξ, η, g) is an almost contact structure on M, that is, it satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0. \tag{1}$$

It follows that

$$\nabla_X \xi = \phi AX.$$

Let $\widetilde{R}$ and R be the Riemannian curvature tensors of $\widetilde{M}_n(4c)$ and M, respectively. ¿From the expression of the curvature tensor $\widetilde{R}$ of $\widetilde{M}_n(4c)$, we have the following *equations of Gauss and Codazzi*:

$$\begin{aligned} R(X,Y)Z &= c(g(Y,Z)X - g(X,Z)Y \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi).$$

To close this section, we recall the following two fundamental results (See *e.g.*, [2]).

Lemma 2.1. *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Lemma 2.2. *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = \lambda X$ for $X \perp \xi$, then we have $(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2c)\phi X$.*

We refer to the reader [2] about general theory of differential geometry of real hypersurfaces in complex space forms.

3. $*$-Einstein real hypersurfaces

Let us denote by S^* the *Ricci $*$-tensor* of a real hypersurface M which is defined by

$$S^*(X,Y) = \frac{1}{2}\text{trace}(Z \mapsto R(X, \phi Y)\phi Z).$$

Then the Gauss equation implies that

$$S^*(X,Y) = 2cn(g(X,Y) - \eta(X)\eta(Y)) - g(\phi A \phi AX, Y), \tag{2}$$

for all $X, Y \in TM$.

The *Ricci $*$-operator* Q^* is the linear endomorphism field associated to S^*;

$$S^*(X,Y) = g(Q^*X, Y), \quad X, Y \in TM.$$

The trace ρ^* of Q^* is called the *$*$-scalar curvature* of M.

Let $T^\circ M$ be a distribution defined by a subspace

$$T_x^\circ M = \{X \in T_x M : X \perp \xi_x\}$$

in the tangent space $T_x M$. The formulas (1) imply that the distribution $T^\circ M$ is invariant under ϕ. The distribution $T^\circ M$ is called the *holomorphic distribution* of

M. If the Ricci $*$-tensor is a constant multiple of the Riemannian metric for the holomorphic distribution, *i.e.*

$$S^*(X,Y) = \frac{\rho^*}{2(n-1)} g(X,Y)$$

for $X, Y \in T^\circ M$ on M, then M is said to be a $*$-*Einstein* real hypersurface.

The first author proved the following results in [1].

Proposition 3.1. *Let M be a connected $*$-Einstein real hypersurface of $P_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c > 0$, whose structure vector field ξ is a principal curvature vector. Then M is an open subset of one of the following:*

- (i) *a geodesic hypersphere of radius r $(0 < r < \pi/(2\sqrt{c}))$,*
- (ii) *a tube over a totally geodesic complex projective space $P_k(\mathbf{C})$ of radius $\pi/(4\sqrt{c})$, where $0 < k < n-1$,*
- (iii) *a tube over a complex quadric Q_{n-1} of radius r $(0 < r < \pi/(4\sqrt{c}))$.*

Proposition 3.2. *Let M be a connected $*$-Einstein real hypersurface of $H_n(\mathbf{C})$ of constant holomorphic sectional curvature $4c < 0$, whose structure vector field ξ is a principal curvature vector. Then M is an open subset of one of the following:*

- (i) *a geodesic hypersphere of radius r $(0 < r < \infty)$,*
- (ii) *a tube over a totally geodesic complex hyperbolic hyperplane of radius r $(0 < r < \infty)$,*
- (iii) *a tube over a totally real hyperbolic space $H^n(\mathbf{R})$ of radius r $(0 < r < \infty)$,*
- (iv) *a horosphere.*

Now we take the symmetric part $\mathrm{Sym}S^*$ and the alternate part $\mathrm{Alt}S^*$ of Ricci $*$-tensor S^* of M;

$$\mathrm{Sym}S^*(X,Y) = \frac{1}{2}(S^*(X,Y) + S^*(Y,X)),$$
$$\mathrm{Alt}S^*(X,Y) = \frac{1}{2}(S^*(X,Y) - S^*(Y,X)),$$

for any $X, Y \in TM$.

Using (2), we see that

$$\begin{aligned} \mathrm{Sym}S^*(X,Y) = {} & 2cn(g(X,Y) - \eta(X)\eta(Y)) \\ & - \frac{1}{2} g((\phi A \phi A + A\phi A \phi)X, Y), \end{aligned} \tag{3}$$

$$\mathrm{Alt}S^*(X,Y) = \frac{1}{2} g((A\phi A\phi - \phi A \phi A)X, Y). \tag{4}$$

4. Proof of main theorems

To prove our theorems, we need the following lemma.

Lemma 4.1. *Let M be a real hypersurface of a non-flat complex space form $\widetilde{M}_n(4c)$. If ξ is a principal curvature vector, then the Ricci $*$-tensor of M is symmetric, i.e. $\mathrm{Alt}S^* = 0$.*

Proof. Let X be a unit principal curvature vector orthogonal to ξ with principal curvature λ. From Lemma 2.2, the tangent vector ϕX is also a principal curvature vector. By calculating (4), we get $\mathrm{Alt}S^*(X,Y) = 0$, for any $X \in T^\circ M$ and $Y \in TM$.

On the other hand, by the assumption, we have $\phi A \phi A \xi = 0$ and (1) shows $A\phi A\phi\xi = 0$. Thus, we get $AltS^*(\xi, Y) = 0$ for any $Y \in TM$. □

Proof of theorems. Now let M be a real hypersurface in $\widetilde{M}_n(4c)$ with $c \neq 0$ whose $\mathrm{Sym}S^*$ is a constant multiple of g over $T^\circ M$. Assume that the structure vector field ξ is principal. Then Lemma 4.1 implies that $S^*(X,Y) = \mathrm{Sym}S^*(X,Y)$ for $X, Y \in TM$. Hence M is $*$-Einstein. This fact, together with Propositions 3.1 and 3.2, yields the required results. □

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