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The Non-existence of Common Models for Certain Classes of Continua

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Abstract
In this paper, we first prove that for each continuum Z, the class of all hereditarily indecomposable continua Y_Z that map onto Z by a light map does not admit a common model. Also, for several topological properties previously shown not to be Whitney reversible, we demonstrate that the corresponding classes of continua forming the counterexamples do not admit a common model.

Keywords Common model · Indecomposable continua · Hereditarily indecomposable continua · Cut point · Wilder continua · Whitney reversible property

Mathematics Subject Classification Primary 54F15 · Secondary 54F16

1 Introduction

Throughout this paper, all spaces are assumed to be metrizable and all maps continuous. I denotes the closed interval [0, 1]. For a space X and a subset $A \subset X$, the interior of A relative to X will be denoted by $\mathrm{Int}_X A$, and the closure of A in X will be denoted by $\mathrm{Cl}_X A$. A *continuum* means a nonempty compact connected metric space, and when such a space is regarded as a subset of a larger space we shall refer to it as a *subcontinuum*. A continuum X is said to be *decomposable* if it can be expressed as the union of two proper subcontinua. If X is not decomposable, it is called an *indecomposable continuum*. Moreover, if every subcontinuum of X is indecomposable, then X is called a *hereditarily indecomposable continuum*.

Given a continuum X, we write $C(X)$ for the family of all nonempty subcontinua of X, endowed with the Hausdorff metric. The space $C(X)$ is called the *hyperspace of subcontinua* of X. A *Whitney map* is a map $\mu : C(X) \to [0, \mu(X)]$ such that

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$\mu(\{x\}) = 0$ for every $x \in X$, and $\mu(A) < \mu(B)$ whenever $A, B \in C(X)$ with $A \subsetneq B$. It is a classical fact that every continuum admits a Whitney map (see [7, Theorem 13.4]). Moreover, if $\mu : C(X) \to [0, \mu(X)]$ is a Whitney map, then for each $t \in [0, \mu(X)]$, the fiber $\mu^{-1}(t)$ is itself a continuum ([7, Theorem 19.9]). Each such set $\mu^{-1}(t)$ is referred to as a *Whitney level*.

It is classical to study topological properties via Whitney maps. A property $\mathcal{P}$ is called a *Whitney property* if, for every continuum X, whenever X itself has $\mathcal{P}$, then each Whitney level of any Whitney map $\mu : C(X) \to [0, \mu(X)]$ also has $\mathcal{P}$. Conversely, $\mathcal{P}$ is said to be *Whitney reversible* if, for every continuum X and every Whitney map μ, whenever each Whitney level has $\mathcal{P}$, then X itself also has $\mathcal{P}$. With respect to research on these properties, for example, see [7].

An *arc* is a continuum homeomorphic to I, while a *tree* is a 1-dimensional polyhedron containing no simple closed curves. A *triod* is a continuum consisting of three arcs meeting at a single common endpoint. A continuum X is said to be *arc-like*, *tree-like*, or *triod-like* if for every $\varepsilon > 0$ there exists a continuous surjection $f : X \to A$ (onto an arc A), $f : X \to T$ (onto a tree T), or $f : X \to Y$ (onto a triod Y), respectively, such that $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for every y in the target continuum. A continuum M is said to be a *common model* for a class $\mathcal{C}$ of continua if, for every $X \in \mathcal{C}$, there exists a continuous surjective map $f : M \to X$. As an illustration, the Hahn–Mazurkiewicz theorem (see [21, Section 8]) implies that the closed interval $[0, 1]$ serves as a common model for the class of all Peano continua. In a similar vein, the pseudo-arc (=the arc-like hereditarily indecomposable continuum) provides a common model for the collection of all arc-like continua (see [5], [13], and [20]).

On the other hand, Waraszkiewicz [23] demonstrated the existence of an uncountable collection of compactifications of the half-open interval $(0, 1]$, each with the circle as its remainder, such that the family as a whole admits no common model. Similarly, in Section 20 of [15], the authors constructed an analogous uncountable family whose remainders are all simple triods, again with no common model for the entire collection. In particular, this shows that there is no common model for the class of all triod-like continua. Section 20 of [15] contains numerous further results regarding common models for various classes of continua. For more recent developments on this subject, the reader may consult [11]. Also, see [19].

The paper is divided as follows:

In Sect. 2, we introduce the definitions and previous results used in this paper.

In Sect. 3, we deal with hereditarily indecomposable continua. In [10, Corollary 5.2], Krasinkiewicz showed that every continuum is a light image of some hereditarily indecomposable continuum. In this section, using this result, we show that for each continuum Z, the class of all hereditarily indecomposable continua Y_Z that map onto Z via a light map does not admit a common model.

In Sect. 4, we consider certain topological properties and the continua that witness the failure of each property to be Whitney reversible. We show that, for each such property, the class of all continua demonstrating its failure does not admit a common model. The properties considered here include having a cut point, being a Wilder continuum, and being decomposable. In [16], the author gave an example of a continuum demonstrating that having cut points is not a Whitney reversible property. In this section, we establish that the class of all such continua has no common model. For other topics involving cut points, Whitney properties, and Whitney reversible properties, see, for example, [2]. We also deal with Wilder continua and strongly Wilder continua. Wilder continua were first studied in [24] and have been extensively investigated in [3, 4, 8, 9, 12, 14, 17, 18] and [22]. The notion of a strongly Wilder continuum was introduced in [3]. In [18], the author constructed a continuum all of whose positive Whitney levels are strongly Wilder, while the continuum itself contains no Wilder subcontinuum. In this section, we demonstrate that the class of all such continua admits no common model. Moreover, it is well known in continuum theory that there exist continua all of whose positive Whitney levels are decomposable, whereas the continua themselves are indecomposable (see, for example, [7, Exercise 44.14]). Here, we also show that the class of such continua admits no common model. More precisely, combining our argument with the result of [19], we prove the following. For each continuum X, the class of continua that contain X as a subcontinuum and for which all positive Whitney levels are decomposable while the continua themselves are indecomposable admits no common model.

2 Preliminaries

Here, we recall some basic definitions and results from [19], which are based on ideas from [15, Section 20]. These will be applied in the subsequent sections.

Definition 2.1 Consider subcontinua $X', X'' \subset I^\infty \times [-1, 1]$ such that $X' \cap X'' \neq \emptyset$ and closed subsets $A' \subset X'$, $A'' \subset X''$. Let

$$X := X' \cup X''.$$

Assume the following conditions hold:

(1) $X' \subset I^\infty \times [0, 1]$;
(2) X'' is obtained from X' by reflection across the hyperplane $t = 0$, i.e.,

$$X'' = \{(x, t) \in I^\infty \times [-1, 0] : (x, -t) \in X'\};$$

(3) A' coincides with the top slice of X', and A'' with the bottom slice of X'':

$$X' \cap (I^\infty \times \{1\}) = A', \quad X'' \cap (I^\infty \times \{-1\}) = A''.$$

In this situation, we refer to X as the (X', A')-(X'', A'') *mirror-glued continuum*.

Definition 2.2 Let X be a mirror-glued continuum as in Definition 2.1. Regard X as a subset of $I^\infty \times \mathbb{R}$ and define

$$X^{+1} := \{(x, t) \in I^\infty \times \mathbb{R} : (x, t - 1) \in X\}.$$

Then set:

$$T_1^X = \{(x, t, 0) : (x, t) \in X^{+1}\},$$
$$T_2^X = \{(x, -t, 0) : (x, t) \in X^{+1}\},$$
$$T_3^X = \{(x, 0, t) : (x, t) \in X^{+1}\}.$$

We define the *tri-fold symmetric extension of* X by

$$Y_X := T_1^X \cup T_2^X \cup T_3^X.$$

Definition 2.3 Let X be the (X', A')-(X'', A'') mirror-glued continuum, and let Y_X be its tri-fold symmetric extension. Define the ambient space

$$Q := I^\infty \times \mathbb{R} \times \mathbb{R},$$

with natural projections

$$p : Q \times [0, 3] \to Q, \quad q : Q \times [0, 3] \to [0, 3].$$

We call $\mathbf{m} = (m(i))_{i=0}^\infty \subset \{1, 2, 3\}$ a *meandering pattern* if $m(2i) = m(2i + 1)$ for each $i \geq 0$. We construct a continuum $M_X^{\mathbf{m}}$ as follows. There exist:

- topological copies $\{Z_i\}_{i=0}^\infty$ of X, and
- a strictly decreasing sequence $(d_i)_{i=1}^\infty \subset (0, 1]$ with $d_1 = 1$ and $\lim_{i\to\infty} d_i = 0$,

such that, if A_i' and A_i'' denote the images of A' and A'' in each copy Z_i,

(1) $Z_0 \subset q^{-1}([1, 3])$, with $Z_0 \cap q^{-1}(3) = A_0'$ and $Z_0 \cap q^{-1}(1) = A_0''$;
(2) for each $i \geq 1$, $Z_i \subset q^{-1}([d_{i+1}, d_i])$;
(3) for each $i \geq 0$, $Z_i \cap Z_{i+1} = A_i'' = A_{i+1}'$;
(4) for each $i \geq 0$, $p|_{Z_i} : Z_i \to T_{m(i)}^X$ is a homeomorphism;
(5) the union

$$M_X^{\mathbf{m}} := (Y_X \times \{0\}) \cup \bigcup_{i=0}^\infty Z_i$$

is a continuum.

We call $M_X^{\mathbf{m}}$ the *meandering continuum with pattern* $\mathbf{m}$ associated with X, and denote the collection of all such continua by $\mathbb{M}_X$.

Theorem 2.4 ([19, Corollary 2.8]) *For each* (X', A')–(X'', A'') *mirror-glued continuum* X*, the family* $\mathbb{M}_X$ *of meandering continua has no common model.*

3 Hereditarily Indecomposable Continua

In [10, Corollary 5.2], Krasinkiewicz showed that every continuum is a light image of some hereditarily indecomposable continuum. In this section, using this result, we show that for each continuum Z, the class of all hereditarily indecomposable continua Y_Z that map onto Z by a light map does not admit a common model.

Let Z be a continuum. We denote by $\mathbb{HI}_Z$ the set of all hereditarily indecomposable continua Y_Z for which there exists a surjective light map from Y_Z onto Z.

Theorem 3.1 *For each continuum Z, the class $\mathbb{HI}_Z$ has no common model.*

Proof Let X be the (X', A')–(X'', A'') mirror-glued continuum, where X' and X'' are both homeomorphic to Z and $|X' \cap X''| = 1$. By [10, Corollary 5.2], for each $M_X^{\mathbf{m}} \in \mathbb{M}_X$, there exist a hereditarily indecomposable continuum $\tilde{M}_X^{\mathbf{m}}$ and a surjective light map $\ell_{\mathbf{m}} : \tilde{M}_X^{\mathbf{m}} \to M_X^{\mathbf{m}}$. Also, by the proof of [19, Theorem 3.3], which corresponds to Theorem 4.10, there exist $Z^{\mathbf{m}} \subset M_X^{\mathbf{m}}$ and an open retraction $r_{\mathbf{m}} : M_X^{\mathbf{m}} \to Z^{\mathbf{m}} \cong Z$ such that each fiber of $r_{\mathbf{m}}$ is totally disconnected. Then, the map $r_{\mathbf{m}} \circ \ell_{\mathbf{m}}$ is a surjective light map from $\tilde{M}_X^{\mathbf{m}}$ onto $Z^{\mathbf{m}} \cong Z$.

Let $\mathbb{HI}_X^{\sharp} = \{\tilde{M}_X^{\mathbf{m}} : \mathbf{m}$ is a meandering pattern$\}$. Then, $\mathbb{HI}_X^{\sharp} \subset \mathbb{HI}_Z$. By Theorem 2.4, $\mathbb{M}_X$ admits no common model. Since $\ell_{\mathbf{m}}$ is a continuous surjection from $\tilde{M}_X^{\mathbf{m}}$ onto $M_X^{\mathbf{m}}$ for each meandering pattern $\mathbf{m}$, it follows that $\mathbb{HI}_X^{\sharp}$ admits no common model. Therefore, $\mathbb{HI}_Z$ also admits no common model. This completes the proof. $\square$

4 Whitney Reversible Properties

Let X be a continuum. A subcontinuum $C \subset X$ is called *terminal* if for every subcontinuum $D \subset X$ that intersects C, we have either $C \subset D$ or $D \subset C$. Let $f : X \to Y$ be a continuous surjection between continua. We say that f is *atomic* if for each point $y \in Y$, the fiber $f^{-1}(y)$ is a terminal subcontinuum of X. A well-known condition, which characterizes atomic maps, asserts that for every subcontinuum $A \subset X$, whenever $f(A)$ is not a singleton, we have $f^{-1}(f(A)) = A$.

A point x in a continuum X is called a *cut point of* X if $X \setminus \{x\}$ is not connected. Let $\mathbb{CNWR}$ be the set of all continua Z satisfying the following properties:

(1) Z does not have a cut point, and
(2) $\mu^{-1}(s)$ has a cut point for each Whitney map $\mu : C(Z) \to [0, \mu(Z)]$ and for each $s \in (0, \mu(Z))$.

By [16, Theorem 2.2], the property of having cut points is not a Whitney reversible property; in other words, the class $\mathbb{CNWR}$ is nonempty. In the sequel, we shall establish a stronger result.

Theorem 4.1 $\mathbb{CNWR}$ *does not admit a common model.*

Proof The proof is obtained by a slight modification of the proof of [16, Theorem 2.2]. By [1, Theorem], there exist a continuum X' and a monotone open map $f : X' \to I$ such that for each $y \in I$, the fiber $f^{-1}(y)$ is a nondegenerate terminal subcontinuum of X'. We may assume

$$X' \subset I^{\infty} \times [0, 1], \quad X' \cap (I^{\infty} \times \{1\}) = f^{-1}(1), \quad X' \cap (I^{\infty} \times \{0\}) = f^{-1}(0).$$

Define

$$X'' = \{(x,t) \in I^\infty \times [-1,0] : (x,-t) \in X'\},$$

$$A' = f^{-1}(1), \qquad A'' = \{(x,t) \in I^\infty \times [-1,0] : (x,-t) \in A'\}.$$

Let X be the (X', A')-(X'', A'') mirror-glued continuum.

For a given meandering pattern $\mathbf{m}$, let $M_X^{\mathbf{m}}$ denote the associated meandering continuum, and let

$$q_{\mathrm{top}} : M_X^{\mathbf{m}} \to Z_{\mathbf{m}}$$

be the quotient map collapsing the top-level set $q^{-1}(3) \cap M_X^{\mathbf{m}}$ to a single point $*$.

By [21, Theorem 13.5] and the construction of $Z_{\mathbf{m}}$, it is not difficult to check that there exists an atomic map $\alpha : Z_{\mathbf{m}} \to [0,3]$ such that:

(1) $\alpha^{-1}(0) = q_{\mathrm{top}}(q^{-1}(0))$,
(2) $\alpha^{-1}(3) = q_{\mathrm{top}}(q^{-1}(3))$,
(3) the map $t \mapsto \alpha^{-1}(t)$ from $(0,3]$ to $C(Z_{\mathbf{m}})$ is continuous.

Step 1. $Z_{\mathbf{m}}$ has no cut point. Let $z \in Z_{\mathbf{m}}$.

If $\{z\} = \alpha^{-1}(3)$, then

$$Z_{\mathbf{m}} \setminus \{z\} = \alpha^{-1}([0,3)),$$

which is connected since α is monotone. Thus z is not a cut point.

If $z \in \alpha^{-1}(t)$ for some $t \in (0,3)$, assume towards a contradiction that z is a cut point. Then $Z_{\mathbf{m}} \setminus \{z\} = O \cup H$ with O, H nonempty open and disjoint. Both $O \cup \{z\}$ and $H \cup \{z\}$ are nondegenerate continua ([21, Proposition 6.3]). Suppose $\alpha^{-1}(t) \cap H \neq \emptyset$. Since $O \cup \{z\}$ is nondegenerate, there exists a sequence $\{z_i\} \subset O$ converging to z. Let $t_i = \alpha(z_i)$. Then $t_i \to t$, and $\alpha^{-1}(t_i) \subset O$ for all i. By continuity of $t \mapsto \alpha^{-1}(t)$, we obtain $\alpha^{-1}(t) \subset O \cup \{z\}$, contradicting $\alpha^{-1}(t) \cap H \neq \emptyset$.

If $z \in \alpha^{-1}(0)$, then

$$\alpha^{-1}((0,1]) \subset Z_{\mathbf{m}} \setminus \{z\} \subset Z_{\mathbf{m}} = \mathrm{Cl}_{Z_{\mathbf{m}}} \alpha^{-1}((0,1]),$$

so $Z_{\mathbf{m}} \setminus \{z\}$ is connected. Thus z is not a cut point.

Hence $Z_{\mathbf{m}}$ has no cut point.

Step 2. Every positive Whitney level has a cut point. Let $\mu : C(Z_{\mathbf{m}}) \to [0, \mu(Z_{\mathbf{m}})]$ be a Whitney map and $s \in (0, \mu(Z_{\mathbf{m}}))$. Choose $a, b \in (0,3)$ such that

$$\mu(\alpha^{-1}([a,b])) = s.$$

(This is possible by (3), the fact that $\alpha^{-1}(3)$ is a point, and $Z_{\mathbf{m}} = \mathrm{Cl}_{Z_{\mathbf{m}}} \alpha^{-1}((0,3])$.)

We claim that

$$(\sharp)\ \mu^{-1}(s) = \{C \in \mu^{-1}(s) : C \subset \alpha^{-1}([0,b])\} \ \cup\ \{C \in \mu^{-1}(s) : C \subset \alpha^{-1}([a,3])\}.$$

Indeed, if not, there exists $D \in \mu^{-1}(s)$ meeting both $\alpha^{-1}([0,a))$ and $\alpha^{-1}((b,3])$. Then $\alpha(D)$ contains $[a,b]$ as a proper subcontinuum. Since α is atomic, $D = \alpha^{-1}(\alpha(D))$, so D properly contains $\alpha^{-1}([a,b])$. However, both D and $\alpha^{-1}([a,b])$ belong to $\mu^{-1}(s)$, a contradiction. Thus $(\sharp)$ holds.

Moreover:

- The two sets in ($\sharp$) intersect exactly in $\{\alpha^{-1}([a, b])\}$.
- Each is a nondegenerate subcontinuum of $\mu^{-1}(s)$ (see [7, Exercise 27.7]).

Hence $\alpha^{-1}([a, b])$ is a cut point of $\mu^{-1}(s)$.

Therefore, $Z_{\mathbf{m}}$ has no cut point, whereas every Whitney level of $Z_{\mathbf{m}}$ does. Hence, $\{Z_{\mathbf{m}} : \mathbf{m}$ is a meandering pattern$\} \subset \mathbb{CNWR}$. Moreover, for each meandering pattern $\mathbf{m}$ there exists a continuous surjection from $Z_{\mathbf{m}}$ onto $M_I^{\mathbf{m}}$. Together with Theorem 2.4, which asserts that $\mathbb{M}_I$ admits no common model, this shows that $\{Z_{\mathbf{m}} : \mathbf{m}$ is a meandering pattern$\}$ does not admit a common model. Consequently, $\mathbb{CNWR}$ admits no common model. $\square$

Our next goal is to prove Theorem 4.4. To this end, we first introduce some definitions. A continuum X is called a *Wilder continuum* if, for any three distinct points $x, y, z \in X$, there exists a subcontinuum L of X such that $x \in L$ and L contains exactly one of y and z. If, in addition, the subcontinuum L can be chosen to contain any two of the three points while excluding the third, then X is called a *strongly Wilder continuum*. Refer to [8, Figure 1] for the relationship between the class of Wilder continua and other classes of continua.

Here, we introduce the following lemma, which will be used in the proof of Theorem 4.4.

Lemma 4.2 ([18, Theorem 5.3]) *Let X be a nondegenerate continuum and let $\mu : C(X) \to [0, \mu(X)]$ be a Whitney map. Assume there exists a totally disconnected subset T of X satisfying the following property:*

- *If a, b and c are mutually distinct points in $X \setminus T$, then there exists a subcontinuum L of X such that $a, b \in L$ and $c \notin L$.*

Then, $\mu^{-1}(t)$ is a strongly Wilder continuum for each $t \in (0, \mu(X))$. In particular, being a strongly Wilder continuum is a Whitney property.

Let $\mathbb{WNWR}$ be the set of all continua Z satisfying the following properties:

(1) For each Whitney map $\mu : C(Z) \to [0, \mu(Z)]$ and each $t \in (0, \mu(Z))$, $\mu^{-1}(t)$ is a strongly Wilder continuum.

(2) Z does not contain a Wilder continuum.

By [18, Example 5.6], we see that $\mathbb{WNWR}$ is nonempty. In fact, to show [18, Example 5.6], the author showed the following:

Example 4.3 *(see the argument in [18, Example 5.6])* There exist a continuum Z and a countable dense subset $T \subset Z$ satisfying the following property:

(1) For any three distinct points $a, b, c \in Z \setminus T$, there exists a subcontinuum $L \subset Z$ such that $a, b \in L$ and $c \notin L$.

(2) Z does not contain a Wilder continuum.

Theorem 4.4 $\mathbb{WNWR}$ *does not admit a common model.*

Proof Let X be the (X', A')–(X'', A'') mirror-glued continuum, where X' and X'' are both homeomorphic to Z as in Example 4.3, and $|X' \cap X''| = 1$.

For each meandering pattern $\mathbf{m}$, it is not difficult to check that $M_X^{\mathbf{m}}$ contains no Wilder continuum, and that there exists a countable dense subset $T \subset M_X^{\mathbf{m}}$ such that for any three distinct points $a, b, c \in M_X^{\mathbf{m}} \setminus T$, there exists a subcontinuum $L \subset M_X^{\mathbf{m}}$ with $a, b \in L$ and $c \notin L$ (each X_i and T_i^X used in the construction of $M_X^{\mathbf{m}}$ carries a subset corresponding to the set T appearing in the statement of Example 4.3. By taking the union of all these subsets together with the collection of all branch points arising in the course of the construction, we obtain the desired set T). Hence, by Lemma 4.2, $\mu^{-1}(t)$ is a strongly Wilder continuum for each Whitney map μ on $C(M_X^{\mathbf{m}})$ and for each $t \in (0, \mu(M_X^{\mathbf{m}}))$. Therefore, $\mathbb{M}_X \subset \mathbb{WNWR}$. By Theorem 2.4, $\mathbb{M}_X$ does not admit a common model, and hence neither does $\mathbb{WNWR}$. $\square$

Our final goal in this section is to prove Theorem 4.11. Before doing so, we present some related results.

Theorem 4.5 (see [6, Theorem 1.1]) *For each continuum X, there exists an indecomposable continuum Y containing X and an open retraction $r : Y \to X$ such that each fiber $r^{-1}(x)$ is homeomorphic to the Cantor set. Moreover, Y is homeomorphic to the closure of a countable union of topological copies of X in Y.*

Remark 4.6 In [6, Theorem 1.1], it is asserted that Y is homeomorphic to the closure, taken in a certain ambient space, of a countable union of topological copies of X. Nevertheless, one readily verifies that the closure may in fact be taken within Y itself.

To prove Theorem 4.5, [6] shows the following.

$(*)$ For each continuum X, there exists an indecomposable continuum Y containing a topological copy X' of X and an open retraction $r : Y \to X'$ such that each fiber of r is homeomorphic to the Cantor set. Moreover, Y is homeomorphic to the closure of a countable union of topological copies of X in Y.

Remark 4.7 Strictly speaking, in the above, Y may not contain X itself but only a topological copy of X. However, by forming a new space in which the union of X and the complement of that copy in Y is endowed with the topology induced by an appropriate neighborhood system, we may regard Y as containing X.

We now recall the construction of Y and r. Let $I_i = I$ for $i \geq 1$, and set $S = X \times \prod_{i=1}^{\infty} I_i$. Let $M_1 = X$ and let $X_1 = M_1 \times \{0\} \times \{0\} \times \cdots \subset S$. Inductively, suppose X_i and M_i have been defined. Choose a countable dense subset $\{a_i^n\}_{n=1}^{\infty} \subset M_i$. For each n, take a continuous surjection $g_i^n : M_i \to [1/(n+1), 1/n]$ such that $(g_i^n)^{-1}(1/n) = \{a_i^n\}$ and $(g_i^n)^{-1}(1/(n+1)) = \{a_i^{n+1}\}$, and let $G_i^n = \{(x, g_i^n(x)) : x \in M_i\} \subset M_i \times I_i$ denote its graph. Then set $M_{i+1} = (M_i \times \{0\}) \cup \bigcup_{n=1}^{\infty} G_i^n \subset M_i \times I_i$ and $X_{i+1} = M_{i+1} \times \{0\} \times \{0\} \times \cdots \subset M_{i+1} \times \prod_{j=i+1}^{\infty} I_j$. Let $f_i : X_{i+1} \to X_i$ be the natural projection. Repeating this construction, define $Y = \varprojlim \{X_i, f_i\}$.

Define $X' = \{((x, \mathbf{0}), (x, \mathbf{0}), \ldots) \in Y : x \in X\}$, which is a topological copy of X in Y, where $\mathbf{0} = (0, 0, \ldots) \in \prod_{i=1}^{\infty} I_i$. To define a retraction $r : Y \to X'$, consider the projection $\pi_1 : Y \to X_1$ and the natural homeomorphism $h : X_1 \to X'$, and set $r = h \circ \pi_1$.

Lemma 4.8 *In the above construction, for each point $\mathbf{x} \in X'$, there exists a point $\mathbf{x}' \in r^{-1}(\mathbf{x})$ such that $\mathbf{x}$ and $\mathbf{x}'$ lie in different composants of Y.*

Proof Let $\mathbf{x} = ((x, \mathbf{0}), (x, \mathbf{0}), (x, \mathbf{0}), ...) \in X'$. Consider a point

$$\mathbf{x}' = ((x, 0, 0, 0, ...), (x, x_1, 0, 0, ...), (x, x_1, x_2, 0, ...), ...) \in Y,$$

where $f_{i+1}(x, x_1, x_2, ..., x_i, x_{i+1}, 0, 0, ...) = (x, x_1, x_2, ..., x_i, 0, 0, ...)$, $x_i > 0$ for each $i \geq 1$, and $f_1(x, x_1, 0, 0, ...) = (x, 0, 0, ...)$. Note that $\mathbf{x}, \mathbf{x}' \in r^{-1}(\mathbf{x})$. We now show that $\mathbf{x}$ and $\mathbf{x}'$ lie in different composants of Y. Let L be a subcontinuum of Y that contains $\mathbf{x}$ and $\mathbf{x}'$. Let $i \geq 1$. Then, $(x, \mathbf{0})$ and $(x, x_1, x_2, ..., x_i, 0, 0, ...)$ are contained in $\pi_{i+1}(L)$. Since $x_i > 0$, $\{a_i^n\}_{n=1}^{\infty}$ is dense in M_i, and by the construction of M_{i+1}, we can easily see that $\pi_i(L) = f_i(\pi_{i+1}(L)) = M_i \times \{0\} \times \{0\} \times \cdots = X_i$. Since $i \geq 1$ is arbitrary, we see that $L = Y$. This completes the proof. □

The following lemma may be known; however, the author could not find any reference containing this result. Hence, for completeness, we provide the proof. The proof is exactly the same as that of [7, Exercise 44.14].

Lemma 4.9 (cf. [7, Exercise 44.14]) *Let X be an indecomposable continuum, and let $a, b \in X$ lie in distinct composants of X. Define X' as the quotient space obtained from X by identifying a and b. Then X' is an indecomposable continuum, and for every Whitney map $\mu : C(X') \to [0, \mu(X')]$ and every $t \in (0, \mu(X'))$, the fiber $\mu^{-1}(t)$ is a decomposable continuum.*

Proof Let $q : X \to X'$ be the quotient map. Since X is indecomposable, it has uncountably many composants (see [21, Theorem 11.15]). Consequently, X' also has uncountably many composants. By [21, Theorem 11.13], this implies that X' is an indecomposable continuum.

Next, let $\mu : C(X') \to [0, \mu(X')]$ be a Whitney map and fix $t \in (0, \mu(X'))$. Define $\nu : C(X) \to [0, \nu(X)]$ by $\nu(C) = \mu(q(C))$ for each $C \in C(X)$. Then ν is a Whitney map. We have

$$\mu^{-1}(t) = \{\, q(C) : C \in \nu^{-1}(t) \,\} \cup \{\, F \in \mu^{-1}(t) : q(a) \in F \,\}.$$

Since the map $\hat{q} : \nu^{-1}(t) \to \mu^{-1}(t)$ defined by $\hat{q}(C) = q(C)$ is continuous and $\{\, q(C) : C \in \nu^{-1}(t) \,\} = \hat{q}(\nu^{-1}(t))$, it follows that $\{\, q(C) : C \in \nu^{-1}(t) \,\}$ is a continuum. Moreover, by [7, Exercise 27.7], the set $\{\, F \in \mu^{-1}(t) : q(a) \in F \,\}$ is a subcontinuum of $\mu^{-1}(t)$. By the continuity of q and ORDER ARC THEOREM ([21, p.85]), it is easy to see that these two subcontinua intersect each other. Also, since a and b lie in distinct componsants of X, it is not difficult to see that neither of them is contained in the other. Hence $\mu^{-1}(t)$ is a decomposable continuum. □

Theorem 4.10 ([19, Theorem 3.3]) *Let Z be a continuum. Define $\mathbb{IC}_Z$ to be the family of indecomposable continua Y satisfying:*

(I) *$Z \subset Y$;*

(II) *There exists an open retraction $r : Y \to Z$ such that each fiber $r^{-1}(z)$ is homeomorphic to the Cantor set;*
(III) *There exists a sequence $\{Z_i'\}_{i=1}^{\infty}$ of subcontinua of Y, each homeomorphic to Z, such that $\mathrm{Cl}_Y\left(\bigcup_{i=1}^{\infty} Z_i'\right) = Y$.*

Then, the following hold:

(1) The family $\mathbb{IC}_Z$ does not admit a common model.

(2) If Z is tree-like, then there exists a subfamily $\mathbb{IC}_Z'$ of $\mathbb{IC}_Z$ such that $\mathbb{IC}_Z'$ does not have a common model, and each member of $\mathbb{IC}_Z'$ is tree-like.

Theorem 4.11 *Let Z be a continuum. Define $\mathbb{DNWR}_Z$ to be the family of indecomposable continua Y satisfying:*

(I) *$Z \subset Y$;*
(II) *There exists a retraction $r : Y \to Z$ such that each fiber $r^{-1}(z)$ is homeomorphic to the Cantor set;*
(III) *There exists a sequence $\{Z_i'\}_{i=1}^{\infty}$ of subcontinua of Y, each homeomorphic to Z, such that $\mathrm{Cl}_Y\left(\bigcup_{i=1}^{\infty} Z_i'\right) = Y$.*
(IV) *For every Whitney map $\mu : C(Y) \to [0, \mu(Y)]$ and every $t \in (0, \mu(Y))$, the fiber $\mu^{-1}(t)$ is a decomposable continuum.*

Then, the family $\mathbb{DNWR}_Z$ does not admit a common model.

Proof Let Z be a continuum. Let X be the (X', A')–(X'', A'') mirror-glued continuum, where X' and X'' are both homeomorphic to Z and $|X' \cap X''| = 1$. By Theorem 4.5, for each $M_X^{\mathbf{m}} \in \mathbb{M}_X$, there exist an indecomposable continuum $C_X^{\mathbf{m}} \supset M_X^{\mathbf{m}}$ and an open retraction $r_{\mathbf{m}} : C_X^{\mathbf{m}} \to M_X^{\mathbf{m}}$ such that each fiber of $r_{\mathbf{m}}$ is homeomorphic to the Cantor set. Also, by the proof of Theorem 4.10 (i.e., the proof of [19, Theorem 3.3]), there exist $Z^{\mathbf{m}} \subset M_X^{\mathbf{m}}$ and an open retraction $r_{\mathbf{m}}' : M_X^{\mathbf{m}} \to Z^{\mathbf{m}} \cong Z$ such that each fiber of $r_{\mathbf{m}}'$ is totally disconnected. Moreover, by Theorem 4.5 and the construction of $M_X^{\mathbf{m}}$, we can see that there exists a sequence $\{Z_i^{\mathbf{m}}\}_{i=1}^{\infty}$ of subcontinua of $C_X^{\mathbf{m}}$, each homeomorphic to Z, such that $\mathrm{Cl}_{C_X^{\mathbf{m}}}\left(\bigcup_{i=1}^{\infty} Z_i^{\mathbf{m}}\right) = C_X^{\mathbf{m}}$.

Fix $x_{\mathbf{m}} \in M_X^{\mathbf{m}}$ for each meandering pattern $\mathbf{m}$. By Lemma 4.8, there exist distinct points $a_{\mathbf{m}}, b_{\mathbf{m}} \in r_{\mathbf{m}}^{-1}(x_{\mathbf{m}})$ that lie in different composants of $C_X^{\mathbf{m}}$. Let $D_X^{\mathbf{m}}$ be the quotient space obtained from $C_X^{\mathbf{m}}$ by identifying $a_{\mathbf{m}}$ and $b_{\mathbf{m}}$. Let $q_{\mathbf{m}} : C_X^{\mathbf{m}} \to D_X^{\mathbf{m}}$ be the quotient map. Then, $q_{\mathbf{m}} \circ r_{\mathbf{m}}' \circ r_{\mathbf{m}} \circ q_{\mathbf{m}}^{-1} : D_X^{\mathbf{m}} \to q_{\mathbf{m}}(Z^{\mathbf{m}}) \cong Z$ is a retraction such that each fiber of it is homeomorphic to the Cantor set.

Since $C_X^{\mathbf{m}}$ is indecomposable, as in the beginning of the proof of Lemma 4.9, we see that $D_X^{\mathbf{m}}$ is also indecomposable. Since $\mathrm{Cl}_{C_X^{\mathbf{m}}}\left(\bigcup_{i=1}^{\infty} Z_i^{\mathbf{m}}\right) = C_X^{\mathbf{m}}$, it follows that $\mathrm{Cl}_{D_X^{\mathbf{m}}}\left(\bigcup_{i=1}^{\infty} q_{\mathbf{m}}(Z_i^{\mathbf{m}})\right) = D_X^{\mathbf{m}}$. By [21, Exercise 6.19], each subcontinuum of $D_X^{\mathbf{m}}$ has an empty interior. Note that for each $i \geq 1$, $Z_i^{\mathbf{m}}$ and $q_{\mathbf{m}}(Z_i^{\mathbf{m}})$ are homeomorphic to each other if $a_{\mathbf{m}} \notin Z_i^{\mathbf{m}}$ and $b_{\mathbf{m}} \notin Z_i^{\mathbf{m}}$. Hence, we can easily see that there exists a sequence $\{Z_i'\}_{i=1}^{\infty}$ of subcontinua of $D_X^{\mathbf{m}}$, each homeomorphic to Z, such that $\mathrm{Cl}_{D_X^{\mathbf{m}}}\left(\bigcup_{i=1}^{\infty} Z_i'\right) = D_X^{\mathbf{m}}$.

By Lemma 4.9, for every Whitney map $\mu : C(D_X^{\mathbf{m}}) \to [0, \mu(D_X^{\mathbf{m}})]$ and every $t \in (0, \mu(D_X^{\mathbf{m}}))$, the fiber $\mu^{-1}(t)$ is a decomposable continuum.

Let $\mathbb{IC}_X^{\sharp} = \{D_X^{\mathbf{m}} : \mathbf{m}$ is a meandering pattern$\}$. From the above arguments and the idea of Remark 4.7, we may regard that $\mathbb{IC}_X^{\sharp} \subset \mathbb{DNWR}_Z$. By Theorem 2.4, $\mathbb{M}_X$

admits no common model. Since $r_{\mathbf{m}} \circ q_{\mathbf{m}}^{-1} : D_X^{\mathbf{m}} \to M_X^{\mathbf{m}}$ is a continuous surjection for each meandering pattern $\mathbf{m}$, it follows that $\mathbb{IC}_X^{\sharp}$ admits no common model. Therefore, $\mathbb{DNWR}_Z$ also admits no common model. This completes the proof. □

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Declarations

Conflict of Interest The author declares no conflict of interest.

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