Mem. Fac. Sci. Shimane Univ.,
20, pp. 9–19 Dec. 20, 1986

Remarks on Canonical Connections of Loops with the Left Inverse Property

Michihiko KIKKAWA
Department of Mathematics, Shimane University, Matsue, Japan
(Received September 6, 1986)

A theorem on the torsion and the curvature of the canonical connection of a homogeneous Lie loop at the unit element, given in our paper [8], is extended to the canonical connection of a differentiable loop with the left inverse property.

§ 1. Introduction

In the preceding paper [8], we have described interrelation between the canonical connections of homogeneous Lie loops and the Chern connections of the corresponding 3-webs of loops, and have given some explicit formulas connecting the tangent Lie triple algebras of homogeneous Lie loops and the multiplication of the loops. In general, the tangent algebra of analytic loops have been considered by M. Akivis [1] which is called Akivis algebra, and a generalized theory of Campbell-Hausdorff series for Lie groups are developed for analytic loops by M. Akivis–A. Shelekhov [2], [3], [4] and others, where they used some fundamental formulas of the coefficients of torsion and curvature of the Chern connection of 3-webs given by S. S. Chern [5]. According to the letters of M. Akivis and V. Goldberg, the tangential equations for loops with the left inverse property (abbrev. left I.P. loops) given in §§2–3 of [8] which are due to S. S. Chern [5] have been published in [2]. To show the relation between these formulas and the tangent Lie triple algebras of homogeneous Lie loops we have introduced there the concept of the canonical connection for differentiable left I.P. loops (cf. §4 of [8]). After the paper [8] appeared, we found the fact that the most of the equations of the canonical connection of homogeneous Lie loops given there are also valid for left I.P. loops, which is the main object to remark in this paper.

In what follows, we use the same notations as those used in [8] except for calculation of partial derivatives of the components of tangent vectors such as;

$$\eta(a, X_b, \eta(a, Y_b, Z_c)) = \frac{\partial^2\eta^k}{\partial b^i \partial c^j}(a, b, \eta(a, b, c))\frac{\partial^2\eta^j}{\partial b^m \partial c^p}(a, b, c)X_b^i Y_b^m Z_c^p \partial_k$$

$$\eta(a, X_b, Z_c \cdot W_c) = \frac{\partial^3\eta^k}{\partial b^i \partial c^j \partial c^m}(a, b, c)X_b^i Z_c^j W_c^m \partial_k$$

and so on, where X_b, Y_b are tangent vectors at b and Z_c, W_c are at c. It is assumed that an arbitrarily fixed local coordinate system is given around each point and that the components of any tangent vector are determined with respect to the corresponding local coordinates by $X_b = X_b^i \partial_i(b)$ and $Z_c = Z_c^j \partial_j(c)$ etc. Note that the dot product appeared in such calculation (for instance, $Z_c \cdot W_c$ in the second equation above) is always symmetric.

Let (G, μ) be a differentiable loop defined by the multiplication $xy = \mu(x, y)$ on a C^∞-class differentiable manifold G. In the rest of this paper, we assume that the loop G has the *left inverse property* (*left I.P. loop*), that is, each element x has a unique inverse x^{-1}, $x^{-1}x = xx^{-1} = e$, such that $L_{x^{-1}} = L_x^{-1}$, where e is the unit element of G and L_x denotes the left translation by x. The loop G is called a *homogeneous Lie loop* if all left inner mappings $L_{x,y} = L_{xy}^{-1} L_x L_y$ are automorphisms of G. Now, we consider the ternary system η: $G \times G \times G \to G$ on G (cf. [8]) given by $\eta(x, y, z) = x((x^{-1}y)(x^{-1}z))$. It satisfies the following equations for any x, y, z in G:

$$(\mathrm{H}_1) \quad \eta(x, x, y) = y$$

$$(\mathrm{H}_2) \quad \eta(x, y, x) = y$$

$$(\mathrm{H}_3') \quad \eta(x, e, \eta(e, x, y)) = \eta(e, x, \eta(x, e, y)) = y$$

and

$$(\mathrm{H}_4') \quad \eta(e, x, \eta(e, y, z)) = \eta(x, \eta(e, x, y), \eta(e, x, z)).$$

It is easy to show that G is a homogeneous loop if and only if the ternary system η satisfies

$$(\mathrm{H}_4) \quad \eta(e, w, \eta(x, y, z)) = \eta(\eta(e, w, x), \eta(e, w, y), \eta(e, w, z)).$$

Indeed, (H_4) is equivalent to the relation ((H_4) of [8])

$$\eta(x, y, \eta(u, v, w)) = \eta(\eta(x, y, u), \eta(x, y, v), \eta(x, y, w))$$

and it implies

$$(\mathrm{H}_3) \quad \eta(x, y, \eta(y, x, z)) = z,$$

that is, (G, η) is a *homogeneous system* on G (cf. [7]) if (H_4) is satisfied. For later use we note that the equation

$$(\mathrm{H}_5') \quad \eta(e, w, \eta(e, x, \eta(e, y, z))) = \eta(w, \eta(e, w, x), \eta(w, \eta(e, w, y), \eta(e, w, z)))$$

is derived from (H_4') on the left I.P. loop.

§2. Tangential formulas

By partial differentiation of the preceding relations (H_1)–(H'_5) we can show the following formulas on tangent vectors of the differentiable left I.P. loop (G, μ).

In the first place, the following formulas are obtained from the equation (H_1):

(2.1.1) $\eta(x, x, Y_y)=Y_y$

(2.1.2) $\eta(X_x, x, y)+\eta(x, X_x, y)=0$

(2.1.3) $\eta(X_x, x, Y_y)+\eta(x, X_x, Y_y)=0$

(2.1.4) $\eta(x, x, Y_y\cdot Z_y)=0$ or, in general, $\eta(x, x, Y_y^1\cdot Y_y^2\cdots Y_y^p)=0 \quad (p\geqq 2)$

(2.1.5) $\eta(W_x\cdot X_x, x, y)+\eta(X_x, W_x, y)+\eta(W_x, X_x, y)+\eta(x, W_x\cdot X_x, y)=0$

(2.1.6) $\eta(W_x\cdot X_x, x, Y_y)+\eta(X_x, W_x, Y_y)+\eta(W_x, X_x, X_y)+\eta(x, W_x\cdot X_x, Y_y)=0.$

Further formulas corresponding partial derivatives of order more than three are also obtained from (H_1), which are omitted here. In the same way, we can show the followings by differentiating the equation (H_2):

(2.2.1) $\eta(x, Y_y, x)=Y_y$

(2.2.2) $\eta(X_x, y, x)+\eta(x, y, X_x)=0$

(2.2.3) $\eta(X_x, Y_y, x)+\eta(x, Y_y, X_x)=0$

(2.2.4) $\eta(x, Y_y\cdot Z_y, x)=0, \quad \eta(x, Y_y^1\cdot Y_y^2\cdots Y_y^p, x)=0 \quad (p\geqq 2)$, etc.

The equations (2.1.1) and (2.2.2) evaluated at $x=y$ imply

(2.2.5) $\eta(X_x, x, x)=-X_x.$

Also, by (2.1.3) and (2.2.3) evaluated at $x=y$, we get

(2.2.6) $\eta(X_x, x, Y_x)=\eta(Y_x, X_x, x)=-\eta(x, X_x, Y_x).$

This equation with (2.1.5) and (2.2.4) implies

(2.2.7) $\eta(x, X_x, Y_x)+\eta(x, Y_x, X_x)=\eta(X_x\cdot Y_x, x, x).$

Each of two equalities in (H'_3) implies respectively the following formulas:

(2.3.1) $\eta(e, x, \eta(x, e, Y_y))=Y_y$

(2.3.2) $\eta(e, X_x, \eta(x, e, y))+\eta(e, x, \eta(X_x, e, y))=0$

(2.3.3) $\eta(e, X_x, \eta(x, e, Y_y))+\eta(e, x, \eta(X_x, e, y)\cdot\eta(x, e, Y_y))$
$+\eta(e, x, \eta(X_x, e, Y_y))=0;$

(2.3.1)′ $\eta(x, e, \eta(e, x, Y_y)) = Y_y$

(2.3.2)′ $\eta(X_x, e, \eta(e, x, y)) + \eta(x, e, \eta(e, X_x, y)) = 0$

(2.3.3)′ $\eta(X_x, e, \eta(e, x, Y_y)) + \eta(x, e, \eta(e, X_x, y) \cdot \eta(e, x, Y_y))$
$$+ \eta(x, e, \eta(e, X_x, Y_y)) = 0.$$

By differentiating (2.3.3) in the direction Z_x and evaluating at $x = y = e$, we get

(2.3.4) $\eta(e, X_e \cdot Z_e, Y_e) - \eta(e, X_e, \eta(e, Z_e, Y_e)) - \eta(e, X_e, Z_e \cdot Y_e)$
$$- \eta(e, Z_e, X_e \cdot Y_e) - \eta(e, Z_e, \eta(e, X_e, Y_e)) + \eta(X_e \cdot Z_e, e, Y_e) = 0,$$

where we used the formulas (2.2.5) and (2.2.6), and, by means of the equation (2.1.6) at $x = y = e$, we get

(2.3.5) $\eta(X_e, Y_e, Z_e) + \eta(Y_e, X_e, Z_e) + \eta(e, X_e, \eta(e, Y_e, Z_e))$
$$+ \eta(e, Y_e, \eta(e, X_e, Z_e)) + \eta(e, X_e, Y_e \cdot Z_e) + \eta(e, Y_e, X_e \cdot Z_e) = 0.$$

Now, differentiating the equation (H_4') at x, y, z in the directions X_x, Y_y and Z_z, respectively, we have the followings:

(2.4.1) $\eta(e, x, \eta(e, Y_y, z)) = \eta(x, \eta(e, x, Y_y), \eta(e, x, z))$

(2.4.2) $\eta(e, x, \eta(e, y, Z_z)) = \eta(x, \eta(e, x, y), \eta(e, x, Z_z))$

(2.4.3) $\eta(e, X_x, \eta(e, y, z)) = \eta(X_x, \eta(e, x, y), \eta(e, x, z))$
$$+ \eta(x, \eta(e, X_x, y), \eta(e, x, z)) + \eta(x, \eta(e, x, y), \eta(e, X_x, z))$$

(2.4.4) $\eta(e, x, \eta(e, Y_y, Z_z)) + \eta(e, x, \eta(e, Y_y, z) \cdot \eta(e, y, Z_z))$
$$= \eta(x, \eta(e, x, Y_y), \eta(e, x, Z_z))$$

(2.4.5) $\eta(e, X_x, \eta(e, Y_y, Z_z)) + \eta(e, X_x, \eta(e, Y_y, z) \cdot \eta(e, y, Z_z))$
$$= \eta(X_x, \eta(e, x, Y_y), \eta(e, x, Z_z)) + \eta(x, \eta(e, X_x, Y_y), \eta(e, x, Z_z))$$
$$+ \eta(x, \eta(e, x, Y_y), \eta(e, X_x, Z_z)) + \eta(x, \eta(e, X_x, y) \cdot \eta(e, x, Y_y), \eta(e, x, Z_z))$$
$$+ \eta(x, \eta(e, x, Y_y), \eta(e, X_x, z) \cdot \eta(e, x, Z_z)).$$

From (2.4.4) and (2.4.5) evaluating at $y = z = e$ and setting

$$Y_x^* = \eta(e, x, Y_e), \quad Z_x^* = \eta(e, x, Z_e)$$

we can obtain

(2.4.6) $\eta(x, Y_x^*, Z_x^*) = \eta(e, x, \eta(e, Y_e, Z_e)) + \eta(e, x, Y_e \cdot Z_e)$

and

$$(2.4.7)\quad \eta(X_x, Y_x^*, Z_x^*)=\eta(e, X_x, \eta(e, Y_e, Z_e))+\eta(e, X_x, Y_e\cdot Z_e)$$
$$-\eta(x, \eta(e, X_x, Y_e), Z_x^*)-\eta(x, X_x\cdot Y_x^*, Z_x^*)$$
$$-\eta(x, Y_x^*, \eta(e, X_x, Z_e))-\eta(x, Y_x^*, X_x\cdot Z_x^*).$$

Here, we used (2.1.1) and (2.2.1).
Furthermore, if we put $x=e$ in (2.4.7), we get

$$(2.4.8)\quad \eta(X_e, Y_e, Z_e)=\eta(e, X_e, \eta(e, Y_e, Z_e))+\eta(e, X_e, Y_e\cdot Z_e)$$
$$-\eta(e, \eta(e, X_e, Y_e), Z_e)-\eta(e, X_e\cdot Y_e, Z_e)$$
$$-\eta(e, Y_e, \eta(e, X_e, Z_e))-\eta(e, Y_e, X_e\cdot Z_e).$$

REMARK 1. Assume that the loop (G, μ) is homogeneous, that is, the equation (H_4) is valid on G. Then, we can show the followings:

$$(2.4.9)\quad \eta(e, w, \eta(X_x, y, z))=\eta(\eta(e, w, X_x), \eta(e, w, y), \eta(e, w, z))$$

$$(2.4.10)\quad \eta(e, w, \eta(x, Y_y, Z_z))+\eta(e, w, \eta(x, Y_y, z)\cdot\eta(x, y, Z_z))$$
$$=\eta(\eta(e, w, x), \eta(e, w, Y_y), \eta(e, w, Z_z))$$

$$(2.4.11)\quad \eta(\eta(e, w, X_x), \eta(e, w, Y_y), \eta(e, w, Z_z))$$
$$=\eta(e, w, \eta(X_x, Y_y, Z_z))+\eta(e, w, \eta(X_x, Y_y, z)\cdot(x, y, Z_z))$$
$$+\eta(e, w, \eta(x, Y_y, z)\cdot\eta(X_x, y, Z_z))$$
$$+\eta(e, w, \eta(X_x, y, z)\cdot\eta(x, Y_y, Z_z))$$
$$+\eta(e, w, \eta(X_x, y, z)\cdot\eta(x, Y_y, z)\cdot\eta(x, y, Z_z))$$

$$(2.4.12)\quad \eta(X_x^*, Y_x^*, Z_x^*)=\eta(e, x, \eta(X_e, Y_e, Z_e))+\eta(e, x, Z_e\cdot\eta(e, X_e, Y_e))$$
$$-\eta(e, x, Y_e\cdot\eta(e, X_e, Z_e))-\eta(e, x, X_e\cdot\eta(e, Y_e, Z_e))$$
$$-\eta(e, x, X_e\cdot Y_e\cdot Z_e).$$

Here, we used (2.1.1), (2.2.1), (2.2.5) and (2.2.6).

Finally, in the same way as above, we can show the following formulas by differentiating (H_5'):

$$(2.5.1)\quad \eta(e, w, \eta(e, X_x, \eta(e, y, z)))=\eta(w, \eta(e, w, X_x), \eta(w, \eta(e, w, y), \eta(e, w, Z_z)))$$

$$(2.5.2)\quad \eta(e, w, \eta(e, X_x, \eta(e, Y_y, z)))+\eta(e, w, \eta(e, X_x, \eta(e, y, z)\cdot\eta(e, x, \eta(e, Y_y, z))$$
$$=\eta(w, \eta(e, w, X_x), \eta(w, \eta(e, w, Y_y), \eta(e, w, z)))$$

$$(2.5.3)\quad \eta(w, \eta(e, w, X_x), \eta(w, \eta(e, w, Y_y), \eta(e, w, Z_z)))$$

$$+\eta(w, \eta(e, w, X_x), \eta(w, \eta(e, w, Y_y), \eta(e, w, z))\cdot\eta(w, \eta(e, w, y), \eta(e, w, Z_z)))$$
$$=\eta(e, w, \eta(e, X_x, \eta(e, Y_y, Z_z)))+\eta(e, w, \eta(e, X_x, \eta(e, Y_y, z)\cdot\eta(e, y, Z_z)))$$
$$+\eta(e, w, \eta(e, x, \eta(e, y, Z_z))\cdot\eta(e, X_x, \eta(e, Y_y, z)))$$
$$+\eta(e, w, \eta(e, X_x, \eta(e, y, z))\cdot\eta(e, x, \eta(e, Y_y, Z_z)))$$
$$+\eta(e, w, \eta(e, X_x, \eta(e, y, Z_z))\cdot\eta(e, x, \eta(e, Y_y, z)))$$
$$+\eta(e, w, \eta(e, X_x, \eta(e, y, z))\cdot\eta(e, x, \eta(e, Y_y, z))\cdot\eta(e, x, \eta(e, y, Z_z)))$$

(2.5.4) $$\eta(x, X_x^*, \eta(x, Y_x^*, Z_x^*))+\eta(x, X_x^*, Y_x^*\cdot Z_x^*)$$
$$=\eta(e, x, \eta(e, X_e, \eta(e, Y_e, Z_e)))+\eta(e, x, \eta(e, X_e, Y_e\cdot Z_e))$$
$$+\eta(e, x, X_e\cdot\eta(e, Y_e, Z_e))+\eta(e, x, Y_e\cdot\eta(e, X_e, Z_e))$$
$$+\eta(e, x, Z_e\cdot\eta(e, X_e, Y_e))+\eta(e, x, X_e\cdot Y_e\cdot Z_e).$$

§3. Canonical connections of left I. P. loops

The canonical connection ∇ on a differentiable left I.P. loop (G, μ) is defined in [8] as follows: Let X, Y be any vector fields on G and x a point of G. In a fixed local coordinate neighborhood of x, set $E_i^*(y)=\eta(x, y, \partial_i(x))$ and $Y_y=\tilde{Y}^i(y)E_i^*(y)$. Then $(\nabla_X Y)_x=(X_x\tilde{Y}^i)\partial_i(x)$. By using the notation in the preceding section, we can describe it as follows:

(3.1) $$(\nabla_X Y)_x=X_xY-\eta(x, X_x, Y_x).$$

Hereafter, we use the notation

$$X_xY=(X_xY^i)\partial_i(x)$$

for the coefficients Y^i of Y in the fixed coordinate system, i.e., for $Y_y=Y^i(y)\partial_i(y)$ in the coordinate neighborhood of x.

PROPOSITION 1. *The canonical connection ∇ of a left I.P. loop (G, μ) satisfies the following equation at the unit e;*

(3.2) $$\eta(e, x, (\nabla_Y Z)_e)=(\nabla_{\tilde{Y}}\tilde{Z})_x, \quad x\in G,$$

for any vector fields Y and Z on a neighborhood of e, where $\tilde{Y}_{\tilde{u}}=\eta(e, x, Y_u)$, $\tilde{Z}_{\tilde{u}}=\eta(e, x, Z_u)$ for $\tilde{u}=\eta(e, x, u)$.

PROOF. Let x be a fixed point and $(\tilde{u}^i)$ a fixed local coordinate system around x. If we set $\tilde{Z}_{\tilde{u}}=\tilde{Z}^i(\tilde{u})\tilde{\partial}_i(\tilde{u})$, then $\tilde{Z}^i(\tilde{u})=\eta^i(e, x, Z_u)$ and

$$\tilde{Y}_x\tilde{Z}=(\tilde{Y}_x\tilde{Z}^i)\tilde{\partial}_i(x)=((d\eta(e, x)Y_e)\eta^i(e, x, Z_u))\tilde{\partial}_i(x)=(Y_e\eta^i(e, x, Z_u))\tilde{\partial}_i(x)$$

$$=\eta(e, x, Y_e\cdot Z_e)+\eta(e, x, Y_eZ).$$

On the other hand, the formula (2.4.6) implies

$$\eta(x, \tilde{Y}_x, \tilde{Z}_x)=\eta(e, x, \eta(e, Y_e, Z_e))+\eta(e, x, Y_e\cdot Z_e).$$

Hence, we have

$$\begin{aligned}(\nabla_{\tilde{Y}}\tilde{Z})_x &= \tilde{Y}_x\tilde{Z}-\eta(x, \tilde{Y}_x, \tilde{Z}_x)\\ &=\eta(e, x, Y_eZ)-\eta(e, x, \eta(e, Y_e, Z_e))\\ &=\eta(e, x, (\nabla_YZ)_e).\end{aligned}$$

q. e. d.

REMARK 2. Assume that (G, μ) is a homogeneous Lie loop. For any vector fields Y, Z and any point w, set $\tilde{Y}_{\tilde{x}}=\eta(e, w, Y_x)$ and $\tilde{Z}_{\tilde{x}}=\eta(e, w, Z_x)$, $x\in G$, where $\tilde{x}=\eta(e, w, x)$. Then, by (2.4.10) in Remark 1, we can show the following equation;

$$\eta(e, w, (\nabla_YZ)_x)=(\nabla_{\tilde{Y}}\tilde{Z})_{\tilde{x}}.$$

This means that the displacement $\eta(e, w)\colon G\to G$ sending each x to $\eta(e, w, x)$ is an affine transformation of the canonical connection on the homogeneous Lie loop.

Let T and R denote the torsion tensor field and the curvature tensor field of the canonical connection of the left I.P. loop G, which are given by

$$T(X, Y)=[X, Y]-\nabla_XY+\nabla_YX$$
$$R(X, Y)Z=\nabla_{[X,Y]}Z-\nabla_X\nabla_YZ+\nabla_Y\nabla_XZ$$

for any vector fields X, Y and Z on G.

PROPOSITION 2. *The torsion tensor T satisfies*

(3.3) $$T_x(\eta(e, x, X_e), \eta(e, x, Y_e))=\eta(e, x, T_e(X_e, Y_e))$$

for any X_e, $Y_e\in T_e(G)$ and $x\in G$.

PROOF. By the definition (3.1) of the canonical connection, we get

(3.4) $$T_x(X_x, Y_x)=\eta(x, X_x, Y_x)-\eta(x, Y_x, X_x)$$

for X_x, $Y_x\in T_x(G)$. If $X_x=\eta(e, x, X_e)$ and $Y_x=\eta(e, x, Y_e)$, then (2.4.6) implies

$$\eta(x, X_x, Y_x)=\eta(e, x, \eta(e, X_e, Y_e))+\eta(e, x, X_e\cdot Y_e).$$

Since the dot product in the last term is symmetric, we have the required equation.

q. e. d.

LEMMA. *Let K be a $(1, s)$-tensor field on G. If K satisfies the following equation* *at each point x on G, then* $(\nabla K)_e=0$:

$$K_x(\eta(e, x, X_e^1),\dots, \eta(e, x, X_e^s))=\eta(e, x, K_e(X_e^1,\dots, X_e^s)) \tag{3.5}$$

for $X_e^1,\dots, X_e^s \in T_e(G)$.

PROOF. For any tangent vector X_e, Y_e at e, we construct the vector fields X^* and Y^* on G by the equations $X_w^*=\eta(e, w, X_e)$, $Y_w^*=\eta(e, w, Y_e)$, $w \in G$. Then, by using the formula (2.4.6), we get

$$\begin{aligned}(\nabla_{Y^*}X^*)_x &= X_x^* Y^* - \eta(x, X_x^*, Y_x^*)\\ &= X_x^*\eta(e, w, Y_e) - \eta(e, x, \eta(e, X_e, Y_e)) - \eta(e, x, X_e\cdot Y_e)\\ &= \eta(e, X_x^*, Y_e) - \eta(e, x, \eta(e, X_e, Y_e)) - \eta(e, x, X_e\cdot Y_e).\end{aligned}$$

If we put $x=e$, then the equation $(\nabla_{Y^*}X^*)_e=0$ follows from (2.1.1) and (2.1.4), which proves the lemma for $s=0$. We now show the lemma for $s\geqq 1$. For any tangent vectors Y_e, $X_e^1,\dots, X_e^s$ at e, let Y^* and X^{*p}, $p=1,\dots, s$, be the vector fields on G given by $Y_w^*=\eta(e, w, Y_e)$ and $X_w^{*p}=\eta(e, w, X_e^p)$ for $w \in G$. Then, we have

$$\begin{aligned}(\nabla_{Y^*}K)_x(X_x^{*1},\dots, X_x^{*s}) &= (\nabla_{Y^*}(K(X^{*1},\dots, X^{*s})))_x\\ &\quad - \textstyle\sum_{p=1}^s K_x(X_x^{*1},\dots, (\nabla_{Y^*}X^{*p})_x,\dots, X_x^{*s})\\ &= Y_x^*(K(X^{*1},\dots, X^{*s})) - \eta(x, Y_x^*, K_x(X_x^{*1},\dots, X_x^{*s}))\\ &\quad - \textstyle\sum_{p=1}^s K_x(X_x^{*1},\dots, (\nabla_{Y^*}X^{*p})_x,\dots, X_x^{*s}).\end{aligned}$$

By the assumption of the lemma, we can use (2.4.6) for $\eta(x, Y_x^*, K_x(X^{*1},\dots, X^{*s}))$ and we get

$$Y_x^*(K(X^{*1},\dots, X^{*s})) = \eta(e, Y_x^*, K_e(X_e^1,\dots, X_e^s))$$

and

$$\begin{aligned}\eta(x, Y_x^*, K_x(X_x^{*1},\dots, X_x^{*s})) &= \eta(e, x, \eta(e, Y_e, K_e(X_e^1,\dots, X_e^s)))\\ &\quad + \eta(e, x, Y_e\cdot K_e(X_e^1,\dots, X_e^s)).\end{aligned}$$

If we put $x=e$, then we have $(\nabla_{Y_e}K)_e(X_e^1,\dots, X_e^s)=0$ since $(\nabla_{Y^*}X^{*p})_e=0$ as shown above. q.e.d.

From this lemma and Proposition 2, we obtain the following;

PROPOSITION 3. *The torsion tensor T of the canonical connection on a left I.P. loop satisfies the equation*

$$(\nabla T)_e=0$$

at the unit element e.

In the following, we describe the curvature R of the canonical connection ∇ of the left I.P. loop (G, μ) by means of the tangential formulas given in §2. From (3.1) we can show the following;

PROPOSITION 4. *The curvature tensor of the canonical connection of a left I.P. loop G is given by*

$$(3.6)\qquad \begin{aligned} R_x(X_x, Y_x)Z_x &= \eta(X_x, Y_x, Z_x) - \eta(Y_x, X_x, Z_x) \\ &\quad - \eta(x, X_x, \eta(x, Y_x, Z_x)) + \eta(x, Y_x, \eta(x, X_x, Z_x)) \\ &\quad - \eta(x, X_x, Y_x \cdot Z_x) + \eta(x, Y_x, X_x \cdot Z_x) \end{aligned}$$

for any tangent vectors X_x, Y_x and Z_x at any point x.

The expression above of the curvature tensor implies the following;

PROPOSITION 5. *For any $X_e, Y_e, Z_e \in T_e(G)$, set $X_x^* = \eta(e, x, X_e)$, $Y_x^* = \eta(e, x, Y_e)$ and $Z_x^* = \eta(e, x, Z_e)$.*
Then,

$$(3.7)\qquad \begin{aligned} R_x(X_x^*, Y_x^*)Z_x^* &- \eta(e, x, R_e(X_e, Y_e)Z_e) \\ &= \eta(X_x^*, Y_x^*, Z_x^*) - \eta(Y_x^*, X_x^*, Z_x^*) \\ &\quad - \eta(e, x, \eta(X_e, Y_e, Z_e)) + \eta(e, x, \eta(Y_e, X_e, Z_e)) \\ &\quad - \eta(e, x, T_e(X_e, Y_e) \cdot Z_e). \end{aligned}$$

PROOF. Apply (3.6) for X_x^*, Y_x^* and Z_x^*. Then, by using (2.5.4) and (3.4), we can easily show the equation. q. e. d.

REMARK 3. If (G, μ) is homogeneous, then, by (2.4.12) in Remark 1, the right hand side of the equality (3.7) vanishes and we have

$$(3.8)\qquad R_x(X_x^*, Y_x^*)Z_x^* = \eta(e, x, R_e(X_e, Y_e)Z_e),$$

from which we can obtain

$$(3.9)\qquad (\nabla R)_e = 0$$

by Lemma given in this section. We have known that the equations $\nabla T = 0$ and $\nabla R = 0$ hold at every point of the homogeneous Lie loop (cf. [7]), which follows also from (3.9) and Proposition 3, under the assumption of homogeneity.

We can describe the value at the unit element e for the curvature tensor R as follows:

$$(3.10)\quad R_e(X_e, Y_e)Z_e = \frac{1}{2}\{\eta(X_e, Y_e, Z_e) - \eta(Y_e, X_e, Z_e) - \eta(e, T_e(X_e, Y_e), Z_e)\}.$$

In fact, by the formula (2.4.8), we get

$$\eta(X_e, Y_e, Z_e)+\eta(e, Y_e, (e, X_e, Z_e))+\eta(e, Y_e, X_e\cdot Z_e)$$
$$=\eta(e, X_e, \eta(e, Y_e, Z_e))+\eta(e, X_e, Y_e\cdot Z_e)$$
$$-\eta(e, \eta(e, X_e, Y_e), Z_e)-\eta(e, X_e\cdot Y_e, Z_e).$$

This implies (3.10) by evaluating (3.6) at e. In [8], we have considered the endomorphisms $dL(X_e, Y_e)$ of the tangent space $T_e(G)$ at e, which is derived from the left inner mappings $L_{x,y}=L_{xy}^{-1}L_xL_y$ of the left I.P. loop (cf. (3.19) and (3.20) of [8]). By using our notation in this paper, the equation (3.20) and (3.21) of [8] are rewritten as follows:

$$(3.11)\quad dL(X_e, Y_e)Z_e=\eta(e, X_e\cdot Y_e, Z_e)-\eta(e, Y_e, X_e\cdot Z_e)$$
$$+\eta(e, \eta(e, Y_e, X_e), Z_e)-\eta(e, Y_e, \eta(e, X_e, Z_e)).$$

The following theorem, which has been given in [8] for homogeneous Lie loops, is shown now for differentiable left I.P. loops:

THEOREM. *Let (G, μ) be a differentiable loop with the left inverse property. The torsion tensor T and the curvature tensor R of the canonical connection of (G, μ) are given at the unit element e by the following equations;*

$$(3.12)\qquad T_e(X_e, Y_e)=d\mu(X_e, Y_e)-d\mu(Y_e, X_e),$$

$$(3.13)\qquad R_e(X_e, Y_e)Z_e=2dL(X_e, Y_e)Z_e$$

for any tangent vectors X_e, Y_e and Z_e at e, where the bilinear operations $d\mu$: $T_e(G)\times T_e(G)\to T_e(G)$ and dL: $T_e(G)\times T_e(G)\to \mathrm{End}\,(T_e(G))$ are those introduced in [8], which are derived respectively from the multiplication μ and left inner mappings of (G, μ).

PROOF. Since $\mu(x, y)=\eta(e, x, y)$, the equation (3.12) is same as (3.4) for $x=e$. As for (3.13), we use the formula (2.4.8) for the equation

$$dL(X_e, Y_e)Z_e-dL(Y_e, X_e)Z_e=\eta(e, X_e, \eta(e, Y_e, Z_e))$$
$$-\eta(e, Y_e, \eta(e, X_e, Z_e))-\eta(e, \eta(e, X_e, Y_e), Z_e)$$
$$+\eta(e, \eta(e, Y_e, X_e), Z_e)+\eta(e, X_e, Y_e\cdot Z_e)$$
$$-\eta(e, Y_e, X_e\cdot Z_e)$$

and get

$$(3.14)\quad dL(X_e, Y_e)Z_e-dL(Y_e, X_e)Z_e=\eta(X_e, Y_e, Z_e)+\eta(e, X_e\cdot Y_e, Z_e)$$
$$+\eta(e, \eta(e, Y_e, X_e), Z_e).$$

The following equation follows from (2.3.5) and (2.4.8) which is equal to a formula shown by S. S. Chern [5] (cf. Cor. 1 to Theorem 1 of [8]);

(3.15) $$dL(X_e, Y_e)+dL(Y_e, X_e)=0.$$

Therefore, the equation (3.13) is proved by (3.14), (3.15) and (3.10). q. e. d.

References

[1] M. A. Akivis, Geodesic loops and local triple systems in an affinely connected space (Russian), Sibirsk. Mat. Zh. **19** (1978), 243–253. English translation: Siberian Math. J. **19** (1978), 171–178.

[2] M. A. Akivis–A. M. Shelekhov, On the computation of the curvature and torsion tensors of a multidimensional three-web and of the associator of the local quasigroup connected with it (Russian), Sibirsk. Mat. Zh. **12** (1971), 12–24. English translation: Siberian Math. J. **12** (1971), 685–689.

[3] ———, Foundations of the theory of webs (Russian), Kalinin Gos. Univ., Kalinin, 1981.

[4] ———, Introduction to the theory of three-webs (Russian), Kalinin Gos. Univ., Kalinin, 1985.

[5] S. S. Chern, Eine Invariantentheorie der Dreigewebe aus r-dimensionalen Mannigfaltigkeiten im R_{2r}, Abh. Math. Sem. Univ. Hamburg **11** (1936), 333–358.

[6] M. Kikkawa, Geometry of homogeneous Lie loops, Hiroshima Math. J. **5** (1975), 141–179.

[7] ———, On homogeneous systems I–V, Mem. Fac. Sci. Shimane Univ. **11** (1977), 9–17; **12** (1978), 5–13; **14** (1980), 41–46; **15** (1981), 1–7; **17** (1983), 9–13.

[8] ———, Canonical connections of homogeneous Lie loops and 3-webs, Mem. Fac. Sci. Shimane Univ. **19** (1985), 37–55.