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NOTES ON HAUSDORFF MEASURE AND CLASSICAL CAPACITY

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1. Introduction

This is an exposition on the Matts Essén's part of the Lecture Notes in Mathematics [2]. We supplement missing conditions and details of proofs in some statements. Sections 2, 3, 4 and 5 in this article correspond to Sections 2, 4, 5 and 7 of that book, respectively. In Section 2 we study comparability, outer and inner relations of the Hausdorff measure and net measures. In Section 3 we discuss the maximum principle and the continuity principle for potentials of measures, and give more details than [2, Section 4]. In Section 4 we define a capacity and discuss the existence and the uniqueness of the equilibrium measure. We mention relationships among capacity, the Chebychev's constant and the generalized diameter in Section 5.

2. Hausdorff measures

2.1. **Definition.** Let h be a measure function, *i.e.* an increasing function from $(0, \infty)$ to $(0, \infty)$ such that $\lim_{r\to 0} h(r) = 0$. We denote $B(x, r) = \{y \in \mathbb{R}^N; |x - y| < r\}$.

Definition 2.1 (Hausdorff measure)**.** For $E \subset \mathbb{R}^N$ we define

$$\Lambda_h^\rho(E) := \inf\left\{\sum_j h(r_j)\ ;\ E \subset \bigcup_j B(x_j, r_j),\ r_j < \rho\right\}$$

when $0 < \rho \le \infty$. Λ_h^0 is defined as the limiting value as $\rho \to 0$.

It is easy to see that Λ_h^ρ decreases when ρ increases, and thus Λ_h^0 is well-defined. Λ_h^0 is called the *Hausdorff measure*.

Theorem 2.1. *Λ_h^ρ is subadditive.*

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Proof. First we assume that $0 < \rho \le \infty$. Let $\{B(x_{jk}, r_{jk})\}_k$ be a covering of a set E_j with $r_{jk} < \rho$. Since $\{B(x_{jk}, r_{jk})\}_{j,k}$ is a covering of $\bigcup_j E_j$,

$$\Lambda_h^\rho\left(\bigcup_j E_j\right) \le \sum_j \sum_k h(r_{jk}),$$

and thus

$$\Lambda_h^\rho\left(\bigcup_j E_j\right) \le \sum_j \Lambda_h^\rho(E_j).$$

Letting $\rho \to 0$ we have the result in the case $\rho = 0$. □

For an integer p let G_p be the collection of cubes represented by a form $\left(n_1 2^{-p}, (n_1+1)2^{-p}\right] \times \cdots \times \left(n_N 2^{-p}, (n_N+1)2^{-p}\right]$ for some integers $n_1, \ldots, n_N$.

Definition 2.2 (Net measures)**.** For $E \subset \mathbb{R}^N$ we define

$$M_h^\rho(E) := \inf\left\{\sum_j h\left(2^{-p_j}\right) \;;\; E \subset \bigcup_j Q_j,\, Q_j \in G_{p_j},\, 2^{-p_j} < \rho\right\}$$

and

$$m_h^\rho(E) := \inf\left\{\sum_j h\left(2^{-p_j}\right) \;;\; E \subset \left(\bigcup_j Q_j\right)^\circ,\, Q_j \in G_{p_j},\, 2^{-p_j} < \rho\right\}.$$

If $\rho = 0$, then we define as the limiting value.

We can prove similarly that M_h^ρ and m_h^ρ are subadditive.

2.2. **Comparability.**

Theorem 2.2. *Λ_h^ρ, M_h^ρ and m_h^ρ are comparable for each ρ and h; comparison constants depend only on the dimension N.*

Proof. We shall show that

$$\Lambda_h^\rho(E) \le c_1 M_h^\rho(E) \le c_2 m_h^\rho(E) \le c_3 \Lambda_h^\rho(E) \quad \text{for any set } E,$$

where c_1, c_2 and c_3 are constants depending only on N. We may assume that $\rho > 0$. The second inequality is trivial.

For the first inequality we take cubes $\{Q_j\}_j$ such that $E \subset \bigcup_j Q_j$ and $\delta_j < \rho$, where δ_j is the side length of Q_j. Then we can find $\{x_{jk}\}_{k=1}^{c_1}$ such that $Q_j \subset \bigcup_{k=1}^{c_1} B(x_{jk}, \delta_j)$ for each j. Since $\{B(x_{jk}, \delta_j)\}_{j,k}$ is a covering of E,

$$\Lambda_h^\rho(E) \le \sum_j \sum_{k=1}^{c_1} h(\delta_j) = c_1 \sum_j h(\delta_j).$$

Therefore

$$\Lambda_h^\rho(E) \le c_1 M_h^\rho(E).$$

Next we shall prove the third inequality. Let $\{B(x_j, r_j)\}_j$ be a covering of E such that $r_j < \rho$. Let p_j be an integer such that $2^{-p_j} \le r_j < 2^{-p_j+1}$. Then we can find cubes $\{Q_{jk}\}_{k=1}^{c_4}$ such that $B(x_j, r_j) \subset (\bigcup_{k=1}^{c_4} Q_{jk})^\circ$ for each j and the side length of Q_{jk} is 2^{-p_j}, where c_4 is a constant depending only on N. Therefore

$$m_h^\rho(E) \le \sum_j \sum_{k=1}^{c_4} h\left(2^{-p_j}\right) \le c_4 \sum_j h(r_j),$$

and thus

$$m_h^\rho(E) \le c_4 \Lambda_h^\rho(E).$$

Hence we conclude the result. □

Theorem 2.3. *If $0 < \rho_1 < \rho_2 < \infty$, then $\Lambda_h^{\rho_1}$ and $\Lambda_h^{\rho_2}$ are comparable; comparison constants depend on ρ_2/ρ_1 and N.*

Proof. Let $\{B(x_j, r_j)\}_j$ be a covering of E such that $r_j < \rho_2$. Then we can find $\{x_{jk}\}_{k=1}^{c}$ such that $B(x_j, r_j) \subset \bigcup_{k=1}^{c} B(x_{jk}, \rho_1 r_j/\rho_2)$, where c is a constant depends on ρ_2/ρ_1 and N. Since $\rho_1 r_j/\rho_2 < \rho_1$,

$$\Lambda_h^{\rho_1}(E) \le \sum_j \sum_{k=1}^{c} h(\rho_1 r_j/\rho_2) \le c \sum_j h(r_j),$$

and thus

$$\Lambda_h^{\rho_1}(E) \le c\Lambda_h^{\rho_2}(E).$$

The opposite is clear, and the theorem is proved. □

Example 2.1. *If $0 < \rho < \infty$, then there is a measure function h such that*

(i) *Λ_h^ρ and Λ_h^∞ are not comparable;*
(ii) *Λ_h^ρ and Λ_h^0 are not comparable.*

Proof. (i) Take $c > \rho$ and let $h(r) = r$ if $0 \le r \le c$ and $h(r) = c$ if $r \ge c$. If E is a line segment of length l, then $\Lambda_h^\infty(E) \le c$ and $\Lambda_h^\rho(E) \ge l/2$. Since l is arbitrarily large, the result follows.

(ii) Let $h(r) = \sqrt{r}$ and let E be a line segment whose length is l with $l < \rho$. Then $\Lambda_h^0(E) = \infty$ and $\Lambda_h^\rho(E) \le \sqrt{l}$. □

Theorem 2.4. *For $0 \le \rho_1 \le \rho_2 \le \infty$, $\Lambda_h^{\rho_1}(E) = 0$ if and only if $\Lambda_h^{\rho_2}(E) = 0$.*

Proof. If $0 \le \rho_1 \le \rho_2 \le \infty$, then $\Lambda_h^\infty(E) \le \Lambda_h^{\rho_2}(E) \le \Lambda_h^{\rho_1}(E) \le \Lambda_h^0(E)$. Therefore we have only to prove that $\Lambda_h^\infty(E) = 0$ implies $\Lambda_h^0(E) = 0$. For given $\varepsilon > 0$ there is a covering $\{B_j\}_j$ of E such that $\sum_j h(r_j) < h(\varepsilon)$. Since $r_j < \varepsilon$,

$$\Lambda_h^\varepsilon(E) \le \sum_j h(r_j) < h(\varepsilon).$$

Letting ε to 0, we have $\Lambda_h^0(E) = 0$. □

Theorem 2.5. *Let h_1 and h_2 be measure functions such that*

$$\lim_{r\to 0} \frac{h_2(r)}{h_1(r)} = 0.$$

If $\Lambda_{h_1}^0(E) < \infty$, then $\Lambda_{h_2}^0(E) = 0$.

Proof. Let $\{B(x_j, r_j)\}_j$ be a covering of E such that $r_j < \rho$. Then

$$\Lambda_{h_2}^{\rho}(E) \le \sum_j h_2(r_j) \le \sup_{0<r<\rho} \frac{h_2(r)}{h_1(r)} \sum_j h_1(r_j),$$

therefore

$$\Lambda_{h_2}^{\rho}(E) \le \sup_{0<r<\rho} \frac{h_2(r)}{h_1(r)} \Lambda_{h_1}^{\rho}(E).$$

The right hand side tends to 0 as $\rho \to 0$, and thus the result follows. □

Theorem 2.2 implies that M_h^{ρ} or m_h^{ρ} satisfies similar relations.

2.3. **Hausdorff dimension.** When $h(r) = r^s$ with $s > 0$, Λ_h^0 is called the outer s-dimensional Hausdorff measure. Theorem 2.5 implies that there exists an $s_0 \ge 0$ such that

$$\Lambda_{r^s}^0(E) = \begin{cases} \infty & \text{if } 0 < s < s_0, \\ 0 & \text{if } s_0 < s. \end{cases}$$

The number s_0 is called the *Hausdorff dimension* of E, denoted by $\dim(E)$.

Example 2.2. $\dim(E) = 1$ *if E is a line segment.*

Proof. Let E be a line segment with length l. Take an integer n such that $l/n < \rho$, and cover E by n balls with radii l/n. If $s > 1$, then

$$\Lambda_{r^s}^{\rho}(E) \le n(l/n)^s \to 0 \quad \text{as } n \to \infty.$$

Thus $\Lambda_{r^s}^0(E) = 0$.

Let $\{B(x_j, r_j)\}_j$ be a covering of E with $r_j < \rho$. Since the length of $B(x_j, r_j) \cap E$ is less than $2r_j$, we have $\sum_j 2r_j \ge l$, and thus $2\Lambda_r^{\rho}(E) \ge l$. Therefore $2\Lambda_r^0(E) \ge l$. Hence $\dim(E) = 1$.

Since Λ_h^0 is subadditive, the conclusion follows immediately even when E is a line with infinite length. □

We can similarly prove that $\dim(E) = 2$ when E is a square, and $\dim(E) = 3$ when E is a cube, and so on. But when E is not such a set, it is intricate.

Example 2.3. $\dim(E) = \log 2/\log 3$ *if E is the 1/3-Cantor set.*

Proof. Let $E_0 = [0, 1]$, $E_1 = [0, 1/3] \cup [2/3, 1]$, ... Then $E = \bigcap_n E_n$. Since E_n is covered by 2^n balls with radii 3^{-n},

$$\Lambda_{r^s}^{\rho}(E) \le \Lambda_{r^s}^{\rho}(E_n) \le 2^n \left(3^{-n}\right)^s = \left(2 \cdot 3^{-s}\right)^n.$$

If $s > \log 2/\log 3$, then the right hand side tends to 0 as $n \to \infty$, and thus $\Lambda_{r^s}^0(E) = 0$, *i.e.*

$$\dim(E) \le \log 2/\log 3.$$

Next we consider the opposite inequality. Let ω be a union of finite number of open intervals contained in $[0, 1]$ and let $A_n(\omega)$ be the number of intervals of E_n which intersects ω. Then, since $A_{n+1}(\omega) \le 2A_n(\omega)$, we have that $A_n(\omega)2^{-n}$ is decreases, and thus

$$\Phi(\omega) := \lim_{n\to\infty} A_n(\omega) 2^{-n}$$

exists. Since $A_n(\omega_1 \cup \omega_2) \le A_n(\omega_1) + A_n(\omega_2)$,

$$\Phi(\omega_1 \cup \omega_2) \le \Phi(\omega_1) + \Phi(\omega_2).$$

If $E \subset \omega$, then $A_n(\omega) = 2^n$, and thus $\Phi(\omega) = 1$. Let I be an interval with length d such that $3^{-(n+1)} \le d < 3^{-n}$. Then $A_n(I) \le 1$. Therefore

$$\Phi(I) \le A_n(I)\, 2^{-n} \le 2^{-n} \le (3d)^\alpha$$

where $\alpha = \log 2/\log 3$.

Let $\{I_j\}_{j=1}^m$ be a covering of E where I_j is an open interval with length d_j. Since E is compact, we may assume that $m < \infty$. Then

$$\sum_{j=1}^m d_j^\alpha \ge 3^{-\alpha} \sum_{j=1}^m \Phi(I_j) \ge 3^{-\alpha}\Phi\left(\bigcup_{j=1}^m I_j\right) = 3^{-\alpha}.$$

Hence

$$\Lambda^0_{r^\alpha}(E) \ge 3^{-\alpha}.$$

Therefore we have the result. □

2.4. **Outer relations.**

Theorem 2.6. *If $0 < \rho \le \infty$, then $\Lambda^\rho_h(E) = \inf\{\Lambda^\rho_h(O)\ ;\ O$ is open, $E \subset O\}$.*

Proof. Let $\{B(x_j, r_j)\}_j$ be a covering of E such that $r_j < \rho$. Then, since $\bigcup_j B(x_j, r_j)$ is an open set containing E,

$$\inf_O \Lambda^\rho_h(O) \le \Lambda^\rho_h\left(\bigcup_j B_j\right) \le \sum_j h(r_j),$$

therefore

$$\inf_O \Lambda^\rho_h(O) \le \Lambda^\rho_h(E).$$

The opposite inequality is clear, and thus the theorem is proved. □

Similar discussion works for m^ρ_h.

Theorem 2.7. *If $0 < \rho \le \infty$ and h satisfies*

$$\liminf_{r\to 0} r^{1-N} h(r) = 0,$$

then $M^\rho_h(E) = \inf\{M^\rho_h(O)\ ;\ O$ is open, $E \subset O\}$.

Proof. First we shall show that $\Lambda^\rho_h(A) = 0$ where A is a face of a cube. From the assumption we can take $\{r_j\}_j$ such that $r_j \searrow 0$ and

$$\lim_{j\to\infty} r_j^{1-N} h(r_j) = 0.$$

Take a covering $\{B(x_{jk}, r_j)\}_{k=1}^{n_j}$ of A, where $n_j \le c r_j^{1-N}$ and c is a constant depending on the side length of A. Then

$$\Lambda^\rho_h(A) \le \sum_{k=1}^{n_j} h(r_j) \le c r_j^{1-N} h(r_j) \to 0 \quad \text{as } j \to \infty.$$

By Theorem 2.6 we can take an open set O containing A such that $\Lambda_h^\rho(O)$ is arbitrarily small. Therefore Theorem 2.2 implies that $M_h^\rho(O)$ is arbitrarily small.

Let $\{Q_j\}_j$ be a covering of E with $Q_j \in G_{p_j}$. For given $\varepsilon > 0$, we can find a covering $\{Q_{jk}\}_k$ of an open set containing the faces of Q_j such that

$$\sum_k h(\delta_{jk}) < 2^{-j}\varepsilon$$

where δ_{jk} is the side length of Q_{jk}. Since $\left(\left(\bigcup_j Q_j\right) \cup \left(\bigcup_j \bigcup_k Q_{jk}\right)\right)^\circ$ is an open set containing E,

$$\inf_O M_h^\rho(O) \le \sum_j h(\delta_j) + \sum_j \sum_k h(\delta_{jk}) \le \sum_j h(\delta_j) + \varepsilon.$$

Therefore

$$\inf_O M_h^\rho(O) \le M_h^\rho(E) + \varepsilon.$$

Since ε is arbitrary, we have the result. □

Example 2.4. *There is a measure function h and a set E such that*

$$\Lambda_h^0(E) \ne \inf\left\{\Lambda_h^0(O)\ ;\ O \text{ is open, } E \subset O\right\}.$$

Proof. Let $h(r) = r^{N-1/2}$ and E a set of one point. Then $\Lambda_h^0(E) = 0$ and $\Lambda_h^0(O) = \infty$ for any non-empty open set O, and thus the result follows. □

Example 2.5. *If h satisfies*

$$\liminf_{r\to 0} r^{1-N} h(r) > 0,$$

then $M_h^\rho(E) \ne \inf\{M_h^\rho(O)\ ;\ O \text{ is open, } E \subset O\}$ for some E.

See [3] for the proof.

2.5. **Inner relations.**

Lemma 2.1. *When $0 \le \rho < \infty$, if $E_n \nearrow E$, then $\lim_{n\to\infty} M_h^\rho(E_n) = M_h^\rho(E)$.*

Proof. First assume that $0 < \rho < \infty$. If $M_h^\rho(E_n) = \infty$ for some n, then the lemma is trivial. Thus we may assume that $M_h^\rho(E_n) < \infty$ for all n.

For given $\varepsilon > 0$ let $\varepsilon_n = 2^{-n}\varepsilon$. For every n there is a covering $\{Q_{nj}\}_j$ of E_n such that $\delta_{nj} < \rho$ and

$$\sum_j h(\delta_{nj}) \le M_h^\rho(E_n) + \varepsilon_n$$

where δ_{nj} is the side length of Q_{nj}. For every cube $Q_{n_0j_0}$, since $\rho < \infty$, we can find the largest cube in $\{Q_{nj}\ ;\ Q_{n_0j_0} \subset Q_{nj}\}$. We denote all such cubes by $\{Q_i\}_i$. Fix n. Let $\left\{Q_k^{(1)}\right\}_k = \{Q_i\}_i \cap \{Q_{1j}\}_j$ and $C_1 = E_n \cap \left(\bigcup_k Q_k^{(1)}\right)$. Also let $\left\{Q_{nk}^{(1)}\right\}_k = \left\{Q_{nj}\ ;\ Q_{nj} \subset \bigcup_i Q_i^{(1)}\right\}$. Take $x \in C_1$. Since $x \in E_n$, there is a j with $x \in Q_{nj}$. Also there is an i with $x \in Q_i^{(1)}$. Since Q_i's are the largest, $Q_{nj} \subset Q_i^{(1)}$. Therefore

$Q_{nj} \in \left\{Q_{nk}^{(1)}\right\}_k$, and thus $x \in \bigcup_k Q_{nk}^{(1)}$, *i.e.* $C_1 \subset \bigcup_k Q_{nk}^{(1)}$. Since $\{Q_{1j}\}_j \setminus \left\{Q_i^{(1)}\right\}_i$ covers $E_1 \setminus C_1$,

$$\sum_k h\left(\delta_{nk}^{(1)}\right) + \varepsilon_1 + M_h^\rho\left(E_1 \setminus C_1\right) \geq M_h^\rho\left(E_1 \cap C_1\right) + \varepsilon_1 + M_h^\rho\left(E_1 \setminus C_1\right)$$
$$\geq M_h^\rho\left(E_1\right) + \varepsilon_1 \geq \sum_j h\left(\delta_{1j}\right) \geq \sum_i h\left(\delta_i^{(1)}\right) + M_h^\rho\left(E_1 \setminus C_1\right).$$

Hence

$$\sum_k h\left(\delta_{nk}^{(1)}\right) + \varepsilon_1 \geq \sum_i h\left(\delta_i^{(1)}\right).$$

Let $\left\{Q_k^{(2)}\right\}_k = \{Q_i\}_i \cap \left(\{Q_{2j}\}_j \setminus \{Q_{1j}\}_j\right)$ and let $\left\{Q_{nk}^{(2)}\right\}_k = \left\{Q_{nj}\ ;\ Q_{nj} \subset \bigcup_i Q_i^{(2)}\right\}$. Then

$$\sum_k h\left(\delta_{nk}^{(2)}\right) + \varepsilon_2 \geq \sum_i h\left(\delta_i^{(2)}\right).$$

Repeat this argument. We have

$$\sum_{m=1}^n \sum_i h\left(\delta_i^{(m)}\right) \leq \sum_{m=1}^n \sum_k h\left(\delta_{nk}^{(m)}\right) + \sum_{m=1}^n \varepsilon_m \leq \sum_j h\left(\delta_{nj}\right) + \sum_{m=1}^n \varepsilon_m$$
$$\leq M_h^\rho\left(E_n\right) + \varepsilon_n + \sum_{m=1}^n \varepsilon_m.$$

Therefore

$$M_h^\rho(E) \leq \sum_i h\left(\delta_i\right) \leq \lim_{n\to\infty} M_h^\rho\left(E_n\right) + \varepsilon.$$

Since ε is arbitrary,

$$M_h^\rho(E) \leq \lim_{n\to\infty} M_h^\rho\left(E_n\right).$$

The opposite inequality is trivial, thus the lemma is proved in this case.

Next we consider the case $\rho = 0$. For $\varepsilon > 0$ there is a $\rho > 0$ such that

$$M_h^0(E) \leq M_h^\rho(E) + \varepsilon.$$

Therefore

$$M_h^0(E) \leq \lim_{n\to\infty} M_h^\rho\left(E_n\right) + \varepsilon \leq \lim_{n\to\infty} M_h^0\left(E_n\right) + \varepsilon.$$

Since ε is arbitrary, we have the lemma. □

If $\rho = \infty$, then the same relation holds for a bounded set E. Also we can prove a similar relation for m_h^ρ.

Question 2.1. *Does Λ_h^ρ satisfy a similar relation?*

2.6. **The Frostman lemma.**

Theorem 2.8 (Frostman). (i) *Let μ be a non-negative and subadditive set function such that $\mu(B(x, r)) \leq h(r)$ for any x, then $\mu(E) \leq \Lambda_h^\infty(E)$.*

(ii) *There is a constant c such that, for any compact set F, there exists a measure μ such that $\operatorname{supp}\mu \subset F$, $\mu(F) \geq cM_h^\infty(F)$, and $\mu(B(x, r)) \leq h(r)$ for any x.*

Proof. (i) Take a covering $\{B(x_j, r_j)\}_j$ of E. Then

$$\mu(E) \leq \sum_j \mu(B(x_j, r_j)) \leq \sum_j h(r_j),$$

and thus we conclude the result.

(ii) Take an integer p sufficiently large such that $F \subset (-2^{p-1}, 2^{p-1}) \times \cdots \times (-2^{p-1}, 2^{p-1})$. For a fixed integer n we define measures $\{\mu_j^n\}_{j=-p}^n$ as follows. Take $Q_n \in G_n$. If $Q_n \cap F = \emptyset$ then $\mu_j^n(Q_n) = 0$ for $j = n, n-1, \ldots, -p$. If $Q_n \cap F \neq \emptyset$, then take a sequence $\{Q_j\}_{j=-p}^n$ such that $Q_j \in G_j$ and $Q_n \subset Q_{n-1} \subset \cdots \subset Q_{-p}$, and let

$$\mu_n^n(Q_n) = h\left(2^{-n}\right),$$

$$\mu_j^n(Q_n) = \min\left(1, \frac{h(2^{-j})}{\mu_{j+1}^n(Q_j)}\right)\mu_{j+1}^n(Q_n) \quad \text{for } j = n-1, n-2, \ldots, -p$$

where μ_j^n distributes uniformly in each Q_n.

Now we assume that $Q_n \cap F \neq \emptyset$. First we have $\mu_n^n(Q_n) = h(2^{-n})$. Next, if $\mu_n^n(Q_{n-1}) \leq h(2^{-n+1})$ then

$$\mu_{n-1}^n(Q_n) = \mu_n^n(Q_n) = h\left(2^{-n}\right).$$

If $\mu_n^n(Q_{n-1}) \geq h(2^{-n+1})$, then every cube $Q_n' \in G_n$ included in Q_{n-1} satisfies

$$\mu_{n-1}^n(Q_n') = \frac{h(2^{-n+1})}{\mu_n^n(Q_{n-1})}\mu_n^n(Q_n'),$$

and thus

$$\mu_{n-1}^n(Q_{n-1}) = \frac{h(2^{-n+1})}{\mu_n^n(Q_{n-1})}\mu_n^n(Q_{n-1}) = h\left(2^{-n+1}\right).$$

After several steps we have similarly that there is a j with $-p \leq j \leq n$ such that

$$\mu_{-p}^n(Q_j) = h\left(2^{-j}\right). \tag{2.1}$$

For every $x \in F$, we take $Q_n \in G_n$ including x and we take the smallest j satisfying (2.1). We denote $\{Q^m\}_m$ for all such cubes, *i.e.* $Q^m \in G_{j_m}$ and $\mu_{-p}^n(Q^m) = h(2^{-j_m})$ for some j_m. Then $F \subset \bigcup_m Q^m$ and $Q^m \cap Q^{m'} = \emptyset$ if $m \neq m'$. Therefore

$$M_h^\infty(F) \leq \sum_m h\left(2^{-j_m}\right) = \sum_m \mu_{-p}^n(Q^m) \leq \mu_{-p}^n\left(\mathbb{R}^N\right). \tag{2.2}$$

Let $-p \leq j \leq n$ and $Q_j \in G_j$. Then

$$\mu_{-p}^n(Q_j) \leq \mu_{-p+1}^n(Q_j) \leq \cdots \leq \mu_j^n(Q_j)$$

and

$$\mu_j^n(Q_j) \le \frac{h(2^{-j})}{\mu_{j+1}^n(Q_j)}\mu_{j+1}^n(Q_j) = h\left(2^{-j}\right).$$

Therefore

$$\mu_{-p}^n(Q_j) \le h\left(2^{-j}\right). \tag{2.3}$$

Since $\operatorname{supp}\mu_{-p}^n \subset (-2^p, 2^p) \times \cdots \times (-2^p, 2^p)$ and the right hand side is included in 2^N cubes of G_{-p}, (2.3) gives

$$\mu_{-p}^n\left(\mathbb{R}^N\right) \le 2^N h(2^p). \tag{2.4}$$

Therefore by taking a subsequence we may assume that $\left\{\mu_{-p}^n\right\}_n$ converges weakly to a measure μ.

Let E be a compact set with $E \cap F = \emptyset$. Since $\operatorname{supp}\mu_{-p}^n$ is disjoint from E for sufficiently large n, $\operatorname{supp}\mu$ is also disjoint from E, *i.e.* $\operatorname{supp}\mu \subset F$.

Also from (2.2)

$$\mu(F) = \lim_{n\to\infty}\mu_{-p}^n\left(\mathbb{R}^N\right) \ge M_h^\infty(F).$$

Finally, let $B = B(a, r)$. If $r \ge 2^p$, then we have by (2.4)

$$\mu(B) \le \mu\left(\mathbb{R}^N\right) = \lim_{n\to\infty}\mu_{-p}^n\left(\mathbb{R}^N\right) \le 2^N h(2^p) \le 2^N h(r).$$

If $r < 2^p$, then we take ρ and j such that $2^{-j} \le r < \rho < 2^{-j+1}$, and we let φ be a continuous function such that $0 \le \varphi \le 1$ and

$$\varphi(x) = \begin{cases} 1 & \text{if } |x-a| < r, \\ 0 & \text{if } |x-a| > \rho. \end{cases}$$

Since $B(a, \rho)$ is covered by at most c cubes of G_j where c is a constant depending only on N, (2.3) gives

$$\mu(B) = \int_B \varphi\, d\mu \le \int \varphi\, d\mu = \lim_{n\to\infty}\int \varphi\, d\mu_{-p}^n \le \lim_{n\to\infty}\mu_{-p}^n(B(a, \rho)) \le ch\left(2^{-j}\right) \le ch(r).$$

The measure $c^{-1}\mu$ satisfies the theorem. □

3. Potential theory

Let $K(r)$ be a non-negative, decreasing and lower semi-continuous function such that $\lim_{r\to 0} K(r) = \infty$, $\lim_{r\to\infty} K(r) = 0$ and

$$\int_0^a K(r)\, r^{N-1}\, dr < \infty \quad \text{for sufficiently small } a > 0.$$

For simplicity we denote $K(x) = K(|x|)$ for $x \in \mathbb{R}^N$. Thus the assumption above can be represented by

$$\int_{|x|<a} K(x)\ dx < \infty.$$

We denote all of Radon measures by $\mathfrak{M}$, and all of non-negative Radon measures by $\mathfrak{M}^+$.

Definition 3.1 (Potential and energy)**.** For $\sigma,\,\tau \in \mathfrak{M}$ we define the *potential* as

$$K\sigma\,(x) := \int K\,(x-y)\;d\sigma\,(y)\,,$$

and the *mutual energy* as

$$I_K\,(\sigma,\,\tau) := \iint K\,(x-y)\;d\sigma\,(y)\,d\tau\,(x) = \int K\sigma\,(x)\;d\tau\,(x)\,,$$

when they can be defined. If $\sigma = \tau$, then we denote simply

$$I_K\,(\sigma) := I_K\,(\sigma,\,\sigma)$$

and we call it the *energy.*

Lemma 3.1. (i) *If $\mu \in \mathfrak{M}^+$ has finite mass, then $K\mu$ is lower semi-continuous.*

(ii) *If $\{\mu_n\}_n \subset \mathfrak{M}^+$ converges weakly to $\mu \in \mathfrak{M}^+$, then*

$$\liminf_{n\to\infty} K\mu_n\,(x) \geq K\mu\,(x)\,.$$

(iii) *If $\{\mu_n\}_n \subset \mathfrak{M}^+$ and $\{\nu_n\}_n \subset \mathfrak{M}^+$ converge weakly to $\mu \in \mathfrak{M}^+$ and $\mu \in \mathfrak{M}^+$ respectively, then*

$$\liminf_{n\to\infty} I_K\,(\mu_n,\,\nu_n) \geq I_K\,(\mu,\,\nu)\,.$$

Proof. (i) Let $\{K_p\}_p$ be an increasing sequence of continuous functions with compact supports which converges to K. Then $\{K_p\mu\}_p$ is an increasing sequence of continuous functions and converges to $K\mu$. Therefore $K\mu$ is lower semi-continuous.

(iii) First we shall prove that $d\mu_n\,(x)\;d\nu_n\,(y) \rightharpoonup d\mu\,(x)\;d\nu\,(y)$. Let $f\,(x,\,y)$ be a continuous function with compact support. Also let B_1, B_2 and B_3 be open balls in $\mathbb{R}^N$ such that $\operatorname{supp} f \subset B_1 \times B_1$ and $\bar{B}_1 \subset B_2 \subset \bar{B}_2 \subset B_3$. The Weierstrass approximation theorem implies that there is a sequence $\{P_m\,(x,\,y)\}_m$ of polynomials which converges uniformly to $f\,(x,\,y)$ in $\bar{B}_2 \times \bar{B}_2$. We can take continuous functions $\{\varphi_{mj}\}_{m,j}$ and $\{\psi_{mj}\}_{m,j}$ such that $\varphi_{mj} = \psi_{mj} = 0$ outside B_3 and $\left\{\sum_j \varphi_{mj}\,(x)\,\psi_{mj}\,(y)\right\}_m$ converges uniformly to $f\,(x,\,y)$ in $\mathbb{R}^N$. Therefore, for given $\varepsilon > 0$ and any $x,\,y \in \mathbb{R}^N$,

$$\left| f\,(x,\,y) - \sum_j \varphi_{mj}\,(x)\,\psi_{mj}\,(y) \right| < \varepsilon \quad \text{for sufficiently large } m.$$

Hence

$$\begin{aligned}
&\limsup_{n\to\infty} \iint f\,(x,\,y)\;d\mu_n\,(x)\,d\nu_n\,(y) \\
&\quad \leq \limsup_{n\to\infty} \left(\iint \sum_j \varphi_{mj}\,(x)\,\psi_{mj}\,(y)\;d\mu_n\,(x)\,d\nu_n\,(y) + \varepsilon\mu_n\left(\bar{B}_2\right)\nu_n\left(\bar{B}_2\right) \right) \\
&\quad = \limsup_{n\to\infty} \left(\sum_j \int \varphi_{mj}\,(x)\;d\mu_n\,(x) \int \psi_{mj}\,(y)\;d\nu_n\,(y) + \varepsilon\mu_n\left(\bar{B}_2\right)\nu_n\left(\bar{B}_2\right) \right)
\end{aligned}$$

$$\leq \sum_j \int \varphi_{mj}(x)\ d\mu(x) \int \psi_{mj}(y)\ d\nu(y) + \varepsilon\mu(B_3)\,\nu(B_3)$$
$$\leq \iint f(x,\, y)\ d\mu(x)\,d\nu(y) + 2\varepsilon\mu(B_3)\,\nu(B_3)\,.$$

Similarly we have the opposite inequality. Therefore

$$d\mu_n(x)\ d\nu_n(y) \rightharpoondown d\mu(x)\ d\nu(y)\,.$$

Take K_p as in (i). Then

$$\liminf_{n\to\infty} I_K(\mu_n,\, \nu_n) \geq \liminf_{n\to\infty} \iint K_p(x-y)\ d\mu_n(x)\,d\nu_n(y)$$
$$= \iint K_p(x-y)\ d\mu(x)\,d\nu(y)$$

Therefore the monotone convergence theorem implies the result.

(ii) Since $I_K(\mu,\, \delta_x) = K\mu(x)$ where δ_x is the Dirac measure at x, (iii) implies (ii). □

Theorem 3.1 (Weak maximum principle)**.** *There exists a constant c such that if $\mu \in \mathfrak{M}^+$ satisfies $K\mu \leq 1$ on $\operatorname{supp}\mu$, then $K\mu \leq c$ everywhere.*

Proof. We can find $\{e_j\}_{j=1}^c \subset \mathbb{R}^N$ such that $|e_j| = 1$ and $\mathbb{R}^N \setminus \{x\} = \bigcup_{j=1}^c \Gamma_j$ for $x \in \mathbb{R}^N \setminus \operatorname{supp}\mu$, where $\Gamma_j = \{y\ ;\ \langle e_j,\, y-x\rangle > |y-x|\cos\pi/6\}$ and $\langle\cdot,\, \cdot\rangle$ denotes the inner product. Let ξ_j be (one of) the closest point to x in $\Gamma_j \cap \operatorname{supp}\mu$. Remark that $|y-\xi_j| \leq |y-x|$ for any $y \in \Gamma_j \cap \operatorname{supp}\mu$. Therefore

$$K\mu(x) \leq \sum_{j=1}^c \int_{\Gamma_j} K(x-y)\ d\mu(y) \leq \sum_{j=1}^c \int_{\Gamma_j} K(\xi_j - y)\ d\mu(y) \leq \sum_{j=1}^c K\mu(\xi_j) \leq c.$$

□

We denote $C_{\mathrm{W}} = C_{\mathrm{W}}(K)$ for the minimal constant satisfying Theorem 3.1.

Lemma 3.2. *Let $\mu \in \mathfrak{M}^+$. Then the following two conditions are equivalent :*

(i) *$K_n\mu$ converges uniformly to $K\mu$ on E where $K_n(x) := \min(K(x),\, n)$;*

(ii) *For any $\varepsilon > 0$, there is an $\eta > 0$ such that*

$$\int_{|x-y|<\eta} K(x-y)\ d\mu(y) < \varepsilon \quad \textit{for any } x \in E.$$

Proof. First suppose that the condition (i) holds, *i.e.* for any $\varepsilon > 0$ and any $x \in E$

$$\int (K(x-y) - K_n(x-y))\ d\mu(y) < \varepsilon \quad \text{for sufficiently large } n.$$

Take η such that $K(\eta) \geq 2n$. Since $K(r) - K_n(r) \geq n$ for $r < \eta$,

$$n\mu(B(x,\, \eta)) \leq \int_{B(x,\eta)} (K(x-y) - K_n(x-y))\ d\mu(y) < \varepsilon,$$

and thus

$$\int_{B(x,\eta)} K(x-y)\, d\mu(y) \le \int_{B(x,\eta)} K_n(x-y)\, d\mu(y) + \varepsilon = n\mu(B(x,\eta)) + \varepsilon < 2\varepsilon,$$

this means the condition (ii) holds.

Next suppose that the condition (ii) holds. If $K(\eta) \le n$, then, since $K(r) - K_n(r) = 0$ for $r \ge \eta$ and $K(r) - K_n(r) \le K(r)$ for all r,

$$\begin{aligned} K\mu(x) - K_n\mu(x) &= \int_{|x-y|<\eta} (K(x-y) - K_n(x-y))\, d\mu(y) \\ &\le \int_{|x-y|<\eta} K(x-y)\, d\mu(y) < \varepsilon, \end{aligned}$$

this means the condition (i) holds. □

Definition 3.2 (Uniform convergence of potentials)**.** Let $\mu \in \mathfrak{M}^+$. Then $K\mu$ *converges uniformly* on a set E if μ satisfies (one of) the conditions of Lemma 3.2.

Lemma 3.3. *Suppose that K is continuous on* $(0, \infty)$*. Let $\mu \in \mathfrak{M}^+$ with finite mass such that $K\mu$ converges uniformly on* $\operatorname{supp}\mu$*. Then $K\mu$ is continuous everywhere.*

Proof. From the assumption, for any $\varepsilon > 0$, there is an η such that

$$\int_{\overline{B(x,2\eta)}} K(x-y)\, d\mu(y) < \varepsilon \quad \text{for any } x \in \operatorname{supp}\mu.$$

Let z be any point and let $\{z_n\}_n$ be a sequence of points converging to z. Also let $\mu_1 = \mu|_{\overline{B(z,\eta)}}$ and $\mu_2 = \mu - \mu_1$. If $x \in \operatorname{supp}\mu_1$, then $x \in \operatorname{supp}\mu$ and $|z-x| \le \eta$. Therefore

$$K\mu_1(x) = \int_{\overline{B(z,\eta)}} K(x-y)\, d\mu(y) \le \int_{\overline{B(x,2\eta)}} K(x-y)\, d\mu(y) < \varepsilon.$$

The weak maximum principle implies that

$$K\mu_1 \le C_{\mathrm{W}}\varepsilon \quad \text{everywhere}.$$

Next we consider

$$K\mu_2(z_n) = \int_{|z-y|\ge\eta} K(z_n - y)\, d\mu(y).$$

We may assume that $|z_n - y| \ge \eta/2$. Since $K(z_n - y)$ is bounded, the bounded convergence theorem implies

$$\lim_{n\to\infty} K\mu_2(z_n) = \int_{|z-y|\ge\eta} K(z-y)\, d\mu(y) \le K\mu(z).$$

Hence

$$\limsup_{n\to\infty} K\mu(z_n) \le C_{\mathrm{W}}\varepsilon + K\mu(z).$$

Using Lemma 3.1 (i), we have the result. □

Theorem 3.2 (Continuity principle)**.** *Suppose that K is continuous on $(0, \infty)$. Let $\mu \in \mathfrak{M}^+$ with compact support. If $K\mu$ is continuous on $\operatorname{supp}\mu$, then $K\mu$ is continuous everywhere.*

Proof. Since $K_n\mu$ is continuous and converges to $K\mu$ as $n \to \infty$, Dini's theorem implies that $K_n\mu$ converges uniformly to $K\mu$ on $\operatorname{supp}\mu$, *i.e.* $K\mu$ converges uniformly on $\operatorname{supp}\mu$. Therefore Lemma 3.3 gives the result. □

Lemma 3.4. *Let $\mu \in \mathfrak{M}^+$ with finite mass such that $K\mu < \infty$ μ-a.e. For any $\varepsilon > 0$ there exists a closed set F such that $\mu\left(\mathbb{R}^N \setminus F\right) < \varepsilon$ and $K\mu$ converges uniformly on F. Moreover, if K is continuous on $(0, \infty)$, then $K\mu|_F$ continuous everywhere.*

Proof. Since $K_n\mu$ converges to $K\mu$, the Egorov theorem implies that there is a set E such that $\mu\left(\mathbb{R}^N \setminus E\right) < \varepsilon$ and $K_n\mu$ converges uniformly to $K\mu$ on E, *i.e.* for any $\delta > 0$ we have

$$\int \left(K(x-y) - K_n(x-y)\right) d\mu(y) < \delta \quad \text{for any } x \in E \text{ and sufficiently large } n.$$

Let F be the closure of E. It is clear that

$$\mu\left(\mathbb{R}^N \setminus F\right) < \varepsilon.$$

Now let $x \in F$ and let $\{x_j\}_j$ be a sequence in E which converges to x. Since K is lower semi-continuous, Fatou's lemma implies

$$\begin{aligned}\int \left(K(x-y) - K_n(x-y)\right) d\mu(y) &\le \int \liminf_{j\to\infty} \left(K(x_j-y) - K_n(x_j-y)\right) d\mu(y)\\ &\le \liminf_{j\to\infty} \int \left(K(x_j-y) - K_n(x_j-y)\right) d\mu(y) \le \delta.\end{aligned}$$

Therefore $K\mu$ converges uniformly on F. Lemma 3.3 implies the remaining part. □

Theorem 3.3 (Strong maximum principle)**.** *Suppose that $K(r)$ is absolutely continuous and that $K'(r)\, r^{N-1}$ is increasing. Then $C_{\mathrm{W}}(K) = 1$.*

Proof. When $N \ge 3$ we let $H(t) = K\left(t^{1/(2-N)}\right)$. Then

$$H'(t) = t^{(N-1)/(2-N)} K'\left(t^{1/(2-N)}\right) / (2-N),$$

which is increasing. Therefore K can be written as

$$K(r) = H(\Phi(r))$$

with a convex function H where $\Phi(r) = r^{2-N}$. Similarly, when $N = 2$, the above holds for $\Phi(r) = -\log r$.

Let $\mu \in \mathfrak{M}^+$ such that $K\mu \le 1$ on $\operatorname{supp}\mu$. First we assume that $\operatorname{supp}\mu$ is compact. From Lemma 3.4, for any $\delta > 0$ there is a closed set $F \subset \operatorname{supp}\mu$ such that $\mu\left(\mathbb{R}^N \setminus F\right) < \delta$ and $K\mu|_F$ is continuous everywhere. We let $\mu_1 = \mu|_F$.

We shall prove that $K\mu_1$ is subharmonic outside F. Let σ be the surface measure of $|y| = \rho$ such that $\|\sigma\| = 1$. Then, by Jensen's inequality,

$$\int_{|y|=\rho} K(x+y)\, d\sigma(y) = \int_{|y|=\rho} H(\Phi(x+y))\, d\sigma(y) \ge H\left(\int_{|y|=\rho} \Phi(x+y)\, d\sigma(y)\right).$$

If $|x| > \rho$, then, since Φ is harmonic except the origin, we have $\int_{|y|=\rho} \Phi(x+y)\, d\sigma(y) = \Phi(x)$. Therefore

$$\int_{|y|=\rho} K(x+y)\, d\sigma(y) \geq H(\Phi(x)) = K(x).$$

Let $x \notin F$ and $0 < \rho < \operatorname{dist}(x, F)$. Then

$$\int_{|y|=\rho} K\mu_1(x-y)\, d\sigma(y) = \int \int_{|y|=\rho} K((z-x)+y)\, d\sigma(y)\, d\mu_1(z).$$

If $z \in F$, then $|z-x| \geq \operatorname{dist}(x, F) > \rho$, and thus

$$\int_{|y|=\rho} K((z-x)+y)\, d\sigma(y) \geq K(z-x).$$

Therefore

$$\int_{|y|=\rho} K\mu_1(x-y)\, d\sigma(y) \geq \int K(z-x)\, d\mu_1(z) = K\mu_1(x),$$

which means $K\mu_1$ is subharmonic.

If x is a boundary point of $\mathbb{R}^N \setminus F$, then $x \in \operatorname{supp}\mu$, and

$$K\mu_1(x) \leq K\mu(x) \leq 1.$$

When $|x|$ tends to ∞, we have $K\mu_1(x) \to 0$. Hence, by the maximum principle of subharmonic functions,

$$K\mu_1(x) \leq 1 \quad \text{for } x \notin F.$$

This inequality also holds for $x \in F$, thus it holds everywhere.

Let $x \notin \operatorname{supp}\mu$ and $\rho = \operatorname{dist}(x, \operatorname{supp}\mu)$. Then, since $\mu\left(\mathbb{R}^N \setminus F\right) < \delta$,

$$K\mu(x) = K\mu_1(x) + \int_{\mathbb{R}^N \setminus F} K(x-y)\, d\mu(y) \leq 1 + \delta K(\rho).$$

Since δ is arbitrary, we have the theorem in this case.

Next we consider the general case. Let $\nu_R = \mu|_{\overline{B(0,R)}}$. Then the previous part implies

$$K\nu_R \leq 1 \quad \text{everywhere.}$$

Therefore the monotone convergence theorem gives

$$K\mu(x) = \lim_{R\to\infty} K\nu_R(x) \leq 1 \quad \text{everywhere,}$$

and thus the result follows. □

Question 3.1. *Find a necessary condition for K to satisfy the strong maximum principle.*

4. Capacity

4.1. Definitions and some properties.

Definition 4.1 (Capacity)**.** For a set E

$$C_K(E) := \sup\left\{\mu(E)\ ;\ \mu \in \mathfrak{M}^+,\ \operatorname{supp}\mu \subset E,\ K\mu \le 1 \text{ everywhere}\right\}.$$

Definition 4.2 (Quasi-everywhere)**.** A property is said to hold *quasi-everywhere*, *q.e.* for short, if it holds except a set E such that $C_K(E) = 0$.

Lemma 4.1. *If E is an F_σ-set with $C_K(E) = 0$ and $K\mu$ is bounded on E, then $\mu(E) = 0$.*

Proof. First we assume that E is compact. Let $M = \sup_E K\mu$ and $\tau = (MC_{\mathrm{W}})^{-1}\,\mu|_E$. Then $K\tau \le C_{\mathrm{W}}^{-1}$ on $\operatorname{supp}\tau$, and thus $K\tau \le 1$ everywhere. Therefore

$$\mu(E) = MC_{\mathrm{W}}\tau(E) \le MC_{\mathrm{W}}C_K(E) = 0.$$

Now we consider the general case. Take a sequence $\{F_n\}_n$ of compact sets which converges increasingly to E. Since $C_K(F_n) \le C_K(E) = 0$, we have $\mu(F_n) = 0$ from the first part, and thus we have the result. □

Lemma 4.2. *If $\{E_n\}_n$ is a sequence of F_σ-sets, then*

$$C_K\left(\bigcup_n E_n\right) \le \sum_n C_K(E_n).$$

Proof. Let $\mu \in \mathfrak{M}^+$ such that $\operatorname{supp}\mu \subset \bigcup_n E_n$ and $K\mu \le 1$ everywhere. First we assume that all E_n are compact. Since $\operatorname{supp}\mu|_{E_n} \subset E_n$ and $K\mu|_{E_n} \le 1$ everywhere, we have $\mu(E_n) \le C_K(E_n)$. Therefore

$$\mu\left(\bigcup_n E_n\right) \le \sum_n \mu(E_n) \le \sum_n C_K(E_n).$$

Hence the lemma follows in this case.

Next we consider the general case. Take a compact set $F_n \subset E_n$ for every n. Then

$$\mu(F_n) \le C_K(F_n) \le C_K(E_n).$$

Since $\mu(E_n) = \sup_{F_n}\mu(F_n)$, we have $\mu(E_n) \le C_K(E_n)$. Therefore the lemma follows similarly to the first part. □

4.2. Equilibrium measure.

Lemma 4.3. *Let F be a non-empty compact set and let*

$$\gamma = \inf\left\{I_K(\mu)\ ;\ \mu \in \mathfrak{M}^+,\ \operatorname{supp}\mu \subset F,\ \mu(F) = 1\right\}.$$

Then $C_K(F) \le \gamma^{-1} \le C_{\mathrm{W}}C_K(F)$ and there is a measure $\mu \in \mathfrak{M}^+$ such that $\operatorname{supp}\mu \subset F$, $\mu(F) = 1$, $I_K(\mu) = \gamma$, $K\mu \ge \gamma$ q.e. on F and $K\mu \le \gamma$ on $\operatorname{supp}\mu$.

Proof. First assume that $\gamma = \infty$. If $C_K(F) > 0$, then there is a measure $\mu \in \mathfrak{M}^+$ such that $\operatorname{supp} \mu \subset F$, $\mu(F) = 1$ and $K\mu$ is bounded. Then $I_K(\mu) < \infty$, which is a contradiction. Therefore $C_K(F) = 0$. Hence any measure $\mu \in \mathfrak{M}^+$ with $\operatorname{supp} \mu \subset F$ and $\mu(F) = 1$ satisfies the conditions.

Next assume that $\gamma < \infty$. Let $\{\mu_n\}_n \subset \mathfrak{M}^+$ be a sequence such that $\operatorname{supp} \mu_n \subset F$, $\mu_n(F) = 1$ and

$$\lim_{n \to \infty} I_K(\mu_n) = \gamma.$$

By taking a subsequence we may assume that μ_n converges weakly to a measure μ. Then it is easy to see that $\operatorname{supp} \mu \subset F$ and $\mu(F) = 1$. Lemma 3.1 (iii) yields

$$\gamma \leq I_K(\mu) \leq \liminf_{n \to \infty} I_K(\mu_n) = \gamma,$$

that is

$$I_K(\mu) = \gamma.$$

Let $T_m = \{x \in F \,;\, K\mu(x) \leq \gamma - m^{-1}\}$ and $T = \{x \in F \,;\, K\mu(x) < \gamma\}$. Suppose that $C_K(T_m) > 0$. Then we can find a measure $\tau \in \mathfrak{M}^+$ such that $\operatorname{supp} \tau \subset T_m$, $\tau(T_m) = 1$ and $K\tau \leq c_0 < \infty$ everywhere. Now let $\mu_t = (1-t)\mu + t\tau$ for $0 < t < 1$. Then $\operatorname{supp} \mu_t \subset F$ and $\mu_t(F) = 1$. Therefore

$$I_K(\mu_t) \geq \gamma.$$

On the other hand, since

$$I_K(\mu, \tau) = \int K\mu \, d\tau \leq \left(\gamma - m^{-1}\right) \tau(T_m) = \gamma - m^{-1}$$

and

$$I_K(\tau) = \int K\tau \, d\tau \leq c_0,$$

we have

$$I_K(\mu_t) \leq (1-t)^2 \gamma + 2t(1-t)\left(\gamma - m^{-1}\right) + t^2 c_0 = \gamma - 2m^{-1}t + \left(2m^{-1} - \gamma + c_0\right) t^2 < \gamma$$

when t is sufficiently small, which is a contradiction. Therefore $C_K(T_m) = 0$. Since $T = \bigcup_m T_m$, Lemma 4.2 gives

$$C_K(T) \leq \sum_m C_K(T_m) = 0,$$

which means that $K\mu \geq \gamma$ q.e. on F. Also, using Lemma 4.1, we have $\mu(T) = 0$.

Next suppose that there is an $x \in \operatorname{supp} \mu$ with $K\mu(x) > \gamma$. Then we can take a neighborhood O of x such that $K\mu > \gamma$ on O. Since $\mu(T) = 0$ and $\gamma = I_K(\mu) = \int_F K\mu \, d\mu$, we have $K\mu = \gamma$ μ-a.e. Hence $\mu(O) = 0$, which is a contradiction. Therefore $K\mu \leq \gamma$ on $\operatorname{supp} \mu$.

Finally let $\nu \in \mathfrak{M}^+$ such that $\operatorname{supp} \nu \subset F$ and $K\nu \leq 1$ everywhere. Let $\nu_1 = \nu(F)^{-1} \nu$. Then $\operatorname{supp} \nu_1 \subset F$ and $\nu_1(F) = 1$. Therefore

$$\gamma \leq I_K(\nu_1) = \nu(F)^{-2} I_K(\nu).$$

Since $I_K(\nu) = \int K\nu \, d\nu \leq \int d\nu = \nu(F)$, we have $\nu(F) \leq \gamma^{-1}$. Hence

$$C_K(F) \leq \gamma^{-1}.$$

On the other hand, let $\mu_1 = (C_{\mathrm{W}}\gamma)^{-1}\mu$. Then $K\mu_1 \le C_{\mathrm{W}}^{-1}$ on $\operatorname{supp}\mu_1$, and thus $K\mu_1 \le 1$ everywhere. Therefore

$$C_K(F) \ge \mu_1(F) = (C_{\mathrm{W}}\gamma)^{-1},$$

and we have the lemma. □

Since $\gamma > 0$, we have $C_K(E) < \infty$ for a bounded set E.

Theorem 4.1 (Equilibrium measure)**.** *Let F be a non-empty compact set. There is a measure $\mu \in \mathfrak{M}^+$ such that* $\operatorname{supp}\mu \subset F$*,* $K\mu \le 1$ *on* $\operatorname{supp}\mu$*,* $K\mu \ge 1$ *q.e. on* F *and* $C_K(F) \le \mu(F) = I_K(\mu) \le C_{\mathrm{W}}C_K(F)$*.*

Proof. Let μ_0 be a measure given by Lemma 4.3, and let $\mu = \gamma^{-1}\mu_0$. Then the conclusion is trivial. □

Theorem 4.2. *Suppose that K is continuous on* $(0, \infty)$*. Let F be a non-empty compact set and suppose that, for every $x \in F$, there is a bounded cone V_x with vertex at x such that $V_x \subset F$. Also suppose that K satisfies the doubling condition, i.e. there is a constant C such that $K(r) \le CK(2r)$. Then there is a measure $\mu \in \mathfrak{M}^+$ such that* $\operatorname{supp}\mu \subset F$*,* $K\mu \le 1$ *on* $\operatorname{supp}\mu$*,* $K\mu \ge 1$ *on F and* $C_K(F) \le \mu(F) = I_K(\mu) \le C_{\mathrm{W}}C_K(F)$*.*

Proof. Let μ be an equilibrium measure for F. We have only to prove that $K\mu(x) \ge 1$ for $x \in F$. Without loss of generality, we can assume that $0 \in F$, and we shall prove $K\mu(0) \ge 1$.

Let $c = \operatorname{Area}(V_0 \cap \partial B(0, R))/\operatorname{Area}(\partial B(0, R))$. Remark that c depends only on V_0. For $\alpha > 0$

$$\int_t^\alpha \frac{K(r)\,r^{N-1}}{\int_0^r K(s)\,s^{N-1}\,ds}\,dr = \left[\log \int_0^r K(s)\,s^{N-1}\,ds\right]_{r=t}^{\alpha}$$

tends to 0 as $t \to \alpha$ and tends to ∞ as $t \to 0$. Therefore we can find $t_0(\alpha)$ such that

$$\int_{t_0(\alpha)}^\alpha \frac{K(r)\,r^{N-1}}{\int_0^r K(s)\,s^{N-1}\,ds}\,dr = c^{-1}.$$

Let

$$q_\alpha(x) = \begin{cases} \frac{K(x)}{\int_{|y|<|x|} K(y)\,dy} & \text{if } x \in V_0 \text{ and } t_0(\alpha) < |x| < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int q_\alpha(x)\,dx = c\int_{t_0(\alpha)}^\alpha \frac{K(r)\,r^{N-1}}{\int_0^r K(s)\,s^{N-1}\,ds}\,dr = 1.$$

The weak maximum principle shows that $\int K(y)\,d\mu(y) = K\mu(0)$ is finite. Therefore, for $\varepsilon > 0$, we can find $\rho > 0$ such that

$$\int_{|y|<\rho} K(y)\,d\mu(y) < \varepsilon.$$

Now we consider the integration

$$\int K(x-y)\, q_\alpha(x)\ dx.$$

If $|x-y| \geq |y|/2$, then $K(x-y) \leq K(y/2)$. Therefore

$$\int_{|x-y|\geq|y|/2} K(x-y)\, q_\alpha(x)\ dx \leq K(y/2)\,.$$

If $|x-y| < |y|/2$, then $|y|/2 \leq |x|$. Therefore

$$\begin{aligned}\int_{|x-y|<|y|/2} K(x-y)\, q_\alpha(x)\ dx &\leq \int_{|x-y|<|y|/2} \frac{K(x-y)\, K(x)}{\int_{|z|<|x|} K(z)\ dz}\, dx \\ &\leq K(y/2) \int_{|x-y|<|y|/2} \frac{K(x-y)}{\int_{|z|<|y|/2} K(z)\ dz}\, dx = K(y/2)\,.\end{aligned}$$

Hence, using the doubling condition, we have

$$\int K(x-y)\, q_\alpha(x)\ dx \leq 2K(y/2) \leq 2CK(y)\,.$$

Therefore

$$\int_{|y|<\rho} \int K(x-y)\, q_\alpha(x)\ dx\, d\mu(y) \leq 2C \int_{|y|<\rho} K(y)\ d\mu(y) \leq 2C\varepsilon.$$

Since K is uniformly continuous in $\{y\,;\,|y| \geq \rho\} \cap \operatorname{supp}\mu$, there is an α such that

$$|K(y-x) - K(y)| < \varepsilon \quad \text{if } |x| \leq \alpha$$

for any $y \in \{y\,;\,|y| \geq \rho\} \cap \operatorname{supp}\mu$. Therefore

$$\left|\int K(y-x)\, q_\alpha(x)\ dx - K(y)\right| \leq \int |K(y-x) - K(y)|\, q_\alpha(x)\ dx \leq \varepsilon$$

for any $y \in \{y\,;\,|y| \geq \rho\} \cap \operatorname{supp}\mu$. Hence

$$\begin{aligned}&\left|\int_{|y|\geq\rho} \int K(y-x)\, q_\alpha(x)\ dx\, d\mu(y) - \int_{|y|\geq\rho} K(y)\ d\mu(y)\right| \\ &\leq \int_{|y|\geq\rho} \varepsilon\, d\mu(y) \leq \varepsilon\mu(F) = \varepsilon C_{\mathrm{W}} C_K(F)\,.\end{aligned}$$

Therefore

$$\left|\iint K(y-x)\, q_\alpha(x)\ dx\, d\mu(y) - K\mu(0)\right| \leq (2C + 1 + C_{\mathrm{W}} C_K(F))\,\varepsilon.$$

Hence

$$\lim_{\alpha\to 0} \iint K(y-x)\, q_\alpha(x)\ dx\, d\mu(y) = K\mu(0)\,.$$

On the other hand, since $K\mu \geq 1$ q.e. on F and the potential of the Lebesgue measure is bounded, we have $K\mu \geq 1$ a.e. on F. Therefore

$$\iint K(y-x)\, q_\alpha(x)\ dx\, d\mu(y) = \int K\mu(x)\, q_\alpha(x)\ dx \geq \int q_\alpha(x)\ dx = 1.$$

Hence the theorem is proved. $\square$

4.3. **Extremal problems.**

Theorem 4.3. *Let F be a compact set with $C_K(F) > 0$. Let*

$$A = \inf\left\{\nu\left(\mathbb{R}^N\right) \ ; \ \nu \in \mathfrak{M}^+,\ K\nu \geq 1 \ \text{q.e. on } F\right\},$$

$$B = \sup\left\{\nu(F) \ ; \ \nu \in \mathfrak{M}^+,\ K\nu \leq 1 \ \text{q.e. on } F,\ K\nu \text{ is bounded}, \operatorname{supp}\nu \subset F\right\}.$$

Then

$$C_K(F)/C_{\mathrm{W}} \leq A \leq C_{\mathrm{W}} C_K(F),$$
$$C_K(F)/C_{\mathrm{W}} \leq B \leq C_{\mathrm{W}} C_K(F).$$

Proof. Let μ be an equilibrium measure for F. Let $\nu_1 \in \mathfrak{M}^+$ such that $K\nu_1 \geq 1$ q.e. on F. Since $K\mu$ is bounded, we have $K\nu_1 \geq 1$ μ-a.e. Therefore

$$C_K(F) \leq \mu(F) \leq \int_F K\nu_1\, d\mu = \int K\mu\, d\nu_1 \leq C_{\mathrm{W}} \nu_1\left(\mathbb{R}^N\right).$$

Hence $C_K(F) \leq C_{\mathrm{W}} A$.

Since μ satisfies $K\mu \geq 1$ q.e. on F,

$$A \leq \mu(F) \leq C_{\mathrm{W}} C_K(F).$$

Let $\nu_2 \in \mathfrak{M}^+$ such that $K\nu_2 \leq 1$ q.e. on F, $K\nu_2$ is bounded and $\operatorname{supp}\nu_2 \subset F$. Then $K\nu_2 \leq 1$ μ-a.e. Also since $K\mu \geq 1$ q.e. on F, we have $K\mu \geq 1$ ν_2-a.e. Therefore

$$C_{\mathrm{W}} C_K(F) \geq \mu(F) \geq \int_F K\nu_2\, d\mu = \int_F K\mu\, d\nu_2 \geq \nu_2(F).$$

Hence $C_{\mathrm{W}} C_K(F) \geq B$.

Since μ/C_{W} satisfies $K(\mu/C_{\mathrm{W}}) \leq 1$ q.e. on F, $K(\mu/C_{\mathrm{W}})$ is bounded and $\operatorname{supp}(\mu/C_{\mathrm{W}}) \subset F$,

$$C_K(F)/C_{\mathrm{W}} \leq \mu(F)/C_{\mathrm{W}} \leq B.$$

$\square$

Lemma 4.4. *If $K\nu \leq 1$ q.e. on $\operatorname{supp}\nu$ and $K\nu$ is bounded, then $K\nu \leq 1$ on $\operatorname{supp}\nu$.*

Proof. Let $E = \{x \in \operatorname{supp}\nu \ ; \ K\nu(x) > 1\}$ and suppose that $x \in E$. We can find a neighborhood O of x such that $K\nu > 1$ on O. Since $C_K(E) = 0$ and $K\nu$ is bounded, Lemma 4.1 implies that $\nu(E) = 0$, and thus $\nu(O) = 0$, which is a contradiction. $\square$

Lemma 4.5. *Suppose that $N \geq 2$. Also suppose that $K(r)$ is absolutely continuous, $K'(r)\, r^{N-1}$ is increasing and that $K(r) = 0$ for sufficiently large r. Then the Fourier transformation of K is strictly positive, i.e.*

$$\hat{K}(\xi) := \int K(x)\, e^{-i\langle \xi, x\rangle}\, dx > 0 \quad \text{for any } \xi$$

where $\langle \cdot, \cdot \rangle$ is the inner product.

Proof. It is easy to see that $\hat{K}(0) > 0$, therefore we may assume that $\xi \neq 0$. Without loss of generality we may assume that $|\xi| = 1$. Let $r = |x|$ and φ the angle between x and ξ. Then

$$\begin{aligned} \hat{K}(\xi) &= c\int_0^\infty \int_0^\pi K(r)\, e^{-ir\cos\varphi} r^{N-1} \sin^{N-2}\varphi \, d\varphi \, dr \\ &= c\int_0^\infty K(r)\, r^{N-1} \int_{-\pi/2}^{\pi/2} e^{ir\sin\theta} \cos^{N-2}\theta \, d\theta \, dr \\ &= 2c\int_0^\infty K(r)\, r^{N-1} \int_0^{\pi/2} \cos(r\sin\theta) \cos^{N-2}\theta \, d\theta \, dr \end{aligned} \tag{4.1}$$

where c is a positive constant.

Let

$$J(r) = \int_0^{\pi/2} \cos(r\sin\theta)\cos^{N-2}\theta \, d\theta.$$

Then

$$J'(r) = -\int_0^{\pi/2} \sin(r\sin\theta)\sin\theta\cos^{N-2}\theta \, d\theta,$$

$$J''(r) = -\int_0^{\pi/2} \cos(r\sin\theta)\sin^2\theta\cos^{N-2}\theta \, d\theta.$$

Therefore

(4.2)

$$\begin{aligned} & rJ''(r) + (N-1)\,J'(r) + rJ(r) \\ &\quad = r\int_0^{\pi/2} \cos(r\sin\theta)\cos^N\theta \, d\theta - (N-1)\int_0^{\pi/2} \sin(r\sin\theta)\sin\theta\cos^{N-2}\theta \, d\theta \\ &\quad = \left[\sin(r\sin\theta)\cos^{N-1}\theta\right]_0^{\pi/2} = 0. \end{aligned}$$

Since $|\sin\varphi| \leq |\varphi|$, we have

$$|J'(r)| \leq \int_0^{\pi/2} r\sin^2\theta\cos^{N-2}\theta \, d\theta \leq \frac{1}{2}\pi r.$$

On the other hand, since

$$\int_0^r K(t)\, t^{N-1}\, dt \geq K(r) \int_0^r t^{N-1}\, dt = \frac{1}{N} K(r)\, r^N,$$

and we assume that $\int_0^r K(t)\, t^{N-1}\, dt < \infty$, we have

$$\lim_{r\to 0} K(r)\, r^N = 0. \tag{4.3}$$

Therefore

$$\lim_{r\to 0} \left|K(r)\, r^{N-1} J'(r)\right| \leq \frac{\pi}{2} \lim_{r\to 0} K(r)\, r^N = 0.$$

We have by (4.1) and (4.2)

$$\hat{K}(\xi) = 2c\int_0^\infty K(r)\, r^{N-1} J(r)\; dr$$

$$
\begin{aligned}
&= -2c \int_0^\infty K(r) r^{N-1} \left(J''(r) + \frac{N-1}{r} J'(r) \right) dr \\
&= -2c \int_0^\infty \left(K(r) r^{N-1} J''(r) + K(r)(N-1) r^{N-2} J'(r) \right) dr \\
&= -2c \left[K(r) r^{N-1} J'(r) \right]_{r=0}^\infty + 2c \int_0^\infty K'(r) r^{N-1} J'(r)\, dr \\
&= 2c \int_0^\infty K'(r) r^{N-1} J'(r)\, dr.
\end{aligned}
$$

Now we shall show that $\liminf_{r\to 0} \left(-K'(r) r^{N+1}\right) = 0$. If not, there are $c_0 > 0$ and $r_0 > 0$ such that

$$-K'(r) r^{N+1} \geq c_0 \quad \text{for } 0 < r < r_0.$$

Therefore

$$K(r) - K(r_0) = -\int_r^{r_0} K'(t)\, dt \geq c_0 \int_r^{r_0} t^{-N-1}\, dt = \frac{c_0}{N} \left(r^{-N} - r_0^{-N} \right),$$

and thus

$$r^N \left(K(r) - K(r_0) \right) \geq \frac{c_0}{N} \left(1 - r^N r_0^{-N} \right).$$

The equation (4.3) shows that $0 \geq c_0/N$, which is a contradiction. Hence we can find a sequence $\{r_j\}_j$ such that $r_j \searrow 0$ and

$$-K'(r_j) r_j^{N+1} \to 0 \quad \text{as } j \to \infty.$$

Since

$$J(0) - J(r) = \int_0^{\pi/2} \left(1 - \cos(r \sin\theta)\right) \cos^{N-2}\theta\, d\theta > 0 \quad \text{for } r > 0,$$

we have

$$\lim_{j\to\infty} \int_{r_j}^\infty \left(J(0) - J(r) \right) d\left(K'(r) r^{N-1} \right) = \int_0^\infty \left(J(0) - J(r) \right) d\left(K'(r) r^{N-1} \right).$$

Since $J'(0) = 0$, we have $J(r) = J(0) + O(r^2)$, and thus

$$\left(J(0) - J(r_j) \right) K'(r_j) r_j^{N-1} = O\left(K'(r_j) r_j^{N+1} \right) \to 0.$$

Since $J'(r) \leq 0$ for sufficiently small r and $K'(r) \leq 0$,

$$\int_{r_j}^\infty J'(r) K'(r) r^{N-1}\, dr \to \int_0^\infty J'(r) K'(r) r^{N-1}\, dr = \frac{1}{2c} \hat{K}(\xi) \quad \text{as } j \to \infty.$$

Therefore

$$
\begin{aligned}
0 &< \int_0^\infty \left(J(0) - J(r) \right) d\left(K'(r) r^{N-1} \right) = \lim_{j\to\infty} \int_{r_j}^\infty \left(J(0) - J(r) \right) d\left(K'(r) r^{N-1} \right) \\
&= \lim_{j\to\infty} \left(\left[\left(J(0) - J(r) \right) K'(r) r^{N-1} \right]_{r_j}^\infty + \int_{r_j}^\infty J'(r) K'(r) r^{N-1}\, dr \right) = \frac{1}{2c} \hat{K}(\xi).
\end{aligned}
$$

Hence we have the lemma. □

Lemma 4.6. *Suppose that $N \geq 2$. Also suppose that $K(r)$ is absolutely continuous, $K'(r)\, r^{N-1}$ is increasing and that $K(r) = 0$ for sufficiently large r. Let $\sigma \in \mathfrak{M}$ with compact support such that $I_K(|\sigma|) < \infty$ and the total variation $\|\sigma\|$ is finite. Then $I_K(\sigma) \geq 0$, and the equality holds if and only if $\sigma \equiv 0$.*

Proof. From the assumption $I_K(\sigma) = \int K\sigma\, d\sigma$ is finite, thus $K\sigma$ can be defined at $|\sigma|$-a.e. points. The Fourier transformation of σ is

$$\hat{\sigma}(\xi) = \int e^{-i\langle \xi, x\rangle}\, d\sigma(x).$$

Now let

$$\Phi_n(x) = (n/\pi)^{N/2} \exp\left(-n|x|^2\right).$$

It is easy to see that

$$\hat{\Phi}_n(\xi) = \exp\left(-|\xi|^2/(4n)\right),$$

which is a positive and integrable function. Since $\left|\hat{K}(\xi)\right| \leq \int K(x)\, dx < \infty$ and $|\hat{\sigma}(\xi)| \leq \int d|\sigma| < \infty$, we have $\hat{\Phi}_n \hat{K} \hat{\sigma}$ is also integrable. Therefore

$$\begin{aligned}\int \hat{\Phi}_n(\xi)\, \hat{K}(\xi)\, |\hat{\sigma}(\xi)|^2\, d\xi &= \int \hat{\Phi}_n(\xi)\, \hat{K}(\xi)\, \hat{\sigma}(\xi) \int e^{i\langle \xi, y\rangle}\, d\sigma(y)\, d\xi \\ &= \iint \hat{\Phi}_n(\xi)\, \hat{K}(\xi)\, \hat{\sigma}(\xi)\, e^{i\langle \xi, y\rangle}\, d\xi\, d\sigma(y).\end{aligned}$$

Here

$$\begin{aligned}\Phi_n * K\sigma(y) &= (2\pi)^{-N} \int \hat{\Phi}_n(\xi)\, \widehat{K\sigma}(\xi)\, e^{i\langle \xi, y\rangle}\, d\xi \\ &= (2\pi)^{-N} \int \hat{\Phi}_n(\xi)\, \hat{K}(\xi)\, \hat{\sigma}(\xi)\, e^{i\langle \xi, y\rangle}\, d\xi \quad \text{for a.e. } y.\end{aligned}$$

Since the potential of the Lebesgue measure is bounded, $K\sigma$ is integrable. Thus the both sides are continuous, therefore the above holds everywhere. Hence

$$\int \hat{\Phi}_n(\xi)\, \hat{K}(\xi)\, |\hat{\sigma}(\xi)|^2\, d\xi = (2\pi)^N \int \Phi_n * K\sigma(y)\, d\sigma(y). \tag{4.4}$$

Now we assume that $K\sigma$ is continuous. Since $\operatorname{supp}\sigma$ is compact, $K\sigma$ is bounded. Therefore $\Phi_n * K\sigma$ converges to $K\sigma$ as $n \to \infty$. Since $\Phi_n * K\sigma$ and $K\sigma$ are continuous, we have that $\Phi_n * K\sigma$ converges locally uniformly to $K\sigma$. Also we have $\hat{\Phi}_n \hat{K} |\hat{\sigma}|^2$ converges increasingly to $\hat{K} |\hat{\sigma}|^2$. Therefore (4.4) becomes

$$\int \hat{K}(\xi)\, |\hat{\sigma}(\xi)|^2\, d\xi = (2\pi)^N \int K\sigma(y)\, d\sigma(y) = (2\pi)^N I_K(\sigma). \tag{4.5}$$

Next we consider the general case. By Lemma 3.4, there exists a closed set F_m such that $|\sigma|\left(\mathbb{R}^N \setminus F_m\right) < 1/m$ and $K|\sigma|\,|_{F_m}$ continuous everywhere. We may assume that F_m increases as m increases. Let $\sigma_m = \sigma|_{F_m}$ and divide it into two parts, $\sigma_m = \sigma_m^+ - \sigma_m^-$. Then, since $K\sigma_m^+$ and $K\sigma_m^-$ continuous everywhere, thus $K\sigma_m = K\sigma_m^+ - K\sigma_m^-$ is also continuous.

Let $\tau_m = \sigma - \sigma_m$. Then

$$2I_K(\sigma, \tau_m) - I_K(\tau_m) = 2I_K(\sigma) - 2I_K(\sigma, \sigma_m) - (I_K(\sigma) - 2I_K(\sigma, \sigma_m) + I_K(\sigma_m))$$
$$= I_K(\sigma) - I_K(\sigma_m).$$

Since $|\tau_m| \leq |\sigma|$,

$$|I_K(\sigma) - I_K(\sigma_m)| \leq 2I_K(|\sigma|, |\tau_m|) + I_K(|\tau_m|) \leq 3I_K(|\sigma|, |\tau_m|).$$

Since $|\sigma|\left(\mathbb{R}^N \setminus F_m\right) \to 0$ and $\int K|\sigma|\, d|\sigma| < \infty$,

$$I_K(|\sigma|, |\tau_m|) = \int_{\mathbb{R}^N \setminus F_m} K|\sigma|\, d|\sigma| \to 0 \quad \text{as } m \to \infty,$$

and thus

$$\lim_{m\to\infty} I_K(\sigma_m) = I_K(\sigma).$$

Since

$$|\hat{\sigma}(\xi) - \hat{\sigma}_m(\xi)| \leq \int d|\tau_m| < 1/m,$$

$\hat{\sigma}_m$ converges uniformly to $\hat{\sigma}$ as $m \to \infty$. We apply (4.5) for σ_m and obtain

$$\int \hat{K}(\xi) |\hat{\sigma}(\xi)|^2\, d\xi = \int \liminf_{m\to\infty} \hat{K}(\xi) |\hat{\sigma}_m(\xi)|^2\, d\xi \leq \liminf_{m\to\infty} \int \hat{K}(\xi) |\hat{\sigma}_m(\xi)|^2\, d\xi$$
$$= (2\pi)^N \liminf_{m\to\infty} I_K(\sigma_m) = (2\pi)^N I_K(\sigma).$$

Therefore $I_K(\sigma) \geq 0$. If $I_K(\sigma) = 0$ and $\hat{\sigma}(\xi) \neq 0$ for some ξ, then $\hat{\sigma} \neq 0$ in some neighborhood of ξ, and thus the above equation implies that $\hat{K}(\xi) = 0$, which contradict Lemma 4.5. Therefore $\hat{\sigma} \equiv 0$, and thus $\sigma \equiv 0$. □

Theorem 4.4. *Suppose that $N \geq 2$. Also suppose that $K(r)$ is absolutely continuous and that $K'(r) r^{N-1}$ is increasing. Let F be a compact set with $C_K(F) > 0$. Then there uniquely exists the measure μ_0 which minimizes*

$$\left\{I_K(\mu)\ ;\ \mu \in \mathfrak{M}^+, \operatorname{supp}\mu \subset F, \mu(F) = 1\right\}$$

(cf. Lemma 4.3). Also there uniquely exists the measure ν_0 which maximizes

$$\left\{\nu(F)\ ;\ \nu \in \mathfrak{M}^+,\ K\nu \leq 1 \textit{ q.e. on } F,\ K\nu \textit{ is bounded}, \operatorname{supp}\nu \subset F\right\}$$

(cf. Theorem 4.3), and they satisfies

$$\nu_0 = C_K(F)\, \mu_0.$$

Proof. Let $\phi(r) = -K'(r) r^{N-1}$. Then ϕ is non-negative decreasing function which satisfies

$$K(r) = \int_r^\infty \phi(t)\, t^{1-N}\, dt.$$

Let $r_0 = 2 \operatorname{diam} F$ and take $r_1 > r_0$ such that

$$\int_{r_0}^{r_1} \phi(r_0)\, t^{1-N}\, dt = \int_{r_0}^\infty \phi(t)\, t^{1-N}\, dt.$$

Let

$$\phi_1(r) = \begin{cases} \phi(r) & \text{if } r < r_0, \\ \phi(r_0) & \text{if } r_0 \le r < r_1, \\ 0 & \text{if } r_1 \le r \end{cases} \qquad \text{and} \qquad K_1(r) = \int_r^\infty \phi_1(t)\, t^{1-N}\, dt.$$

Then $K_1(r) = K(r)$ if $r < r_0$ and $K_1(r) = 0$ if $r_1 < r$. Then we see easily that $C_K(F) = C_{K_1}(F)$ and $K\mu = K_1\mu$ on F and $I_K(\mu) = I_{K_1}(\mu)$ for any measure μ whose support is in F. Note that $C_{\mathrm{W}} = 1$ by Theorem 3.3

Let μ_1 and μ_2 be measures which minimize

$$\left\{ I_K(\mu)\ ;\ \mu \in \mathfrak{M}^+,\, \operatorname{supp}\mu \subset F,\, \mu(F) = 1 \right\}.$$

By Lemma 4.3

$$K_1\mu_1 = C_{K_1}(F)^{-1} \quad \text{q.e. on } F$$

and $K_1\mu_2$ is bounded, thus the above holds μ_2-a.e. Therefore

$$I_{K_1}(\mu_1, \mu_2) = \int C_{K_1}(F)^{-1}\, d\mu_2 = C_{K_1}(F)^{-1}.$$

Hence

$$I_{K_1}(\mu_1 - \mu_2) = I_{K_1}(\mu_1) - 2I_{K_1}(\mu_1, \mu_2) + I_{K_1}(\mu_2) = 0.$$

Also we have

$$I_{K_1}(|\mu_1 - \mu_2|) \le I_{K_1}(\mu_1 + \mu_2) = I_{K_1}(\mu_1) + 2I_{K_1}(\mu_1, \mu_2) + I_{K_1}(\mu_2) < \infty.$$

Therefore Lemma 4.6 implies $\mu_1 = \mu_2$, *i.e.* the minimizing measure is unique.

Let ν be a measure which maximizes

$$\left\{ \nu(F)\ ;\ \nu \in \mathfrak{M}^+,\, K\nu \le 1 \text{ q.e. on } F,\, K\nu \text{ is bounded, } \operatorname{supp}\nu \subset F \right\}.$$

And let

$$\mu = C_{K_1}(F)^{-1}\,\nu.$$

Then $\mu \in \mathfrak{M}^+$ such that $\operatorname{supp}\mu \subset F$ and $\mu(F) = 1$. Since $K_1\nu \le 1$ q.e. on F, we have $K_1\nu \le 1$ ν-a.e. Therefore

$$I_{K_1}(\mu) = C_{K_1}(F)^{-2}\, I_{K_1}(\nu) \le C_{K_1}(F)^{-1}.$$

This means μ minimizes

$$\left\{ I_K(\mu)\ ;\ \mu \in \mathfrak{M}^+,\, \operatorname{supp}\mu \subset F,\, \mu(F) = 1 \right\}.$$

Hence we have the theorem. □

Example 4.1. *A measure which minimizes*

$$\left\{ \nu\left(\mathbb{R}^N\right)\ ;\ \nu \in \mathfrak{M}^+,\, K\nu \ge 1 \text{ q.e. on } F \right\}$$

need not be unique (cf. Theorem 4.3).

Proof. Let $F = \{x \in \mathbb{R}^2\ ;\ |x| = 1\}$ and $K(r) = \log^+(2/r)$. Let μ be a measure on F such that

$$d\mu = d\theta / (2\pi \log 2).$$

Then

$$K\mu(x) = (2\pi \log 2)^{-1} \int_0^{2\pi} \log^+ \left(2/\left|x - e^{i\theta}\right|\right) d\theta$$

depends only on $|x|$. Since $K\mu$ is harmonic in $|x| < 1$,

$$K\mu(0) = (2\pi)^{-1} \int_0^{2\pi} K\mu\left(re^{it}\right) dt = K\mu\left(re^{i\alpha}\right) \quad \text{for any } 0 < r < 1 \text{ and any } \alpha.$$

Thus $K\mu$ is constant in $|x| < 1$. Also we have

$$K\mu(0) = (2\pi \log 2)^{-1} \int_0^{2\pi} \log 2\, d\theta = 1.$$

Therefore $K\mu = 1$ on $|x| < 1$. Since $K\mu$ is lower semi-continuous, we have $K\mu \le 1$ on $|x| = 1$. On the other hand, since

$$K'(r)\, r = \begin{cases} -1 & \text{if } r < 2, \\ 0 & \text{if } r > 2, \end{cases}$$

K satisfies the strong maximum principle. Hence $K\mu = 1$ on $|x| = 1$. This means that $\mu \in \mathfrak{M}^+$ satisfies $K\mu \ge 1$ q.e. on F. Thus Theorem 4.3 implies

$$C_K(F) \le \mu\left(\mathbb{R}^N\right) = (\log 2)^{-1}.$$

Also, since $\mu \in \mathfrak{M}^+$, $\operatorname{supp} \mu \subset E$ and $K\mu \le 1$ on $\operatorname{supp} \mu$,

$$C_K(F) \ge \mu(F) = (\log 2)^{-1}.$$

Therefore

$$C_K(F) = (\log 2)^{-1}$$

and μ is a minimizing measure.

Next let

$$\nu = \delta / \log 2$$

where δ is the Dirac measure at the origin. When $|x| \le 1$, we have

$$K\nu(x) = K(x) / \log 2 \ge 1,$$

i.e. $\nu \in \mathfrak{M}^+$ such that $K\nu \ge 1$ q.e. on F. Also we have

$$\nu\left(\mathbb{R}^N\right) = (\log 2)^{-1}.$$

This means that ν is also a minimizing measure. □

4.4. **The Choquet capacity.**

Definition 4.3 (Choquet capacity)**.** A set function c is called a *Choquet capacity* if it satisfies the following :

(i) $0 \le c(E) \le \infty$ for any E.
(ii) if $E_1 \subset E_2$, then $c(E_1) \le c(E_2)$.
(iii) if $E_n \nearrow E$, then $c(E_n) \to c(E)$.
(iv) if E_n is compact and $E_n \searrow E$, then $c(E_n) \to c(E)$.

Definition 4.4 (Capacitable)**.** A set E is called to be *c-capacitable* if

$$c(E) = \sup\{c(F)\ ;\ F \text{ is compact, } F \subset E\}.$$

Definition 4.5 (C_K^*)**.** For a set E

$$C_K^*(E) := \inf\{C_K(O)\ ;\ O \text{ is open, } E \subset O\}.$$

We shall show that C_K^* is a Choquet capacity under some assumptions. It is clear that C_K^* satisfies the conditions (i) and (ii) of Definition 4.3.

Lemma 4.7. *For any set E*

$$C_K(E) = \sup\{C_K(F)\ ;\ F \textit{ is compact, } F \subset E\}.$$

Proof. Let $\mu \in \mathfrak{M}^+$ such that $\operatorname{supp}\mu \subset E$ and $K\mu \le 1$ everywhere. Also let $\nu = \mu|_{\overline{B(0,R)}}$ and $F = \operatorname{supp}\nu$. Then F is a compact set in E and $K\nu \le 1$ everywhere. Therefore

$$\sup_F C_K(F) \ge C_K(F) \ge \nu(F) = \mu\left(E \cap \overline{B(0,R)}\right).$$

Letting $R \to \infty$, we have

$$\sup_F C_K(F) \ge \mu(E).$$

Hence

$$\sup_F C_K(F) \ge C_K(E).$$

The opposite is trivial, and we have the lemma. □

Lemma 4.8. *For any compact set F*

$$C_K(F) = C_K^*(F).$$

Proof. Let $O_n = \{x\ ;\ \operatorname{dist}(x, F) < 1/n\}$. We can find $\mu_n \in \mathfrak{M}^+$ such that $\operatorname{supp}\mu_n \subset O_n$, $K\mu_n \le 1$ everywhere and

$$\mu_n(O_n) > C_K(O_n) - 1/n.$$

Since O_1 is bounded, we have $C_K(O_1) < \infty$, and thus $\left\{\mu_n\left(\mathbb{R}^N\right)\right\}_n$ is bounded. Therefore by taking a subsequence we may assume that $\{\mu_n\}_n$ converges weakly to a measure μ. Then Lemma 3.1 (ii) gives

$$K\mu(x) \le \liminf_{n\to\infty} K\mu_n(x) \le 1.$$

Since $\operatorname{supp}\mu \subset F$,

$$C_K(F) \geq \mu(F) = \lim_{n\to\infty} \mu_n(O_n) \geq \lim_{n\to\infty} (C_K(O_n) - 1/n) = \lim_{n\to\infty} C_K(O_n) \geq C_K^*(F).$$

The opposite is clear, and we have the lemma. □

Theorem 4.5. *Let $\{F_n\}_n$ be a decreasing sequence of compact sets which converges to E. Then*

$$C_K^*(F_n) \to C_K^*(E),$$

i.e. C_K^* satisfies the condition (iv) *of Definition 4.3.*

Proof. Let O be an open set containing E. Then $F_n \subset O$ for sufficiently large n. Therefore

$$\lim_{n\to\infty} C_K^*(F_n) \leq C_K^*(F_n) \leq C_K(O).$$

Hence

$$\lim_{n\to\infty} C_K^*(F_n) \leq C_K^*(E).$$

The opposite is clear, and we have the theorem. □

Lemma 4.9. *For any sets $\{E_n\}_n$*

$$C_K^*\left(\bigcup_n E_n\right) \leq \sum_n C_K^*(E_n).$$

Proof. For any $\varepsilon > 0$ we find an open set O_n containing E_n such that

$$C_K(O_n) \leq C_K^*(E_n) + 2^{-n}\varepsilon.$$

Then Lemma 4.2 implies

$$C_K^*\left(\bigcup_n E_n\right) \leq C_K\left(\bigcup_n O_n\right) \leq \sum_n C_K(O_n) \leq \sum_n C_K^*(E_n) + \varepsilon.$$

Since ε is arbitrary, we have the lemma. □

Lemma 4.10. *Suppose that K is continuous on $(0, \infty)$. Let $\mu \in \mathfrak{M}^+$ with finite mass such that $K\mu < \infty$ μ-a.e. For given $\varepsilon > 0$ there is an open set O such that $C_K(O) < \varepsilon$ and $K\mu$ is continuous outside O.*

Proof. Take $\{n_j\}_j$ and $\{\delta_j\}_j$ such that $n_j \to \infty$, $\delta_j \to 0$ and $\sum_j n_j\delta_j < \varepsilon/C_{\mathrm{W}}$. For each j Lemma 3.4 gives that there exists a restricted measure μ_j of μ such that $K\mu_j$ is continuous and $\nu_j(\mathbb{R}^N) < \delta_j$ where $\nu_j = \mu - \mu_j$. Let

$$O_j = \{x\ ;\ K\nu_j(x) > 1/n_j,\ |x| < n_j\},$$

and let F be a compact set $\subset O_j$. Since $K(n_j\nu_j) = n_jK\nu_j > 1$ on F, Theorem 4.3 yields

$$C_K(F)/C_{\mathrm{W}} \leq n_j\nu_j(\mathbb{R}^N) < n_j\delta_j.$$

Therefore Lemma 4.7 implies

$$C_K(O_j) \leq C_{\mathrm{W}} n_j\delta_j.$$

If we set $O = \bigcup_j O_j$, then by Lemma 4.2

$$C_K(O) \le \sum_j C_K(O_j) \le C_{\mathrm{W}} \sum_j n_j \delta_j < \varepsilon.$$

Now let $x \notin O$. Then $x \notin O_j$ for each j. Since $|x| < n_j$ for sufficiently large j,

$$K\nu_j(x) \le 1/n_j \quad \text{for sufficiently large } j.$$

Hence for any $x_0 \notin O$

$$\begin{aligned}&\limsup_{x\to x_0,\, x\notin O} |K\mu(x) - K\mu(x_0)| \\ &\quad\le \limsup_{x\to x_0,\, x\notin O} |K\mu_j(x) - K\mu_j(x_0)| + \limsup_{x\to x_0,\, x\notin O} |K\nu_j(x) - K\nu_j(x_0)| \\ &\quad\le 0 + 2/n_j.\end{aligned}$$

Letting $j \to \infty$ we have the lemma. □

Lemma 4.11. *Suppose that K is continuous on $(0, \infty)$. Let $\{\mu_n\}_n \subset \mathfrak{M}^+$ and $\mu \in \mathfrak{M}^+$ such that $\{\mu_n(\mathbb{R}^N)\}_n$ is bounded, $\bigcup_n \operatorname{supp}\mu_n$ is bounded, $K\mu_n < \infty$ μ_n-a.e., $K\mu < \infty$ μ-a.e. and $\mu_n \rightharpoonup \mu$. Then there is a set E such that $C_K^*(E) = 0$ and*

$$\liminf_{n\to\infty} K\mu_n(x) = K\mu(x) \quad \textit{for } x \notin E.$$

Proof. For each m we can find an open set O_m such that $K\mu$ and $K\mu_n$'s are continuous outside O_m and $C_K(O_m) < 1/m$ (By Lemma 4.10 we find an open set for each of μ and μ_n's, and we set O_m to the union of them).

Let

$$F_{nr\rho m} = \{x\,;\ K\mu(x) \le r,\ K\mu_n(x) \ge \rho,\ x \notin O_m\}$$

for rational numbers r and ρ with $r < \rho$, and let

$$G_{nr\rho m} = \bigcap_{k=n}^{\infty} F_{kr\rho m}.$$

Take x outside the closure of $\bigcup_n \operatorname{supp}\mu_n$. Since $K(x-\cdot)$ is continuous in $\bigcup_n \operatorname{supp}\mu_n$,

$$\lim_{n\to\infty} K\mu_n(x) = \lim_{n\to\infty} \int K(x-y)\, d\mu_n(y) = \int K(x-y)\, d\mu(y) = K\mu(x).$$

Therefore $x \notin F_{nr\rho m}$ for sufficiently large n, and thus $x \notin G_{nr\rho m}$. Hence $G_{nr\rho m}$ is compact.

If $C_K(G_{nr\rho m}) > 0$, then, using Lemma 3.4, we can find a positive measure $\nu \in \mathfrak{M}^+$ such that $\operatorname{supp}\nu \subset G_{nr\rho m}$ and $K\nu$ is continuous. Since $\mu_k \rightharpoonup \mu$,

$$0 = \lim_{k\to\infty} \int K\nu\, d(\mu_k - \mu) = \lim_{k\to\infty} \int (K\mu_k - K\mu)\, d\nu \ge (\rho - r)\,\nu(\mathbb{R}^N),$$

which is a contradiction. Therefore $C_K(G_{nr\rho m}) = 0$. Hence Lemma 4.8 implies

$$C_K^*(G_{nr\rho m}) = 0.$$

Let

$$E = \left(\bigcup_{n,\, r,\, \rho,\, m} G_{nr\rho m} \right) \cup \left(\bigcap_m O_m \right).$$

Since $C_K^* \left(\bigcap_m O_m\right) \le C_K\left(O_m\right) < 1/m$ for any m, we have $C_K^* \left(\bigcap_m O_m\right) = 0$. Therefore by Lemma 4.9 we have

$$C_K^*\left(E\right) \le \sum_{n,\, r,\, \rho,\, m} C_K^*\left(G_{nr\rho m}\right) + C_K^*\left(\bigcap_m O_m\right) = 0.$$

Let x be a point such that $\liminf_{n\to\infty} K\mu_n\left(x\right) > K\mu\left(x\right)$. Then there are r and ρ such that

$$K\mu_n\left(x\right) \ge \rho > r \ge K\mu\left(x\right) \quad \text{for sufficiently large } n.$$

If $x \in O_m$ for any m, then $x \in \bigcap_m O_m \subset E$. Otherwise we can find an m with $x \notin O_m$, thus $x \in F_{nr\rho m}$. Therefore $x \in G_{nr\rho m} \subset E$. Hence if $x \notin E$ then $\liminf_{n\to\infty} K\mu_n\left(x\right) \le K\mu\left(x\right)$. Lemma 3.1 (ii) implies the result. □

Lemma 4.12. *Suppose that the strong maximum principle holds and K is continuous on $(0, \infty)$. Let O be a bounded open set. Then there exist a measure $\mu \in \mathfrak{M}^+$ and a set E such that $\operatorname{supp}\mu \subset \bar{O}$, $K\mu \le 1$, $\mu\left(\mathbb{R}^N\right) = C_K\left(O\right)$, $C_K^*\left(E\right) = 0$ and $K\mu = 1$ on $O \setminus E$, .*

Proof. By Lemma 4.7 we can find a sequence $\{F_n\}_n$ of compact sets such that $F_n \nearrow O$ and $\mu_n\left(F_n\right) \to C_K\left(O\right)$ where μ_n is a equilibrium measure for F_n. By taking a subsequence we may assume that $\{\mu_n\}_n$ converges weakly to a measure μ. We have

$$K\mu\left(x\right) \le \liminf_{n\to\infty} K\mu_n\left(x\right) \le 1$$

and

$$\mu\left(\mathbb{R}^N\right) = \lim_{n\to\infty} \mu_n\left(\mathbb{R}^N\right) = C_K\left(O\right).$$

Let $U_{nk} = \{x \in F_n\ ;\ K\mu_n\left(x\right) \le 1 - 1/k\}$ and $U = \bigcup_{n,\, k} U_{nk}$. Then, since U_{nk} is compact and $C_K\left(U_{nk}\right) = 0$,

$$C_K^*\left(U\right) \le \sum_{n,\, k} C_K^*\left(U_{nk}\right) = \sum_{n,\, k} C_K\left(U_{nk}\right) = 0.$$

Also let V be an exceptional set of Lemma 4.11, and let $E = U \cup V$. Then

$$C_K^*\left(E\right) \le C_K^*\left(U\right) + C_K^*\left(V\right) = 0,$$

and

$$K\mu\left(x\right) = \liminf_{n\to\infty} K\mu_n\left(x\right) = 1 \quad \text{for } x \in O \setminus E.$$

Thus we have the lemma. □

Theorem 4.6. *Suppose that the strong maximum principle holds and K is continuous on $(0, \infty)$. Then C_K^* is a Choquet capacity.*

Proof. We have only to prove that C_K^* satisfies the condition (iii) of Definition 4.3, *i.e.* if $E_n \nearrow E$, then $C_K^*(E) = \lim_{n\to\infty} C_K^*(E_n)$.

We can find an open set $O_n \supset E_n$ and

$$C_K(O_n) \le C_K^*(E_n) + 1/n.$$

By Lemma 4.12 we find a measure μ_n and a set U_n such that $C_K^*(U_n) = 0$ and $K\mu_n = 1$ on $O_n \setminus U_n$. We find a subsequence $\{\mu_{n_k}\}_k$ of $\{\mu_n\}_n$ converges weakly to a measure μ. By Lemma 4.11 we can find a set V such that $C_K^*(V) = 0$ and $K\mu(x) = \liminf_{k\to\infty} K\mu_{n_k}(x)$ outside V. Then

$$K\mu(x) = \liminf_{k\to\infty} K\mu_{n_k}(x) = 1 \quad \text{for } x \in E \setminus (U \cup V)$$

where $U = \bigcup_n U_n$.

For any $\varepsilon > 0$ we set $O_\varepsilon = \{x\ ;\ K\mu(x) > 1-\varepsilon\}$. Then $E \setminus (U \cup V) \subset O_\varepsilon$. Let $\nu \in \mathfrak{M}^+$ with $\operatorname{supp}\nu \subset O_\varepsilon$ and $K\nu \le 1$ everywhere. Then

$$\nu(O_\varepsilon) \le (1-\varepsilon)^{-1} \int_{O_\varepsilon} K\mu\, d\nu = (1-\varepsilon)^{-1} \int K\nu\, d\mu \le (1-\varepsilon)^{-1} \mu\left(\mathbb{R}^N\right).$$

Therefore

$$C_K(O_\varepsilon) \le (1-\varepsilon)^{-1} \mu\left(\mathbb{R}^N\right).$$

Hence

$$\begin{aligned} C_K^*(E) &\le C_K^*(E \setminus (U \cup V)) + C_K^*(U) + C_K^*(V) \le C_K^*(O_\varepsilon) \le (1-\varepsilon)^{-1} \mu\left(\mathbb{R}^N\right) \\ &= (1-\varepsilon)^{-1} \lim_{k\to\infty} \mu_{n_k}\left(\mathbb{R}^N\right) = (1-\varepsilon)^{-1} \lim_{k\to\infty} C_K(O_{n_k}) \\ &\le (1-\varepsilon)^{-1} \lim_{k\to\infty} \left(C_K^*(E_{n_k}) + 1/n_k\right) = (1-\varepsilon)^{-1} \lim_{k\to\infty} C_K^*(E_{n_k}). \end{aligned}$$

Since $\{E_n\}_n$ is monotone increasing,

$$C_K^*(E) \le (1-\varepsilon)^{-1} \lim_{n\to\infty} C_K^*(E_n).$$

Letting $\varepsilon \to 0$,

$$C_K^*(E) \le \lim_{n\to\infty} C_K^*(E_n).$$

The opposite is trivial, and we have the theorem. □

Theorem 4.7. *A set E is C_K^*-capacitable if and only if $C_K^*(E) = C_K(E)$.*

Proof. Lemmas 4.7 and 4.8 imply that

$$C_K(E) = \sup\left\{C_K^*(F)\ ;\ F \text{ is compact, } F \subset E\right\}.$$

Thus the theorem is easily proved. □

Theorem 4.8. *If $C_K^*(E) = 0$ and $\nu \in \mathfrak{M}^+$ such that $K\nu$ is bounded, then $\nu(E) = 0$.*

Proof. For any $\varepsilon > 0$ there is an open set O containing E such that

$$C_K(O) < \varepsilon.$$

Take a compact set F in O. Let $M = \sup K\nu$ and $\mu = M^{-1}\nu|_F$. Then $\operatorname{supp}\mu \subset F$ and $K\mu \le 1$ everywhere. Therefore

$$M^{-1}\nu(F) = \mu(F) \le C_K(F) \le C_K(O) < \varepsilon.$$

Hence

$$\nu(E) \le \nu(O) = \sup_F \nu(F) \le M\varepsilon.$$

Since ε is arbitrary, we have the theorem. □

5. Extremal Problems

Let F be a compact set.

Definition 5.1 (Chebychev's constant)**.**

$$M_n(F) := n^{-1} \sup_{x_1,\ldots,x_n} \inf_{x\in F} \sum_{j=1}^{n} K(x - x_j).$$

Definition 5.2 (Generalized diameter)**.**

$$D_n(F) := \frac{2}{n(n-1)} \inf_{x_1,\ldots,x_n\in F} \sum_{i<j} K(x_i - x_j) = \frac{1}{n(n-1)} \inf_{x_1,\ldots,x_n\in F} \sum_{i\neq j} K(x_i - x_j).$$

Theorem 5.1. *$D_n(F)$ is increasing and*

$$\lim_{n\to\infty} D_n(F) = \gamma, \qquad D_{n+1}(F) \le M_n(F) \le C_K(F)^{-1},$$

Where γ is the number defined in Lemma 4.3.

Proof. Since K is lower semi-continuous, we can find $\xi_1^{(n)}, \ldots, \xi_n^{(n)} \in F$ such that

$$D_n = \frac{2}{n(n-1)} \sum_{i<j} K\left(\xi_i^{(n)} - \xi_j^{(n)}\right).$$

Then

$$\begin{aligned} D_{n+1} &= \frac{2}{n(n+1)(n-1)} \sum_{k=1}^{n+1} \sum_{i<j}^{(k)} K\left(\xi_i^{(n+1)} - \xi_j^{(n+1)}\right) \\ &\ge \frac{2}{n(n+1)(n-1)} \sum_{k=1}^{n+1} \frac{n(n-1)}{2} D_n = D_n \end{aligned}$$

where $\sum_{i<j}^{(k)}$ means the summation over i and j such that $i < j$, $i \neq k$ and $j \neq k$.

Let $\mu \in \mathfrak{M}^+$ such that $\operatorname{supp}\mu \subset F$ and $\mu(F) = 1$. Since

$$\frac{n(n-1)}{2} D_n \le \sum_{i<j} K(x_i - x_j) \quad \text{for } x_1, \ldots, x_n \in F,$$

we have

$$\frac{n(n-1)}{2} D_n \le \int \cdots \int \sum_{i<j} K(x_i - x_j)\, d\mu(x_1) \cdots d\mu(x_n)$$

$$= \sum_{i<j} \iint K(x_i - x_j)\, d\mu(x_i)\, d\mu(x_j) = \frac{n(n-1)}{2} I_K(\mu).$$

Therefore $D_n(F) \le \gamma$.

Let $\mu_n = n^{-1} \sum_j \delta_{\xi_j^{(n)}}$ where δ_ξ is the Dirac measure at ξ. Also let $K_m(x) = \min(K(x), m)$. Then

$$I_{K_m}(\mu_n) = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} K_m\left(\xi_i^{(n)} - \xi_j^{(n)}\right) = n^{-2} \sum_{i \ne j} K_m\left(\xi_i^{(n)} - \xi_j^{(n)}\right) + n^{-1} m$$
$$\le n^{-1}(n-1) D_n + n^{-1} m.$$

Take a subsequence $\{\mu_{n_k}\}_k$ which converges weakly to a measure μ. Then

$$I_{K_m}(\mu) \le \liminf_{k\to\infty} I_{K_m}(\mu_{n_k}) \le \liminf_{k\to\infty} D_{n_k} = \lim_{n\to\infty} D_n.$$

Letting $m \to \infty$, by the monotone convergence theorem we have

$$I_K(\mu) \le \lim_{n\to\infty} D_n.$$

Therefore we have the first part.

For j with $1 \le j \le n+1$ and $x \in F$ we define

$$A_j(x) = \sum_{i \ne j} K\left(x - \xi_i^{(n+1)}\right).$$

Then

$$M_n \ge n^{-1} \inf_{x \in F} A_j(x) = n^{-1} A_j\left(\xi_j^{(n+1)}\right).$$

Therefore

$$D_{n+1} = \frac{1}{n(n+1)} \sum_{i \ne j} K\left(\xi_i^{(n+1)} - \xi_j^{(n+1)}\right) = \frac{1}{n(n+1)} \sum_j A_j\left(\xi_j^{(n+1)}\right)$$
$$\le \frac{1}{n(n+1)} \sum_j n M_n = M_n.$$

Now assume that $C_K(F) > 0$. Let $\nu \in \mathfrak{M}^+$ such that $\operatorname{supp}\nu \subset F$ and $K\nu \le 1$ everywhere and let $\mu = C_K(F)^{-1} \nu$. Then

$$K\mu = C_K(F)^{-1} K\nu \le C_K(F)^{-1} \quad \text{everywhere}.$$

Therefore

$$\frac{\nu(F)}{C_K(F)} n^{-1} \inf_{x\in F} \sum_{j=1}^{n} K(x - x_j) = \mu(F)\, n^{-1} \inf_{x \in F} \sum_j K(x - x_j)$$
$$\le \int_F n^{-1} \sum_j K(x - x_j)\, d\mu(x) = n^{-1} \sum_j K\mu(x_j) \le C_K(F)^{-1}.$$

Take supremum of the left hand side with varying ν, then

$$n^{-1}\inf_{x\in F}\sum_{j=1}^{n}K\left(x-x_j\right)\leq C_K\left(F\right)^{-1}.$$

Take supremum of the left hand side with varying $x_1, \ldots, x_n$, then

$$M_n\leq C_K\left(F\right)^{-1}.$$

This is trivial when $C_K\left(F\right)=0$. Thus the theorem follows. □

Now we go back to the classical case. This takes place in the complex plane $\mathbb{C}$, *i.e.* $N=2$. We shall show a classical result. For example see [1].

Definition 5.3 (Diameter of order n)**.**

$$d_n\left(F\right):=\sup_{x_1,\ldots,x_n\in F}\prod_{i<j}\left|x_i-x_j\right|^{2/n(n-1)}.$$

Definition 5.4 (Chebychev polynomial of order n)**.**

$$\rho_n\left(F\right)=\inf\left\{\sup_{x\in F}\left|x^n+a_{n-1}x^{n-1}+\cdots+a_0\right|^{1/n}\ ;\ a_0,\ \ldots,\ a_{n-1}\in\mathbb{C}\right\}.$$

Theorem 5.2. *d_n is decreasing and*

$$\lim_{n\to\infty}d_n\left(F\right)=\lim_{n\to\infty}\rho_n\left(F\right).$$

Proof. Let $\tilde{F}$ be the convex hull of F. Take $a\geq\operatorname{diam}F$ and let $K\left(r\right)=\log^+\left(a/r\right)$. Then Theorem 3.3 implies that K satisfies the strong maximum principle. Also

$$\log\frac{a}{d_n\left(F\right)}=\inf_{x_1,\ldots,x_n\in F}\frac{2}{n\left(n-1\right)}\sum_{i<j}\log\frac{a}{\left|x_i-x_j\right|}=D_n\left(F\right).$$

Since a polynomial can be represented by $\prod_{j=1}^{n}\left(x-x_j\right)$,

$$\log\frac{a}{\rho_n\left(F\right)}=\frac{1}{n}\sup_{x_1,\ldots,x_n}\inf_{x\in F}\sum_j\log\frac{a}{\left|x-x_j\right|}.$$

Take $x_1\notin\tilde{F}$ and let $x_1'\in\tilde{F}$ be the closest point to x_1 and $x_j'=x_j$ for $j=2,\ldots,n$. Then it is easy to see that $\left|x-x_1'\right|\leq\left|x-x_1\right|$ for any $x\in F$. Therefore

$$\sum_j\log\frac{a}{\left|x-x_j'\right|}\geq\sum_j\log\frac{a}{\left|x-x_j\right|}.$$

Hence

$$\inf_{x\in F}\sum_j\log\frac{a}{\left|x-x_j'\right|}\geq\inf_{x\in F}\sum_j\log\frac{a}{\left|x-x_j\right|}.$$

This means that

$$\log\frac{a}{\rho_n\left(F\right)}=\frac{1}{n}\sup_{x_1,\ldots,x_n\in\tilde{F}}\inf_{x\in F}\sum_j\log\frac{a}{\left|x-x_j\right|}=M_n\left(F\right).$$

Hence Theorem 5.1 implies the result. □

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