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HAUSDORFF MEASURES AND PACKING PREMEASURES

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Abstract. We estimate the Hausdorff measures and the packing premeasures of symmetric generalized Cantor sets in the d-dimensional Euclidean space R^d. Two simple estimations will be obtained. Let ϕ_1 and ϕ_2 be two measure functions. Suppose $\lim_{t\to 0}\phi_2(t)/\phi_1(t) = 0$, $\lim_{t\to 0}\phi_2(t)/t^d = \infty$, and $\phi_1(t)/t^d$ is strictly decreasing as t increases. Then we can construct a compact set K in R^d such that $0 < \Lambda_{\phi_1}(K) < \infty$ and $0 < \phi_2 - P(K) < \infty$ with the aid of the above estimations.

1. Introduction

In [2] we evaluated the upper and lower estimates of values of $\alpha-$dimensional Hausdorff measures of symmetric generalized Cantor sets. The following estimation similar to this result is obtained for a slightly generalized measure function ϕ which is used in [3]. Falconer obtained the precise value of the $\alpha-$dimensional Hausdorff measure of a generalized Cantor set in R^1 in [1, Theorem 1.15].

Theorem 1. *Let K^d be the d-dimensional symmetric generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=0}^{\infty}]$. Then we have*

$$\begin{aligned}&2^{-3d}\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q)\\ &\le \Lambda_\phi(K^d)\\ &\le \varliminf_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q).\end{aligned}$$

A Hausdorff dimensional result follows from this theorem.

Corollary 1. *Under the same assumption in Theorem 1*

$$\dim(K^d) = \varliminf_{q\to\infty}\frac{\log(k_1k_2\cdots k_q)^d}{-\log\lambda_q}.$$

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In [4, Examples 2 and 7] Tricot determined the Hausdorff and packing dimensions of Cantor type sets and also these dimensions of their product sets. In the next theorem we obtain the upper and lower estimates of packing premeasures of symmetric generalized Cantor sets.

Theorem 2. *Let K^d be as in Theorem 1. For a positive integer L let $\mathcal{N}_1 = \{q; k_q \leq L\}$ and $\mathcal{N}_2 = \{q; k_q > L\}$. Assume that there is a positive number C such that $\delta_q \leq C\lambda_q$ for all $q \in \mathcal{N}_2$. Then*

$$\begin{aligned}&\overline{\lim}_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q)\\ &\quad\leq \phi - P(K^d)\\ &\quad\leq M \varlimsup_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q),\end{aligned}$$

where M is a constant.

We obtain a packing dimensional result from this theorem.

Corollary 2. *Under the same assumptions of Theorem 2,*

$$\Delta(K^d) = \mathrm{Dim}(K^d) = \varlimsup_{q\to\infty} \frac{\log(k_1k_2\cdots k_q)^d}{-\log\lambda_q}.$$

By Theorem 2 it is easily seen that Δ is uniform on K^d, (for the definition, see [4] thus it follows from Corollary 3 in [4] that $\Delta(K^d) = \mathrm{Dim}(K^d)$.

In Section 2 we give the definition of the symmetric generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=0}^\infty]$. Then Theorem 1 is proved in Section 3 and the proof of Theorem 2 is given in Section 4. Finally, using these estimates we can construct a d-dimensional symmetric generalized Cantor set K such that $0 < \Lambda_{\phi_1}(K) < \infty$ and $0 < \phi_2 - P(K) < \infty$ for ϕ_1, ϕ_2, which are measure functions defined in Section 3, such that $\lim_{t\to0}\frac{\phi_2(t)}{\phi_1(t)} = 0$, $\phi_1(t)/t^d$ is strictly decreasing as t increases and $\lim_{t\to0}\phi_2(t)/t^d = \infty$.

2. Construction of symmetric generalized Cantor sets

Let $\{k_q\}_{q=1}^\infty$ be a sequence of positive integers with $k_q \geq 2$ and $\{\lambda_q\}_{q=0}^\infty$ be a sequence of positive numbers with $k_q\lambda_q < \lambda_{q-1}(q = 1, 2, \cdots)$. Take a positive number δ_q such that $k_q\lambda_q + (k_q - 1)\delta_q = \lambda_{q-1}(q = 1, 2, \cdots)$. According to [2] we shall define a symmetric generalized Cantor set in R^d constructed by the system $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=0}^\infty]$.

In R^1, in the first step, from a given closed interval with the length λ_0 remove $(k_1 - 1)$ open intervals with the length δ_1. This leaves k_1 closed intervals with the length λ_1, denoted by $I_1, \cdots, I_{q_1}$. Then there are a remained closed interval with the length λ_1, a removed open interval with the length δ_1, a remained closed interval with the length λ_1 and so are chosen in turn. Let $J_1 = \cup_{j_1=1}^{k_1} I_{j_1}$. In the next step, from each remained closed interval with the length λ_1 remove $(k_2 - 1)$ open intervals with the length δ_2. Then k_2 closed intervals with the length λ_2 are obtained as above. These are denoted by I_{j_1,j_2}. Let $J_2 = \cup_{j_1=1}^{k_1} \cup_{j_2=1}^{k_2} I_{j_1,j_2}$. We continue this process and in the q-th step obtain $k_1k_2\cdots k_q$ closed intervals with the length λ_q, denoted by $I_{j_1,j_2,\cdots,j_q}$ and denote their union by J_q. Then let $K = \cap_{q=1}^{\infty} J_q$. We call this set a one-dimensional generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=0}^{\infty}]$.

In $R^d (d \geq 2)$, we denote by $K^d (K^1 = K)$ the product set of d copies K in R^d, and call it the d-dimensional symmetric generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=0}^{\infty}]$. Let F_q be the product set of d copies J_q. Then it is easily seen that F_q is the union of $(k_1 k_2 ... k_q)^d$ closed cubes with the side λ_q, each of which is symbolically denoted by $Q^{(q)}$, and $K^d = \cap_{q=1}^{\infty} F_q$.

3. Estimation of Hausdorff measures

Let $\mathcal{M}$ be the family of functions ϕ which is continuous increasing on $[0, t_0)$ for some $t_0 > 0$ with $\phi(0) = 0$ and $\phi(t)/t^d$ is decreasing on $[0, t_0)$. We call an element of $\mathcal{M}$ a measure function. In this paper we assume that ϕ (defined on $[0, t_0)$) $\in \mathcal{M}$.

Since $\phi(t)/t^d$ is decreasing, we can see that ϕ satisfies the doubling condition:

$$\phi(2t) \leq 2^d \phi(t) \text{ for } 0 < t < t_0/2,$$

more precisely,

$$\phi(st) \leq s^d \phi(t) \text{ for } s > 1 \text{ and } 0 < st < t_0$$

and also

$$s^d \phi(t) \leq \phi(st) \text{ for } 0 < s < 1 \text{ and } 0 < t < t_0.$$

It is easy to see that if ϕ is continuous increasing with $\phi(0) = 0$ and $\phi(t)^{1/d}$ is concave on $[0, t_0)$, then $\phi(t)/t^d$ is decreasing, and thus $\phi \in \mathcal{M}$.

For a set $E \subset R^d$ we put $\Lambda_\phi^{(\varepsilon)}(E) = \inf\{\sum_i \phi(r_i)\}$, where the infimum is taken over all coverings of E by at most countable open cubes I_i with the side $r_i \leq \varepsilon < t_0$. Then the limit

$$\Lambda_\phi(E) = \lim_{\varepsilon \to 0} \Lambda_\phi^{(\varepsilon)}(E)$$

exists. We call it the $\phi-$Hausdorff measure of E. We note that if K is a compact set, then we may consider only finite coverings in the definition of $\Lambda_\phi^{(\varepsilon)}(K)$.

In case $\phi(t) = t^\alpha$ for $0 < \alpha \leq d$, which is in $\mathcal{M}$, we use Λ_α instead of Λ_ϕ and call it α-dimensional Hausdorff measure.

The following lemma is slightly generalized in [2, Lemma].

Lemma 1. *Let K be a compact set in R^d. Let Φ be a nonnegative set function defined on every open subset ω which satisfies the following conditions:*

(1) $\Phi(\omega_1 \cup \omega_2) \leq \Phi(\omega_1) + \Phi(\omega_2)$.
(2) *There exists a positive number b such that $\Phi(\omega) \geq b$ for every $\omega \supset K$.*
(3) *There exist positive numbers a and $r_0 (< t_0)$ such that $\Phi(I) \leq a\phi(r)$ for every open cube I with the side $r \leq r_0$.*

Then $\Lambda_\phi(K) \geq b/a$.

Remark 1. Since K is compact and only open cubes are used in the definition of $\phi-$Hausdorff measure, to obtain the conclusion, the set function Φ which satisfies the above three conditions is defined only on sets each of which is a finite union of open cubes.

Proof. Let ε be a positive number with $\varepsilon \le r_0$. Since K is a compact set, for any its finite covering of open cubes $\{I_i\}$ with the side $r_i \le \varepsilon$ by (3), (1) and (2) we have

$$\begin{aligned}\sum_i \phi(r_i) &\ge a^{-1}\sum_i \Phi(I_i)\\ &\ge a^{-1}\Phi(\cup_i I_i) \ge b/a,\end{aligned}$$

which implies the desired inequality. □

Using this lemma we prove Theorem 1.

Proof of Theorem 1. The right-hand inequality can be easily obtained. Hence to obtain the left-hand one, we can assume that $\varliminf_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q) > 0$. Let b be a positive number such that $b < \varliminf_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q)$. Then there is a positive integer q_0 such that $\lambda_{q_0} < t_0$ and $(k_1k_2\cdots k_q)^d\phi(\lambda_q) > b$ for all $q \ge q_0$. We take positive numbers λ'_q such that $b = (k_1k_2\cdots k_q)^d\phi(\lambda'_q)$ for $q \ge q_0$. Then we have $\lambda'_q < \lambda_q$ and $k_{q+1}^d\phi(\lambda'_{q+1}) = \phi(\lambda'_q)$.

For any open set ω we denote by $N_q(\omega)$ the number of closed cubes $Q^{(q)}$ which meet ω. Since $N_{q+1}(\omega) \le k_{q+1}^d N_q(\omega)$ and $k_{q+1}^d\phi(\lambda'_{q+1}) = \phi(\lambda'_q)$, the sequence $\{N_q(\omega)\phi(\lambda'_q)\}$ is decreasing. Hence we define a nonnegative set function

$$\Phi(\omega) = \lim_{q\to\infty} N_q(\omega)\phi(\lambda'_q).$$

It is easy to check that this Φ satisfies conditions (1) and (2) of Lemma 1 with the above b. Therefore we shall prove that Φ satisfies condition (3) of the same lemma with $r_0 = \lambda_{q_0}$ and $a = 2^{3d}$.

Let I be an open cube with the side $r < r_0$. Then there exists positive integers $q > q_0$ and j such that $1 \le j < k_{q+1}$,

$$\lambda_{q+1} \le r < \lambda_q$$

and

$$j\lambda_{q+1} + (j-1)\delta_{q+1} \le r < (j+1)\lambda_{q+1} + j\delta_{q+1}.$$

Since $N_{q+1}(I) \le 2^d(j+1)^d \le 2^{2d}j^d$, we have

$$\Phi(I) \le 2^{2d}j^d\phi(\lambda'_{q+1}).$$

In case $j = 1$,

$$\Phi(I) \le 2^{2d}\phi(\lambda'_{q+1}) \le 2^{2d}\phi(\lambda_{q+1}) \le 2^{2d}\phi(r) \le a\phi(r),$$

because $\lambda'_{q+1} < \lambda_{q+1} \le r$.

In case $2 \le j < k_{q+1}$, since $(k_1k_2\cdots k_{q+1})^d\phi(\lambda'_{q+1}) = b$ and $\phi(t)/t^d$ is decreasing, we obtain

$$\begin{aligned}j^d\phi(\lambda'_{q+1}) &= (j/(k_1k_2\cdots k_{q+1}))^d b\\ &= (j/k_{q+1})^d\phi(\lambda'_q) \le \phi(j\lambda'_q/k_{q+1}).\end{aligned}$$

Since $2 \le j, \lambda_q < k_{q+1}(\lambda_{q+1} + \delta_{q+1})$ and $\lambda'_q < \lambda_q$, we note that

$$\begin{aligned} j\lambda'_q/k_{q+1} < j\lambda_q/k_{q+1} &< j(\lambda_{q+1} + \delta_{q+1}) \le 2(j\lambda_{q+1} + (j-1)\delta_{q+1}) \\ &\le 2r \end{aligned}$$

which implies

$$\begin{aligned} \Phi(I) &\le 2^{2d} j^d \phi(\lambda'_{q+1}) \le 2^{2d}\phi(2r) \\ &\le 2^{3d}\phi(r) = a\phi(r). \end{aligned}$$

Therefore by Lemma 1 we obtain

$$\Lambda_\phi(K^d) \ge a^{-1}b.$$

Since b is an arbitrary number such that $b < \underline{\lim}_{q\to\infty}(k_1 \cdots k_q)^d \phi(\lambda_q)$, the assertion is proved. □

4. Estimation of packing premeasures

In this section we denote a d-dimensional cube by $I(x, r)$ where x is its center and $2r$ is the length of its side, or simply the side, if necessary.

According to [3] and [4] for a bounded set $E \subset R^d$ we define its packing premeasure, denoted by $\phi - P(E)$, as follows :

$$\begin{aligned} \phi - P(E) = \lim_{\varepsilon\to 0} \sup\{&\sum \phi(2r_i); \\ &\text{disjoint finite open cubes } \{I(x_i, r_i)\}, \\ &x_i \in E, r_i \le \varepsilon\}. \end{aligned}$$

In case $\phi(t) = t^\alpha$ for $0 < \alpha \le d$, we denote $\phi - P(E)$ by $\alpha - P(E)$ and call it α-packing premeasure. Corresponding these premeasures we define a rarefaction index as in [4]:

$$\Delta(E) = \inf\{\alpha > 0; \alpha - P(E) = 0\}.$$

Properties and results of these premeasures refer to [4]. As usual, from these premeasures packing measures are defined. See [3] and [4].

Lemma 2. *Let K be a compact set in R^d. Let Ψ be a nonnegative set function defined on every open subset ω which satisfies the following conditions:*

(1) *If $\omega_1 \cap \omega_2 = \emptyset$, then $\Psi(\omega_1 \cup \omega_2) \ge \Psi(\omega_1) + \Psi(\omega_2)$.*
(2) *There exists a positive number B such that $\Psi(\omega) \le B$ for every ω.*
(3) *There exist positive numbers a and $r_0(< t_0/2)$ such that $\phi(2r) \le a\Psi(I(x, r))$ for every $x \in K$ and $0 < r < r_0$.*

Then $\phi - P(K) \le aB$.

Proof. Let ε be a positive number with $\varepsilon < r_0$ and let $\{I(x_i, r_i)\}$ be disjoint finite open cubes with $x_i \in K$ and $r_i \le \varepsilon$. Since $\{I(x_i, r_i)\}$ are disjoint, thus by (2), (1) and (3) we have

$$\begin{aligned} B \ge \Psi(\cup I(x_i, r_i)) &\ge \sum \Psi(I(x_i, r_i)) \\ &\ge a^{-1} \sum \phi(2r_i). \end{aligned}$$

Therefore
$$\sum \phi(2r_i) \leq aB$$
which proves the assertion. □

To prove Theorem 2 we prepare a technical lemma.

Lemma 3. *Let K be the one-dimensional generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=0}^{\infty}]$. Let I be an open interval of length ℓ such that $2\lambda_q < \ell \leq 2\lambda_{q-1}$ for $q \geq 1$. Denote the number of $I_{j_1,\cdots,j_q}$ by $\tilde{N}_q(I)$ which are completely contained in I, where $I_{j_1,\cdots,j_q}$ is a remained closed interval in the q-th step of the construction of K. Assume that the center of I is contained in K. Then following two estimates of $\tilde{N}_q(I)$ are obtained:*

(1) *If $\ell \leq \lambda_{q-1}$, then*
$$\{\tilde{N}_q(I) - 1\}(\lambda_q + \delta_q) < \ell \leq 2\lambda_q + 2\{\tilde{N}_q(I) + 1\}(\lambda_q + \delta_q).$$

(2) *If $\ell > \lambda_{q-1}$, then*
$$\tilde{N}_q(I) \geq \max\{(k_q - 4)/2, 1\}.$$

Proof. Since $x \in K$, there exists a closed interval, denoted by $I_{j_1,\cdots,j_q}$, in its q-th step which contains x. It follows from the condition $2\lambda_q < \ell$ that $\tilde{N}_q(I) \geq 1$.

At first we prove the case (1). Let y be one of the end points of $I_{j_1,\cdots,j_{q-1}}$ which has a longer (or equal) distance from the point x. Then $|y - x| \geq \lambda_{q-1}/2 \geq \ell/2$ and so $J = (x, x + \ell/2)$ (or $J = (x - \ell/2, x)$) is included in $I_{j_1,\cdots,j_{q-1}}$. Thus we have
$$\tilde{N}_q(J)(\lambda_q + \delta_q) < \ell/2 \leq \lambda_q + (\tilde{N}_q(J) + 1)(\lambda_q + \delta_q)$$
and
$$\tilde{N}_q(J) \leq \tilde{N}_q(I) \leq 2\tilde{N}_q(J) + 1.$$
These imply the conclusion of (1).

Next, for (2) considering the special case of (1) with $\ell = \lambda_{q-1}$, we obtain
$$2(\tilde{N}_q(I) + 1)(\lambda_q + \delta_q) \geq \lambda_{q-1} - 2\lambda_q \geq (k_q - 2)(\lambda_q + \delta_q)$$
so
$$\tilde{N}_q(I) \geq (k_q - 4)/2.$$
Thus the desired result is obtained, because $\tilde{N}_q(I) \geq 1$. □

Using above two lemmas we prove Theorem 2.

Proof of Theorem 2. Since the left bottom corner of each closed cube $Q^{(q)}$ is in K^d, it is easily seen that $(k_1k_2\cdots k_q)^d$ open cubes centered at these points with the side λ_q are disjoint. Hence by the definition of packing premeasure we obtain the left-hand side inequality of the theorem.

Therefore in the following we assume that $\overline{\lim}_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q)$ is finite, and we prove the right-hand side inequality. Because otherwise the desired inequality is evident.

Let B be a number such that $B > \overline{\lim}_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q)$. Then as in the case of the Hausdorff measure there is a positive integer q_0 such that $\lambda_{q_0} < t_0, B/(k_1\cdots k_{q_0})^d < \phi(\lambda_0)$ and $(k_1k_2\cdots k_q)^d\phi(\lambda_q) < B$ for $q \geq q_0$. We take λ'_q such that $(k_1\cdots k_q)^d\phi(\lambda'_q) = B$ for $q \geq q_0$. Then for $q \geq q_0$ $\lambda_q < \lambda'_q < \lambda_0$ and $k_{q+1}^d\phi(\lambda'_{q+1}) = \phi(\lambda'_q)$.

For an open set ω we denote by $M_q(\omega)$ the number of closed cubes $Q^{(q)}$ which are completely included in ω. Then the sequence $M_q(\omega)\phi(\lambda'_q)$ is increasing, because $M_{q+1}(\omega) \geq k_{q+1}^d M_q(\omega)$ and $k_{q+1}^d\phi(\lambda'_{q+1}) = \phi(\lambda'_q)$. Thus we define a nonnegative set function $\Psi(\omega) = \lim_{q\to\infty} M_q(\omega)\phi(\lambda'_q)$. It is easy to see that this set function satisfies the conditions (1) and (2) of Lemma 2 with the above number B. It remains to prove that condition (3) of the lemma is fulfilled.

Set $r_0 = \lambda_{q_0}$. Let $I(x,r)$ be an open cube centered at $x \in K^d$ with the side $2r \leq r_0$. Then there exists an integer $q(> q_0)$ such that $\lambda_q < r \leq \lambda_{q-1}$. Since $x \in K^d \subset F_q = \cup Q^{(q)}$ and $\lambda_q < r$, then $x \in Q^{(q)}$ for some $Q^{(q)}$ and thus $Q^{(q)} \subset I(x,r)$. So $M_q(I(x,r)) \geq 1$.

In the following we divide two cases and estimate $\Psi(I(x,r))$ from below.

Case 1: $q \in \mathcal{N}_1$. Since $k_q \leq L$ and $M_q(I(x,r)) \geq 1$, we have by the doubling condition

$$\begin{aligned}\phi(2r) &\leq \phi(2\lambda_{q-1}) \leq 2^d\phi(\lambda_{q-1}) \leq 2^dB/(k_1\cdots k_{q-1})^d \\ &\leq (2L)^dB/(k_1\cdots k_q)^d = (2L)^d\phi(\lambda'_q) \leq MM_q(I(x,r))\phi(\lambda'_q) \\ &\leq M\Psi(I(x,r)).\end{aligned}$$

Next the case 2: $q \in \mathcal{N}_2$, projecting K^d and I on each x_i-axis, we denote them by K_i and I_i, respectively. We note that each K_i is the one-dimensional generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=0}^\infty]$. Thus we write $\tilde{N}_q(I)$ in Lemma 3 with respect to K_i as $\tilde{N}_{q,i}(I_i)$. In case $q \in \mathcal{N}_2$ and $2r \leq \lambda_{q-1}$, then by (1) of Lemma 3 with $\ell = 2r$, we obtain

$$\begin{aligned}2r &\leq 2\lambda_q + 2(\tilde{N}_{q,i}(I_i)+1)(\lambda_q+\delta_q) \\ &\leq 6(1+C)\tilde{N}_{q,i}(I_i)\lambda_q,\end{aligned}$$

since $\delta_q \leq C\lambda_q$ and $\tilde{N}_{q,i}(I_i) \geq 1$, and hence

$$\begin{aligned}\phi(2r) = \psi(2r)^d &\leq \psi(6(1+C)\tilde{N}_{q,1}(I_1)\lambda_q)\cdots\psi(6(1+C)\tilde{N}_{q,d}(I_d)\lambda_q) \\ &\leq \{6(1+C)\}^d\tilde{N}_{q,1}(I_1)\psi(\lambda_q)\cdots\tilde{N}_{q,d}(I_d)\psi(\lambda_q) \\ &\leq \{6(1+C)\}^dM_q(I)\phi(\lambda_q) \leq \{6(1+C)\}^d\Psi(I),\end{aligned}$$

where $\psi = \phi^{1/d}$. In case $q \in \mathcal{N}_2$ and $2r > \lambda_{q-1}$, then similarly by using (2 of Lemma 3, we have

$$k_q^d \leq 6^dM_q(I),\ 2r \leq 2\lambda_{q-1} \leq 2(1+C)k_q\lambda_q$$

and

$$\phi(2r) \le \phi(2\lambda_{q-1}) \le \{2(1+C)k_q\}^d \phi(\lambda_q)$$
$$\le 6^d \{2(1+C)\}^d M_q(I)\phi(\lambda_q) \le M\Psi(I).$$

Therefore we have proved that Ψ satisfies the three conditions of Lemma 2. Hence it follows from Lemma 2 that $\phi - P(K^d) \le MB$. Since B is an arbitrary number such that $B > \overline{\lim}_{q\to\infty}(k_1k_2\cdots k_q)^d\phi(\lambda_q)$, the proof is complete. □

Remark 2. Suppose that ϕ satisfies only the doubling condition: $\phi(2t) \le C\phi(t)$ for $t > 0$. If the sequence $\{k_q\}$ is bounded, by using the case 1 of the above proof of Theorem 2 we can obtain a similar estimation to the result of Theorem 2.

The next example shows that the condition of Theorem 2: $\delta_q \le C\lambda_q$ for all $q \in \mathcal{N}_2$ is necessary for its conclusion.

Example 1. *Let α be a number with $0 < \alpha < d$. Let $\{k_q\}$ be an increasing sequence of integers such that $k_1 \ge 2$, $\lim_{q\to\infty} k_q = \infty$ and*

$$\frac{(k_q^{d/\alpha} - k_q)}{(k_q - 1)k_q^{d/\alpha}} \le \frac{(k_{q-1}^{d/\alpha} - k_{q-1})}{(k_{q-1} - 1)},$$

and let two sequences $\{\lambda_q\}$ and $\{\delta_q\}$ be determined by

$$\lambda_0 = 1, (k_1k_2\cdots k_q)^d\lambda_q^\alpha = 1$$

and

$$k_q\lambda_q + (k_q - 1)\delta_q = \lambda_{q-1} \text{ for } q \ge 1.$$

Let K^d be the d-dimensional symmetric generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=0}^\infty]$. Then we obtain

$$\lim_{q\to\infty} \delta_q/\lambda_q = \infty,\ \delta_q \le \delta_{q-1},\ \lim_{q\to\infty}(k_1k_2\cdots k_q)^d{\delta_q}^\alpha = \infty$$

and so $\alpha - P(K^d) = \infty$, because there are $(k_1\cdots k_q)^d$ disjoint open cubes with the side δ_q centered at the left bottom corner of each remaining closed cube $Q^{(q)}$ in the q-th step which is in K^d.

Example 2. *For given $\phi_1, \phi_2 \in \mathcal{M}$ such that $\lim_{t\to 0}\frac{\phi_2(t)}{\phi_1(t)} = 0$, $\phi_1(t)/t^d$ is strictly decreasing as t increases and $\lim_{t\to 0}\phi_2(t)/t^d = \infty$. Then there exists a compact set K such that $0 < \Lambda_{\phi_1}(K) < \infty$ and $0 < \phi_2 - P(K) < \infty$.*

In the following we inductively determine two sequences $\{k_q\}_{q=1}^\infty$ and $\{\lambda_q\}_{q=0}^\infty$ which satisfy the following conditions:

$$\lambda_0 > 0,\ k_1 = k_3 = \cdots = k_{2q-1} = \cdots = 2,\ k_{2q} \ge 2$$
$$(k_1k_2\cdots k_{2q-1})^d\phi_1(\lambda_{2q-1}) = \phi_1(\lambda_0),$$
$$\phi_1(\lambda_0) \le (k_1k_2\cdots k_{2q})^d\phi_2(\lambda_{2q}) < 2^d\phi_1(\lambda_0),\ \delta_{2q} \le 2\lambda_{2q},$$

where $k_q\lambda_q + (k_q - 1)\delta_q = \lambda_{q-1}$.

In fact set $\psi_1 = \phi_1^{1/d}$ and $\psi_2 = \phi_2^{1/d}$. Then by the assumptions we obtain that $\psi_2(t) < \psi_1(t)$ for $0 < t \le t_0$, $\psi_1(t)/t$ is strictly decreasing as t increases and $\psi_1(st) < s\psi_1(t)$ for $s > 1, st \le t_0$ and $t > 0$.

At first we take λ_0 such that $0 < \lambda_0 < t_0$ and next, $k_1 = 2$ and λ_1 with $k_1\psi_1(\lambda_1) = \psi_1(\lambda_0)$. Since $\psi_1(t)/t$ is strictly decreasing, we have $2\lambda_1 < \lambda_0$ and so $\delta_1 = \lambda_0 - 2\lambda_1 > 0$.

In the third step, we divide two cases:

Case 1: $\psi_1(\lambda_0) \le 2k_1\psi_2(\lambda_1/4)$ and *Case 2*: $2k_1\psi_2(\lambda_1/4) < \psi_1(\lambda_0)$. In case 1 we let $k_2 = 2, \lambda_2 = \lambda_1/4$ and $\delta_2 = \lambda_1/2$. Then we have

$$\psi_1(\lambda_0) \le k_1k_2\psi_2(\lambda_2) = 2^2\psi_1(\lambda_1)\frac{\psi_2(\lambda_1/4)}{\psi_1(\lambda_1)} < 2\psi_1(\lambda_0) \text{ and } \delta_2/\lambda_2 \le 2,$$

because $k_1\psi_1(\lambda_1) = \psi_1(\lambda_0)$ and $\psi_2(\lambda_1/4) \le \psi_2(\lambda_1) < \psi_1(\lambda_1)$. In case 2 let p be an integer (≥ 2) which is determined later, and put $k_2 = p, \lambda_2 = \lambda_1/(2p)$ and $\delta_2 = \lambda_1/(2p-2)$. Since $2k_1\psi_2(\lambda_1/4) < \psi_1(\lambda_0)$,

$$k_1k_2\psi_2(\lambda_2) = 2p\psi_2(\lambda_1/(2p)) = \lambda_1\psi_2(\lambda_1/(2p))/(\lambda_1/(2p))$$

is increasing as p increases and $\lim_{t\to 0}\psi_2(t)/t = \infty$, we take the smallest integer $p(> 2)$ which satisfies $2p\psi_2(\lambda_1/(2p)) \ge \psi_1(\lambda_0)$. Then we obtain

$$\begin{aligned}\psi_1(\lambda_0) &\le k_1k_2\psi_2(\lambda_2)\\ &= (2p-2)\psi_2(\lambda_1/(2p-2))\frac{p}{p-1}\psi_2(\lambda_1/(2p))/\psi_2(\lambda_1/(2p-2))\\ &< \frac{p}{p-1}\psi_1(\lambda_0) < 2\psi_1(\lambda_0)\end{aligned}$$

and $\delta_2/\lambda_2 \le 2$. Therefore in both cases we have determined k_2, λ_2 and δ_2 satisfying the given conditions:

$$k_2 \ge 2,\ \psi_1(\lambda_0) \le k_1k_2\psi_2(\lambda_2) < 2\psi_1(\lambda_0),\ \delta_2/\lambda_2 \le 2.$$

In the fourth step, since $\psi_1(\lambda_0) \le k_1k_2\psi_2(\lambda_2) < k_1k_2\psi_1(\lambda_2)$, we take λ_2' as $k_1k_2\psi_1(\lambda_2') = \psi_1(\lambda_0)$. Then

$$0 < \lambda_2' < \lambda_2 \text{ and so } k_3 = 2,\ 2\psi_1(\lambda_3) = \psi_1(\lambda_2') \text{ and } \delta_3 = \lambda_2 - 2\lambda_3 > 0.$$

Hence we have obtained

$$k_3 = 2,\ k_1k_2k_3\psi_1(\lambda_3) = \psi_1(\lambda_0).$$

Considering two cases similar to the above third step we can take k_4, λ_4 and δ_4 which satisfy

$$k_4 \ge 2,\ \delta_4 \le 2\lambda_4, \psi_1(\lambda_0) \le k_1\cdots k_4\psi_2(\lambda_4) < 2\psi_1(\lambda_0).$$

Repeating these processes we can obtain desired two sequences $\{k_q\}$ and $\{\lambda_q\}$, because $\psi_1 = \phi_1^{1/d}$ and $\psi_2 = \phi_2^{1/d}$.

Let K be the d-dimensional symmetric generalized Cantor set constructed by the system $[\{k_q\}_{q=1}^{\infty}, \{\lambda_q\}_{q=0}^{\infty}]$. Then by Theorems 1and 2 it is seen that $0 < \Lambda_{\phi_1}(K) <$

$\infty, 0 < \phi_2 - P(K) < \infty$, because

$$\underline{\lim}_{q\to\infty}(k_1k_2\cdots k_q)^d\phi_1(\lambda_q) = \phi_1(\lambda_0)$$

and

$$\phi_1(\lambda_0) \leq \overline{\lim}_{q\to\infty}(k_1k_2\cdots k_q)^d\phi_2(\lambda_q) \leq 2^d\phi_1(\lambda_0).$$

Remark 3. This construction may be considered as a mutual embedding of Cantor sets of two different types.

Remark 4. Let $\phi_2(t) = t^d, \phi_1 \in \mathcal{M}$ such that $\lim_{t\to 0}\frac{\phi_2(t)}{\phi_1(t)} = 0$. Since $\phi_2(t) = t^d$ and so $\phi_2 - P(E)$ is comparable to $\Lambda_d(E) = \Lambda_{\phi_2}(E)$ for every bounded set E and $\lim_{t\to 0}\frac{\phi_2(t)}{\phi_1(t)} = 0$, if a bounded set E is a set of finite Λ_{ϕ_1} measure, then it is a set of zero $\phi_2 - P$ premeasure. Thus there does not exist a compact set K such that $\Lambda_{\phi_1}(K) < \infty$ and $\phi_2 - P(K) > 0$.

Example 3. *For given $\phi \in \mathcal{M}$ such that $\phi(t)/t^d$ is strictly decreasing as t increases, there exists a compact set K such that $0 < \Lambda_\phi(K) < \infty$ and $0 < \phi - P(K) < \infty$.*

To see this let λ_0 be a positive number such that $\lambda_0 < t_0$. Then for any bounded sequence of integers with $k_q \geq 2$ we determine the sequence of positive number λ_q such that $(k_1k_2\cdots k_q)^d\phi(\lambda_q) = \phi(\lambda_0)$. Hence we have

$$\phi(k_q\lambda_q) < k_q^d\phi(\lambda_q) = \phi(\lambda_{q-1})$$

which implies

$$k_q\lambda_q < \lambda_{q-1}.$$

Therefore we can construct the d-dimensional symmetric generalized Cantor set K constructed by the system $[\{k_q\}_{q=1}^\infty, \{\lambda_q\}_{q=0}^\infty]$. By Theorems 1 and 2 this K is the desired one.

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