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SURFACES WHICH CONTAIN HELICAL GEODESICS IN THE 3-SPHERE

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Introduction

A *helical curve* (or a *helix*) is a curve in 3-dimensional space form $\mathcal{M}^3(c)$ of constant curvature c whose both curvature and torsion are constants. It reduces to a *Riemannian circle* or a *geodesic*, if its curvature is constant and torsion is zero, or if its curvature is zero, respectively. A helical curve is said to be a *proper helix* if both curvaure and torsion are non zero constants.

As is well known, circular cylinders in Euclidean 3-space E^3 contain these curves as geodesics.

On the other hand, although a helicoid in E^3 contains ordinary helices, they are not geodecics. Furthermore, (meridian) circles on a surface of revolution in E^3 are not always geodesics. Based on these facts, we mean by a *helical geodesic* on a surface M in $\mathcal{M}^3(c)$ a curve which is helical as a curve in $\mathcal{M}^3(c)$ and a geodesic as a curve on M.

In our previous paper [12], we have shown that complete surfaces of constant mean curvature in E^3 on which there exist two helical geodesics through each point are planes, spheres or circular cylinders.

In this paper we generalize this characterization obtained in [12] to Riemannian space forms of non-negative curvature. More precisely we show the following result for surfaces in the 3-sphere. We assume that all surfaces in $\mathcal{M}^3(c)$ are smooth and connected in this paper.

Theorem *Let M be a complete surface of constant mean curvature in the 3-sphere S^3. If there exist two helical geodesics on M through each point of M, then M is either a great sphere, a small sphere, or a Hopf torus over a circle.*

1. Preliminaries

Throughout this paper, we denote by $\mathcal{M}^3(c)$ the simply connected 3-dimensional Riemannian space form of constant curvature c with metric $\langle \cdot, \cdot \rangle$.

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Without loss of generality, we may choose $c = 0, \pm 1$. Namely $\mathcal{M}^3(0) = E^3$ (Euclidean 3-space), $\mathcal{M}^3(1) = S^3$ (unit 3-sphere) or $\mathcal{M}^3(-1) = H^3$ (unit hyperbolic 3-space).

Let M be surface in space form $\mathcal{M}^3(c)$. Let $\mathfrak{X}(M)$ be the Lie algebra of all smooth tangent vector fields to M. Further, let D be the Levi-Civita connection of $\mathcal{M}^3(c)$, and let ∇ be the Levi-Civita connection of M with the metric induced by $\langle \cdot, \cdot \rangle$. Let ξ be a unit normal vector field to M. Then the *second fundamental form* $\mathbb{I}$ of M derived from ξ is defined by the *Gauss formula*:

$$\mathbb{I}(X, Y)\xi = D_X Y - \nabla_X Y \tag{1}$$

for all $X,\ Y \in \mathfrak{X}(M)$. The *shape operator* S of M derived from ξ is a (1,1)-tensor field on M given by $\mathbb{I}(X, Y) = \langle S(X), Y \rangle$ for all $X,\ Y \in \mathfrak{X}(M)$. It is well known that $D_X \xi = -S(X)$ for all $X \in \mathfrak{X}(M)$.

The shape operator S satisfies the *Codazzi equation*:

$$(\nabla_V S)W = (\nabla_W S)V \tag{2}$$

for all vector fields V and W on M.

The Gaussian curvature K and the mean curvature H are computed by the formulas:

$$K = c + \det S,\ \ H = \frac{1}{2} \operatorname{tr} S.$$

The determinant $\det S$ of S is called the *Gauss-Kronecker curvatue* of M in $\mathcal{M}^3(c)$ and denoted by K_e.

Let γ be a helical curve in $\mathcal{M}^3(c)$ parametrized by the arc length. Then, by the *Frenet-Serret formula*, there exist unit vector fields $X,\ Y$ along γ and constants κ, τ such that

$$\begin{aligned} D_{\gamma'}\gamma' &= \kappa X, \\ D_{\gamma'}X &= -\kappa\gamma' + \tau Y, \\ D_{\gamma'}Y &= -\tau X, \end{aligned}$$

where γ' denotes the unit tangent vector field of γ. A helical curve with non-zero curvature and zero torsion is called a *Riemannian circle*. A helical curve is said to be *proper* if both κ and τ are non-zero.

Example 1. *(Helices in S^3) Let S^3 be the unit 3-sphere imbedded in the Euclidean 4-space E^4. A model helix in $S^3 \subset E^4$ is given by*

$$\gamma(s) = (\cos\phi\cos(as), \cos\phi\sin(as), \sin\phi\cos(bs), \sin\phi\sin(bs)),$$

with

$$a^2\cos^2\phi + b^2\sin^2\phi = 1.$$

Here s is the arclength parameter. It is easy to see that γ lies in the flat torus:

$$x_1^2 + x_2^2 = \cos^2\phi,\ \ x_3^2 + x_4^2 = \sin^2\phi.$$

Note that this flat torus has constant mean curvature $H = \cot(2\phi)$. The curvature κ and torsion τ are given by

$$\kappa = \sqrt{(a^2-1)(1-b^2)},\ \ \tau = ab.$$

Every proper helix in S^3 is congruent to one of these helices.

The following lemma due to Liouville (see *e.g.*, p.291 in Spivak [11]) plays basic role in our proof of Theorems 1 and 2.

Lemma 1.1. *Let M be a Riemannian 2-manifold. If two families of geodesics intersect at a constant angle everywhere on M, then M is flat.*

To close this section, here we recall the classification of *isoparametric surfaces* (surfaces with constant principal curvatures) in $\mathcal{M}^3(c)$ with $c \geq 0$ and flat surfaces in E^3.

Proposition 1.1. ([2], [5], [8]) *Let M be a complete flat surface in Euclidean 3-space E^3. Then M is a cylinder over a plane curve.*

Let us denote by $\pi : S^3 \to S^2(4)$ be the Hopf fibering of S^3 onto the 2-sphere of curvature 4 and let $\bar{\gamma}$ be a curve in $S^2(4)$ with curvature $\bar{\kappa}$. Then the inverse image $M = \pi^{-1}\{\gamma\}$ is a flat surface in S^3. This flat surface has mean curvature $H = (\bar{\kappa} \circ \pi)/2$ and called the *Hopf cylinder* over $\bar{\gamma}$. In particular, if $\bar{\gamma}$ is closed, then M is diffeomorphic to torus and called the *Hopf torus* over $\bar{\gamma}$ (H. B. Lawson. See Pinkall [7]). The Hopf cylinder over a geodesic in $S^2(4)$ is the Clifford (minimal) torus. Flat tori in S^3 are classified by Kitagawa [3].

Proposition 1.2. ([4]) *Let M be an isoparametric surface in E^3. Then M is either a* (*totally geodesic*) *plane, a* (*totally umbilical*) *sphere or a circular cylinder.*

Proposition 1.3. (*cf.* [1]) *Let M be an isoparametric surface in S^3. Then M is either a totally geodesic 2-sphere, or a totally umbilical 2-sphere or a Hopf tori over circles.*

2. Proof of Theorem

To prove Theorem, we give the following two results.

Theorem 2.1. *Let M be a complete surface of constant mean curvature in space form $\mathcal{M}^3(c)$. If M has no umbilic points, and there exists a helical geodesic on M through each point of M whose curvature* (*as a curve in $\mathcal{M}^3(c)$*) *is never zero, then M is a "circular cylinder".*

Here by a "circular cylinder" in $\mathcal{M}^3(c)$, $c \neq 0$, we mean a Hopf cylinder (torus) over a circle in S^3, and tubes (equidistant surface) around geodesics in H^3. Note that Theorem 2.1 holds for negative c.

Lemma 2.1. *Let M be a surface of constant mean curvature in $\mathcal{M}^3(c)$ and U be an open set in M. Assume that there exist two families of asymptotic curves on U all of which are geodesics in the ambient space. Then U is totally geodesic or congruent to an open portion of a circular cylinder in E^3 or a Hopf torus over a circle in S^3.*

Proof. If U is totally geodesic then U admits two families of asymptotic curves which are ambient geodesics.

Thus without loss of generality, we may restrict our attention to the case U is non totally geodesic.

Let α_1 and α_2 be the asymptotic curves on U through a point $p \in U$ which are geodesics as a curve in $\mathcal{M}^3(c)$.

Let λ and $2H - \lambda$ be the principal curvatures of M with corresponding principal vector fields E_1 and E_2. Here H is the mean curvature of M which is constant by our assumption. Further let θ be the angle between E_1 and α_1' so that

$$\begin{aligned} \alpha_1' &= \cos\theta E_1 + \sin\theta E_2, \\ \alpha_2' &= -\cos\theta E_1 + \sin\theta E_2, \end{aligned}$$

where α_1' and α_2' denote the unit tangent vector fields of α_1 and α_2, respectively. Put $\nabla_{E_1}E_1 = \alpha E_2$ and $\nabla_{E_2}E_1 = \beta E_2$, then $\nabla_{E_1}E_2 = -\alpha E_1$ and $\nabla_{E_2}E_2 = -\beta E_1$. Then the Codazzi equation (2) implies

$$\nabla_{E_1}\lambda = -2\beta(\lambda - H), \quad \nabla_{E_2}\lambda = 2\alpha(\lambda - H). \tag{3}$$

Since both asymptotic curves α_1 α_2 are geodesics in M,

$$\nabla_{\alpha_1'}\theta + \alpha\cos\theta + \beta\sin\theta = 0, \tag{4}$$

$$\nabla_{\alpha_2'}\theta + \alpha\cos\theta - \beta\sin\theta = 0. \tag{5}$$

Since, α_1 and α_2 are asymptotic curves and $\mathrm{II}(E_1, E_1) = \lambda$, $\mathrm{II}(E_2, E_2) = 2H - \lambda$,

$$\lambda\cos^2\theta + (2H - \lambda)\sin^2\theta = 0. \tag{6}$$

Differentiating (6) with respect to α_1' and α_2', and using (4) and (5), respectively,

$$\begin{aligned} \alpha\sin\theta(3\cos^2\theta - \sin^2\theta) - \beta\cos\theta(\cos^2\theta - 3\sin^2\theta) = 0, \\ \alpha\sin\theta(3\cos^2\theta - \sin^2\theta) + \beta\cos\theta(\cos^2\theta - 3\sin^2\theta) = 0. \end{aligned}$$

Hence α or β is zero, $3\cos^2\theta = \sin^2\theta$, or $\cos^2\theta = 3\sin^2\theta$. The equations (3) and (6) imply that curvature lines are geodesics on U, since U is not totally geodesic. As is well known, two families of curvature lines intersect at a constant angle $\pi/2$. Therefore by Lemma 1.1, U is flat. Thus $\det S = -c$. On the other hand, since U admits two family of asymptotic curves, $\det S \leq 0$. Hence $c \geq 0$. Moreover (3) implies that U has constant principal curvatures. □

Theorem 2.2. *Let M be a complete surface of constant mean curvature in $\mathcal{M}^3(c)$ of non-negative curvature. If there exist two helical geodesics on M through each point of M, then M is either a totally geodesic surface, a totally umbilical surface, a circular cylinder ($c = 0$) or a Hopf torurs ($c > 0$).*

Proof. The case $\mathcal{M}^3(c) = E^3$ is proved in [12]. It suffices to consider the case $\mathcal{M}^3(c) = S^3$.

If γ is a helical geodesic of M, then the following three cases will be occurred,

CASE 1. $\kappa \neq 0$ and $\tau \neq 0$. In this case we can take $X = \xi$ in the Frenet-Serret formula, because $D_{\gamma'}\gamma' = \mathrm{II}(\gamma', \gamma')\xi$ is normal to M. Then

$$D_{\gamma'}\xi = -\kappa\gamma' + \tau Y \quad \text{and} \quad D_{\gamma'}Y = -\tau\,\xi$$

CASE 2. $\kappa \neq 0$ and $\tau = 0$. Also we can take $X = \xi$ in the Frenet-Serret formula. Then

$$D_{\gamma'}\xi = -\kappa\gamma'.$$

CASE 3. $\kappa = 0$. By the Gauss formula (1) and the Frenet-Serret formula,

$$\mathrm{I\!I}(\gamma', \gamma') = 0.$$

Now, let γ_1 and γ_2 be helical geodesics on M through a point $p \in M$. Then, by Cases 1-3, we have following possibilities:

(i) γ_1 and γ_2 are ambient geodesics,
(ii) γ_1 and γ_2 are Riemannian circles,
(iii) γ_1 and γ_2 are proper helices,
(iv) γ_1 is an ambient geodesic and γ_2 is a Riemannian circle,
(v) γ_1 is an ambient geodesic and γ_2 is a proper helix,
(vi) γ_1 is a Riemannian circle and γ_2 is a proper helix.

Firstly suppose that the Gauss-Kronecker curvature K_e is positive at least one point, and put

$$M_1 = \{p \in M \mid K_e(p) > 0\}.$$

Then each point of M_1 is a point of types (ii), (iii) or (vi). Let M_{11} be the set of all umbilic points of M_1 and $M_{12} = M_1 - M_{11}$. If $M_{12} \neq \emptyset$, then $K = 0$ on M_{12} by Theorem 2.1. This contradicts $K_e > 0$ on M_{12}, so M_1 is totally umbilic. Therefore, M is a totally umbilic surface since M_1 is open and closed.

Secondly suppose that $K_e \leq 0$ on M and put

$$M_2 = \{p \in M \mid K_e(p) < 0\}.$$

Then each point of M_2 can be a point of all types (i)–(vi). Put

$$M_{21} = \{p \in M_2 \mid \text{there exists a circle or a proper helix through } p\}$$

and $M_{22} = M_2 - M_{21}$. Then $M_2 = M_{21} \cup M_{22}$ and it is easily seen that $M_2 = \mathrm{Cl}\, M_{21} \cup \mathrm{Int}\, M_{22}$ and $M_2 = \mathrm{Cl}\, M_{22} \cup \mathrm{Int}\, M_{21}$. Here, for a set A, $\mathrm{Cl}\, A$ is the closure of A, and $\mathrm{Int}\, A$ denotes the interior of A. Hence M_{21} or M_{22} has interior points; or else M_{21} or M_{22} is dense in M_2. Now we show that M_2 is flat. If $\mathrm{Int} M_{21} \neq \emptyset$ or M_{21} is dense in M_2, then M_2 is flat by Theorem 2.1. Next, if $\mathrm{Int}\, M_{22} \neq \emptyset$ and M_{22} is dense in M_2. Then all asymptotic curves on M_2 are ambient geodesics. Hence, by Lemma 2.1, $K = 0$ on M_2. Thus $M_2 = \{p \in M \mid K_e(p) = -1\}$. Hece M_2 is closed. This implies that M is flat since M_2 is open and closed.

Therefore M is a Hopf torus over a circle. Because M is complete flat and isoparametric.

This completes the proof of Theorem 2.2. □

APPENDIX

Theorem 2.1 can be proved in much the same way in our previous paper [12]. For completeness and reader's convinence (making the paper to be selfcontained), we give the proof.

First we recall the classification of complete flat surfaces and isoparametric surfaces in H^3.

Proposition A.1 ([10],[13]) *Let M be a complete flat surface in hyperbolic 3-space H^3. Then M is either a (totally umbilical) horosphere or an equidistant tube of a geodesic in H^3.*

Proposition A.2 ([1]) *Let M be an isoparametric surface in H^3. Then M is either a totally geodesic hyperbolic 2-space, or a totally umbilical surface or an equidistant tube around a geodesic.*

Proof of Theorem 2.1. Let γ be a helical geodesic on M through a point $p \in M$. Then the Gauss formula (1) implies $D_{\gamma'}\gamma' = \mathrm{II}(\gamma', \gamma')\xi$, which is normal to M. Hence we can take $X = \xi$ in the Frenet-Serret formula, that is,

$$\mathrm{II}(\gamma', \gamma') = \kappa, \quad \mathrm{II}(\gamma', Y) = -\tau. \tag{7}$$

Let λ and $2H - \lambda$ be the principal curvatures of M with corresponding principal vector fields on M, as before. Let θ be the angle between γ' and E_1 so that

$$\begin{cases} \gamma' = \cos\theta E_1 + \sin\theta E_2, \\ Y = -\sin\theta E_1 + \cos\theta E_2. \end{cases}$$

Then, since $\mathrm{II}(E_1, E_1) = \lambda$ and $\mathrm{II}(E_2, E_2) = 2H - \lambda$,

$$\mathrm{II}(\gamma', \gamma') = \lambda \cos^2\theta + (2H - \lambda)\sin^2\theta, \tag{8}$$

$$\mathrm{II}(Y, Y) = \lambda \sin^2\theta + (2H - \lambda)\cos^2\theta. \tag{9}$$

The equations (7) and (8) imply $\kappa = \lambda\cos^2\theta + (2H - \lambda)\sin^2\theta$, hence by (9), $\mathrm{II}(Y, Y) = 2H - \kappa$. Then the Gaussian curvature K of M is

$$\begin{aligned} K &= c + \mathrm{II}(\gamma', \gamma') \cdot \mathrm{II}(Y, Y) - \{\mathrm{II}(\gamma', Y)\}^2, \\ &= c + \kappa(2H - \kappa) - \tau^2. \end{aligned}$$

Since this must equal to $c + \lambda(2H - \lambda)$, the function λ is constant along γ (and hence $2H - \lambda$ is also constant along γ).

Now, differentiating (8) with respect to γ',

$$\nabla_{\gamma'}\theta = 0 \tag{10}$$

because M has no umbilic points. Put $\nabla_{E_1}E_1 = \alpha E_2$ and $\nabla_{E_2}E_1 = \beta E_2$. Then $\nabla_{E_1}E_2 = -\alpha E_1$ and $\nabla_{E_2}E_2 = -\beta E_1$. Since γ is a geodesic, from (10),

$$\alpha\cos\theta + \beta\sin\theta = 0. \tag{11}$$

On the other hand, using the Codazzi equation, we obtain (3). Hence, from the fact that the normal part of $D_{\gamma'}(\lambda\xi)$ vanishes and the equation (3),

$$\alpha\sin\theta - \beta\cos\theta = 0. \tag{12}$$

Therefore, from (11) and (12), $\alpha = \beta = 0$ along γ. This implies all lines of curvature on M are geodesics on M. Hence, as in the proof of Lemma 2.1, M is flat by Lemma 1.1 and hence M has constant principal curvatures λ and $2H - \lambda$. Therefore M is either a totally geodesic surface, totally umbilic surfaces or "circular cylinders". However by our assumtion, M is umbilic free. Thus M is a "circular cylinder".

This completes the proof of Theorem 2.1. □

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