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Journal

Mathematical Methods in the Applied Sciences 48 (18) , 16690 - 16699

Published

2025-09-24

URL (The Version of Record)

<https://doi.org/10.1002/mma.70119>

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ARTICLE TYPE

Oscillation criteria of the Leighton–Wintner-type and the Hille–Kneser-type for linear differential equations with a proportional derivative controller

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Summary

In this study, we investigated the oscillatory and nonoscillatory behavior of solutions to linear differential equations with proportional–derivative (PD) control. In particular, we aimed to extend classical oscillation theorems, such as the Leighton–Wintner oscillation theorem and the Hille–Kneser oscillation and nonoscillation criteria, to make them applicable to this class of equations. The Leighton–Wintner oscillation theorem was proved using the Riccati technique, while the Hille–Kneser oscillation and nonoscillation criteria were derived by clarifying the oscillation constant of Euler-type equations with PD control.

KEYWORDS:

Oscillation theory, Leighton–Wintner-type, Hille–Kneser-type, Euler-type equation, proportional–derivative controller

1 | INTRODUCTION

The PD controller (proportional–derivative controller) appearing in the field of control engineering

$$u(t) = \kappa_p E(t) + \kappa_d \frac{d}{dt} E(t)$$

is used to improve control response and has applications in robotics analysis (see, for example, ^{1,2}). Here, for the control output u , κ_p is the proportional control gain, κ_d is the derivative control gain, and E is the error (input deviation) between the state variable and the process variable. Recently, Anderson and Ulness ³ defined the differential operator D^α described below, inspired by PD control.

Definition 1. Let α be a fixed real number from interval $(0, 1)$. Let two continuous functions $\kappa_0 : (0, 1) \times \mathbb{R} \rightarrow [0, \infty)$ and $\kappa_1 : (0, 1) \times \mathbb{R} \rightarrow [0, \infty)$ satisfy

$$\begin{cases} \lim_{\alpha \rightarrow 0+} \kappa_0(\alpha, t) = 0, & \lim_{\alpha \rightarrow 0+} \kappa_1(\alpha, t) = 1, \\ \lim_{\alpha \rightarrow 1-} \kappa_0(\alpha, t) = 1, & \lim_{\alpha \rightarrow 1-} \kappa_1(\alpha, t) = 0, \end{cases} \quad (1.1)$$

and

$$\begin{cases} \kappa_0(\alpha, t) \neq 0, \alpha \in (0, 1), \\ \kappa_1(\alpha, t) \neq 0, \alpha \in (0, 1). \end{cases} \quad (1.2)$$

Then, the differential operator D^α is defined as follows:

$$D^\alpha f(t) = \kappa_1(\alpha, t)f(t) + \kappa_0(\alpha, t)\frac{d}{dt}f(t), \quad (1.3)$$

where κ_0 and κ_1 are the functions that satisfy (1.1) and (1.2).

For the reason mentioned above, Equation (1.3) in Definition 1 is also called the *proportional-derivative controller*. From condition (1.1), we have

$$\lim_{\alpha \rightarrow 0^+} D^\alpha f(t) = D^0 f(t) = f(t) \quad \text{and} \quad \lim_{\alpha \rightarrow 1^-} D^\alpha f(t) = D^1 f(t) = f'(t),$$

where $' = d/dt$. Thus, D^0 is the identity operator, and D^1 is the classical differential operator. For further details on Definition 1, see^{3,4,5}.

In recent years, research related to the differential operator D^α in Definition 1 has been actively conducted. For example, for $\alpha \in (0, 1]$, Khalil et al.⁶ introduced the following definition, known as the conformable derivative:

$$\begin{aligned} T^\alpha f(t) &= \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \\ T^\alpha f(0) &= \lim_{t \rightarrow 0^+} T^\alpha f(t). \end{aligned} \quad (1.4)$$

If f is differentiable, then $T^\alpha f(t) = t^{1-\alpha} f'(t)$. Based on Definition (1.4), many researchers have attempted further extensions (see, for example, page 150 of⁷, Section 6). However, it should be noted that Definition (1.4) does not satisfy the identity operator when $\alpha \rightarrow 0$. Anderson and Ulness³ introduced Definition 1 with the aim of addressing this issue. Indeed, according to³, Definition 1 can encompass (1.4).

This is because, for $\alpha \in (0, 1)$, by relaxing the nonzero condition on κ_1 in condition (1.2) of Definition 1 (setting $\kappa_1(\alpha, t) \equiv 0$) and defining $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, we obtain the following result. By substituting $\alpha \frac{-1}{1-\alpha} s$, it follows that $\kappa_0(\alpha, t) = s^{1-\alpha}$, which coincides with the definition given by Khalil et al.⁶ when f is differentiable. Definition 1 partially extends (1.4), and it is also referred to as a *conformable derivative*.

We consider the oscillation of the linear differential equations with a proportional derivative controller

$$D^\alpha[r(t)D^\alpha u] + c(t)u = 0, \quad t \geq t_0, \quad (1.5)$$

where $\alpha \in (0, 1)$ and $r, c : [t_0, \infty) \rightarrow \mathbb{R}$ are continuous functions with $t_0 \geq 0$ and $r(t) > 0$ for $t \geq t_0$. Note that if $\alpha \rightarrow 1$, Equation (1.5) reduces to the second-order linear differential equation

$$(r(t)u')' + c(t)u = 0, \quad t \geq t_0. \quad (1.6)$$

Furthermore, when $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, Equation (1.5) can be transformed into the following equation:

$$\hat{D}^\alpha[p(s)\hat{D}^\alpha v] + q(s)v = 0, \quad s \geq s_0 = \alpha^{-\frac{1}{\alpha}} t_0. \quad (1.7)$$

Here $\hat{D}^\alpha v$ is defined as

$$\hat{D}^\alpha v(s) := \kappa_1\left(\alpha, \alpha^{\frac{1}{\alpha}} s\right)v(s) + s^{1-\alpha}\dot{v}(s),$$

where $\cdot = d/ds$, $p(s) = r(\alpha^{\frac{1}{\alpha}} s)$, and $q(s) = c(\alpha^{\frac{1}{\alpha}} s)$. Indeed, let $u(t)$ be a solution of Equation (1.5), and set $s = \alpha^{-\frac{1}{\alpha}} t$ and $v(s) = u(t)$. Then, we obtain

$$\begin{aligned} \hat{D}^\alpha v(s) &= \kappa_1(\alpha, t)u(t) + s^{1-\alpha}u'(t)\frac{dt}{ds} \\ &= \kappa_1(\alpha, t)u(t) + \alpha t^{1-\alpha}u'(t) = D^\alpha u(t). \end{aligned}$$

Thus, we have

$$\begin{aligned} \hat{D}^\alpha[p(s)\hat{D}^\alpha v(s)] &= \kappa_1(\alpha, t)r(t)(D^\alpha u(t)) + s^{1-\alpha}(r(t)(D^\alpha u(t)))'\frac{dt}{ds} \\ &= \kappa_1(\alpha, t)r(t)(D^\alpha u(t)) + \alpha t^{1-\alpha}(r(t)(D^\alpha u(t)))' \\ &= D^\alpha[r(t)D^\alpha u(t)] = -c(t)u(t) = -q(s)v(s). \end{aligned}$$

In particular, for $\alpha \in (0, 1)$, if we relax the nonzero condition on κ_1 in (1.2) of Definition 1 (i.e., setting $\kappa_1(\alpha, t) \equiv 0$), then Equation (1.7) reduces to an equation that adopts the special case of the definition given in (1.4). Therefore, by investigating the properties of solutions to Equation (1.5) when $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, we can also gain insight into the properties of solutions to linear differential equations that utilize the definition in (1.4).

The uniqueness of solutions to the initial value problem for Equation (1.5) is guaranteed (see⁴). We now define the oscillatory behavior of solutions to Equation (1.5), which will be discussed in this study. A non-trivial solution u of (1.5) is said to be *nonoscillatory* on $[t_0, \infty)$ if it is eventually either positive or negative. Otherwise, the non-trivial solution u of (1.5) is said to be *oscillatory*. The Sturm comparison theorem and separation theorem for Equation (1.5) have already been established (see⁴, Theorem 7.2 and⁸, Theorem 8.3.6 for details). However, many aspects of classical oscillation and nonoscillation theorems applicable to Equation (1.5) remain unexplored. The purpose of this study is to customize the classical oscillation and nonoscillation theorems introduced below so that they can also be applied to Equation (1.5).

Theorem 1. If

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \frac{1}{r(s)} ds = \infty \quad (1.8)$$

and

$$\lim_{t \rightarrow \infty} \int_{t_0}^t c(s) ds = \infty, \quad (1.9)$$

then all non-trivial solutions of Equation (1.6) are oscillatory.

Theorem 2. Let $r(t) \equiv 1$. Suppose that

$$\omega_* = \liminf_{t \rightarrow \infty} t^2 c(t), \quad \omega^* = \limsup_{t \rightarrow \infty} t^2 c(t). \quad (1.10)$$

Then, all non-trivial solutions of (1.6) are oscillatory if $\omega_* > 1/4$ and all non-trivial solutions of (1.6) are nonoscillatory if $\omega^* < 1/4$, but it can be either oscillatory or nonoscillatory if either ω_* or ω^* equals $1/4$.

Remark 1. Theorem 1 is the Leighton–Wintner oscillation theorem (see, for example,^{9,10} and¹¹, Theorem 2.4). A notable feature of Theorem 1 is that it remains applicable even when the coefficient function c in Equation (1.6) changes sign. In fact, by utilizing Theorem 1, oscillation criteria have been derived for equations with periodic coefficients, such as the Mathieu equation and the Hill equation (see, for example^{12,13,14}). Moreover, the widespread recognition of Theorem 1 is evident from the numerous studies that have extended it, leading to new theorems applicable to other types of equations, including half-linear differential equations, nonlinear differential equations, impulsive differential equations, and differential equations involving the $p(t)$ -Laplacian (see, for example^{15,16,17,18,19,20,21}). Furthermore, in recent years, a Leighton–Wintner-type oscillation theorem has been proposed for nonlinear differential equations that employ the definition in (1.4), as introduced by Khalil et al. However, in this case, it is assumed that the coefficient function c is always positive (see²², Theorem 1). Additionally, Çetinkaya and Cuchta²³ have proposed a Wintner-type oscillation theorem for Equation (1.5) on time scales, where the coefficient function r is assumed to be bounded. The extent to which these assumptions can be relaxed remains an open problem.

Remark 2. Theorem 2 is the Hille–Kneser criterion (see, for example,²⁴, Theorem 9 and¹¹, Theorem 2.41). The proof of Theorem 2 can be derived using Sturm’s comparison theorem by considering the Euler equation with a real parameter λ ,

$$u'' + \frac{\lambda}{t^2} u = 0, \quad t \geq t_0 = 1 \quad (1.11)$$

along with Equation (1.6) when $r(t) \equiv 1$. In fact, the general solution of Equation (1.11) can be explicitly obtained. Specifically, when $\lambda > 1/4$, all non-trivial solutions of Equation (1.11) are oscillatory. On the other hand, when $\lambda \leq 1/4$, all non-trivial solutions of Equation (1.11) are nonoscillatory. The threshold value $1/4$, which separates oscillatory and nonoscillatory solutions, is called the *oscillation constant*. The value $1/4$ appearing in Theorem 2 corresponds to the oscillation constant of Equation (1.11). Recently, Ishibashi²⁵ studied the Euler-type equation

$$D^\alpha[D^\alpha u] + \frac{\lambda}{t^{2\alpha}} u = 0, \quad t \geq t_0 = 1, \quad (1.12)$$

where $\alpha \in (0, 1)$, $\kappa_1(\alpha, t) \equiv 0$, $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$. Ishibashi²⁵ demonstrated that if $\lambda \leq \alpha^4/4$, then all non-trivial solutions of Equation (1.12) are nonoscillatory. Additionally, based on numerical simulations, it was conjectured that if $\lambda > \alpha^4/4$, then all non-trivial solutions of Equation (1.12) might be oscillatory. A rigorous proof of this conjecture remains an open problem.

In this study, we focus on the following two points to address the issues raised in Remarks 1 and 2:

- (i) Deriving an oscillation theorem similar to Theorem 1, which is applicable to Equation (1.5), while allowing the coefficient function r to be unbounded and c to change sign.

- (ii) Clarifying the oscillation constant for Equation (1.12). Furthermore, we aim to establish a version of Theorem 2 that can be applied to Equation (1.5) when $r(t) \equiv 1$ and $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$.

The structure and details of this paper are as follows.

First, the fundamental concepts of calculus using the differential operator D^α defined in Definition 1 are summarized in Section 5 and readers are encouraged to refer to it as needed. In Section 2, we address Problem (i). We also provide examples where Equation (1.5) is applicable, even when the coefficient c changes sign or when the coefficient function r is unbounded. In Section 3, we clarify the oscillation constant of Equation (1.12). To achieve this, we propose an Euler-type equation that includes Equation (1.11) and derive its oscillation constant, thereby resolving Problem (ii). Additionally, we establish Theorem 2, which is applicable to Equation (1.5) when $r(t) \equiv 1$ and $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$. Finally, in Section 4, we summarize the findings of this study.

2 | OSCILLATION CRITERION OF LEIGHTON–WINTNER TYPE

The Leighton–Wintner oscillation criterion applicable to Equation (1.5) is given as follows.

Theorem 3. If

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \frac{e_0(t, t_0)}{r(s)} d_\alpha s = \infty \quad (2.1)$$

and

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s = \infty, \quad (2.2)$$

then all non-trivial solutions of Equation (1.5) are oscillatory. Here, $e_0(t, s) = e^{-\int_s^t \kappa_1(\alpha, \tau) d_\alpha \tau}$, where $d_\alpha s = ds/\kappa_0(\alpha, s)$ (for details, see Theorem 6 in Section 5 and Definition 2).

Proof. To prove by contradiction, suppose that there exists $t_1 \geq t_0$ such that $u(t) > 0$ for all $t \geq t_1$ (the proof for negative $u(t)$ is similar and thus omitted). Then, we define a differentiable function z as follows:

$$z(t) = -\frac{r(t)D^\alpha u(t)}{u(t)}.$$

Then, we have

$$\begin{aligned} 0 &= D^\alpha [r(t)D^\alpha u(t)] + c(t)u(t) \\ &= D^\alpha [-z(t)u(t)] + c(t)u(t) \\ &= -(D^\alpha z(t))u(t) - z(t)(D^\alpha u(t)) + z(t)u(t)\kappa_1(\alpha, t) + c(t)u(t) \\ &= -(D^\alpha z(t))u(t) + \frac{z^2(t)u(t)}{r(t)} + z(t)u(t)\kappa_1(\alpha, t) + c(t)u(t) \\ &= \left(\frac{z^2(t)}{r(t)} - D^\alpha z(t) + \kappa_1(\alpha, t)z(t) + c(t) \right) u(t). \end{aligned}$$

From this, the corresponding first-order nonlinear equation for Equation (1.5) is given by

$$\begin{aligned} D^\alpha z(t) &= \frac{z^2(t)}{r(t)} + \kappa_1(\alpha, t)z(t) + c(t) \\ &= \frac{1}{r(t)} \left(z^2(t) + r(t)\kappa_1(\alpha, t)z(t) \right) + c(t) \\ &= \frac{1}{r(t)} \left(\left(z(t) + \frac{r(t)\kappa_1(\alpha, t)}{2} \right)^2 - \frac{r^2(t)\kappa_1^2(\alpha, t)}{4} \right) + c(t) \\ &= \frac{1}{r(t)} \left(z(t) + \frac{r(t)\kappa_1(\alpha, t)}{2} \right)^2 + c(t) - \frac{r(t)\kappa_1^2(\alpha, t)}{4}. \end{aligned} \quad (2.3)$$

Multiplying both sides of (2.3) by $e_0(\tau, t)$, we obtain

$$(D^\alpha z(t))e_0(\tau, t) = \frac{1}{r(t)} \left(z(t) + \frac{r(t)\kappa_1(\alpha, t)}{2} \right)^2 e_0(\tau, t) + \left(c(t) - \frac{r(t)\kappa_1^2(\alpha, t)}{4} \right) e_0(\tau, t).$$

Further integrating both sides of this equation with respect to the α -integral from t_1 to t the left-hand side becomes

$$z(t) - z(t_1)e_0(t, t_1),$$

while the right-hand side is given by

$$\int_{t_1}^t \frac{1}{r(s)} \left(z(s) + \frac{r(s)\kappa_1(\alpha, s)}{2} \right)^2 e_0(t, s) d_\alpha s + \int_{t_1}^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s.$$

Thus, we obtain

$$\begin{aligned} z(t) &= z(t_1)e_0(t, t_1) + \int_{t_1}^t \frac{1}{r(s)} \left(z(s) + \frac{r(s)\kappa_1(\alpha, s)}{2} \right)^2 e_0(t, s) d_\alpha s \\ &\quad + \int_{t_1}^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s. \end{aligned} \quad (2.4)$$

Here, for any $t \geq t_1$, we have

$$0 < e_0(t, t_1) = e^{-\int_{t_1}^t \frac{\kappa_1(\alpha, s)}{\kappa_0(\alpha, s)} d_\alpha s} \leq 1.$$

Using this, we obtain the following inequality:

$$\begin{aligned} z(t) &\geq -|z(t_1)| + \int_{t_1}^t \frac{1}{r(s)} \left(z(s) + \frac{r(s)\kappa_1(\alpha, s)}{2} \right)^2 e_0(t, s) d_\alpha s \\ &\quad + \int_{t_1}^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s. \end{aligned}$$

For simplicity, let

$$I(t) = \int_{t_1}^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s.$$

From condition (2.2), we have

$$\lim_{t \rightarrow \infty} z(t) = \infty. \quad (2.5)$$

Furthermore, there exists $t_2 \geq t_1$ such that for any $t \geq t_2$, we have $I(t) \geq 0$ and $z(t) > 0$. Here, since $r(t) > 0$ for any $t \geq t_2$, we obtain

$$D^\alpha u(t) = -\frac{z(t)u(t)}{r(t)} < 0 \quad (2.6)$$

as well as

$$\int_{t_2}^t c(s)e_0(t, s) d_\alpha s \geq 0. \quad (2.7)$$

Now, multiplying both sides of Equation (1.5) by $e_0(\tau, t)$ and then integrating with respect to the α -integral, we obtain

$$\begin{aligned} &\int_{t_2}^t D^\alpha [r(s)D^\alpha u(s)]e_0(t, s) d_\alpha s + \int_{t_2}^t c(s)u(s)e_0(t, s) d_\alpha s \\ &= r(t)(D^\alpha u(t)) - r(t_2)(D^\alpha u(t_2))e_0(t, t_2) + \int_{t_2}^t c(s)u(s)e_0(t, s) d_\alpha s = 0. \end{aligned}$$

Thus, we derive

$$r(t)(D^\alpha u(t)) = r(t_2)(D^\alpha u(t_2))e_0(t, t_2) - \int_{t_2}^t c(s)u(s)e_0(t, s)d_\alpha s. \quad (2.8)$$

Using (v) and (vi) of Theorem 7 (see Section 5), we obtain

$$\begin{aligned} \int_{t_2}^t c(s)u(s)e_0(t, s)d_\alpha s &= u(s) \int_{t_2}^s c(\tau)e_0(s, \tau)d_\alpha \tau \Big|_{s=t_2}^t \\ &\quad - \int_{t_2}^t \left(\int_{t_2}^s c(\tau)e_0(s, \tau)d_\alpha \tau \right) (D^\alpha u(s) - \kappa_1(\alpha, s)u(s))e_0(t, s)d_\alpha s \\ &= u(t) \int_{t_2}^t c(s)e_0(t, s)d_\alpha s \\ &\quad - \int_{t_2}^t \left(\int_{t_2}^s c(\tau)e_0(s, \tau)d_\alpha \tau \right) (D^\alpha u(s) - \kappa_1(\alpha, s)u(s))e_0(t, s)d_\alpha s. \end{aligned}$$

Then, using (2.6), (2.7), and (2.8), we obtain

$$\begin{aligned} r(t)(D^\alpha u(t)) &= r(t_2)(D^\alpha u(t_2))e_0(t, t_2) - u(t) \int_{t_2}^t c(s)e_0(t, s)d_\alpha s \\ &\quad + \int_{t_2}^t \left(\int_{t_2}^s c(\tau)e_0(s, \tau)d_\alpha \tau \right) (D^\alpha u(s) - \kappa_1(\alpha, s)u(s))e_0(t, s)d_\alpha s \\ &\leq r(t_2)(D^\alpha u(t_2))e_0(t, t_2). \end{aligned}$$

Thus, for any $t \geq t_2$, we obtain

$$D^\alpha u(t) \leq \frac{r(t_2)}{r(t)}(D^\alpha u(t_2))e_0(t, t_2).$$

Furthermore, multiplying both sides of this inequality by $e_0(\tau, t)$ gives

$$(D^\alpha u(t))e_0(\tau, t) \leq \frac{r(t_2)}{r(t)}(D^\alpha u(t_2))e_0(t, t_2)e_0(\tau, t) = \frac{r(t_2)}{r(t)}(D^\alpha u(t_2))e_0(\tau, t_2).$$

Integrating both sides with respect to the α -integral yields

$$u(t) - u(t_2)e_0(t, t_2) \leq r(t_2)(D^\alpha u(t_2)) \int_{t_2}^t \frac{e_0(t, t_2)}{r(s)}d_\alpha s.$$

Therefore, from conditions (2.1) and (2.6), we obtain

$$u(t) \leq u(t_2)e_0(t, t_2) + r(t_2)(D^\alpha u(t_2)) \int_{t_2}^t \frac{e_0(t, t_2)}{r(s)}d_\alpha s \leq u(t_2) + r(t_2)(D^\alpha u(t_2)) \int_{t_2}^t \frac{e_0(t, t_2)}{r(s)}d_\alpha s.$$

Thus, we find that $u(t) \rightarrow -\infty$ as $t \rightarrow \infty$, which contradicts that $u(t) > 0$ for all $t \geq t_1$. $\square$

Remark 3. When $\alpha \rightarrow 1$, the conditions (2.1) and (2.2) in Theorem 3 coincide with the conditions (1.8) and (1.9) in Theorem 1. Moreover, the coefficient function c in Equation (1.5) does not need to be positive. Equation (2.3), which was used in the proof of Theorem 3, is a Riccati-type equation corresponding to Equation (1.5) (see^{25, Theorem 3.1}).

We present an example where the coefficient function c in Equation (1.5) changes sign, as well as an example where the coefficient function r is unbounded.

Example 1. Let $\kappa_0(\alpha, t) = \alpha$ and $\kappa_1(\alpha, t) = 1 - \alpha$. If $1 > \alpha > (-3 + \sqrt{741})/61 = 0.397071 \dots$, then all non-trivial solutions of

$$D^\alpha \left[\frac{1}{e^{\frac{1}{\alpha}t}} D^\alpha u \right] + \left(\frac{1}{2} e^{\frac{1}{\alpha}t} + e^{\frac{1}{\alpha}t} \cos(8t) \right) u = 0, \quad t \geq t_0 = 0 \quad (2.9)$$

are oscillatory. Comparing the coefficient functions of Equation (2.9) with those of Equation (1.5), we obtain

$$r(t) = \frac{1}{e^{\frac{1}{\alpha}t}} \quad \text{and} \quad c(t) = \frac{1}{2} e^{\frac{1}{\alpha}t} + e^{\frac{1}{\alpha}t} \cos(8t).$$

Here, the coefficient function c is a sign-changing function. Now, we verify the oscillation conditions for Equation (2.9). Noting the following functions:

$$e_0(t, 0) = e^{-\int_0^t \kappa_1(\alpha, \tau) d_\alpha \tau} = e^{-\int_0^t \frac{\kappa_1(\alpha, \tau)}{\kappa_0(\alpha, \tau)} d\tau} = \frac{1}{e^{\frac{1-\alpha}{\alpha}t}} \quad \text{and} \quad e_0(t, s) = \frac{e^{\frac{1-\alpha}{\alpha}s}}{e^{\frac{1-\alpha}{\alpha}t}},$$

we obtain

$$\int_0^t \frac{e_0(t, 0)}{r(s)} d_\alpha s = \frac{1}{\alpha e^{\frac{1-\alpha}{\alpha}t}} \int_0^t e^{\frac{1}{\alpha}s} ds = e^t - \frac{1}{e^{\frac{1-\alpha}{\alpha}t}} \rightarrow \infty$$

as $t \rightarrow \infty$. Thus, condition (2.1) is satisfied. On the other hand,

$$\begin{aligned} \int_0^t e^{\frac{2-\alpha}{\alpha}s} \cos(8s) ds &= \frac{(2-\alpha)\alpha \cos(8t) + 8\alpha^2 \sin(8t)}{(2-\alpha)^2 + 64\alpha^2} e^{\frac{2-\alpha}{\alpha}t} - \frac{(2-\alpha)\alpha}{(2-\alpha)^2 + 64\alpha^2} \\ &\geq -\frac{\alpha e^{\frac{2-\alpha}{\alpha}t}}{\sqrt{(2-\alpha)^2 + 64\alpha^2}} - \frac{(2-\alpha)\alpha}{(2-\alpha)^2 + 64\alpha^2}. \end{aligned}$$

Considering that $1 > \alpha > (-3 + \sqrt{741})/61 = 0.397071 \dots$, we obtain

$$\begin{aligned} &\int_0^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s \\ &= \frac{1}{2\alpha e^{\frac{1-\alpha}{\alpha}t}} \int_0^t e^{\frac{2-\alpha}{\alpha}s} ds + \frac{1}{\alpha e^{\frac{1-\alpha}{\alpha}t}} \int_0^t e^{\frac{2-\alpha}{\alpha}s} \cos(8s) ds - \frac{(1-\alpha)^2}{4\alpha e^{\frac{1-\alpha}{\alpha}t}} \int_0^t e^{-s} ds \\ &\geq \left(\frac{1}{2(2-\alpha)} - \frac{1}{\sqrt{(2-\alpha)^2 + 64\alpha^2}} \right) e^{\frac{1}{\alpha}t} - \frac{1}{2(2-\alpha)e^{\frac{1-\alpha}{\alpha}t}} \\ &\quad - \frac{2-\alpha}{((2-\alpha)^2 + 64\alpha^2)e^{\frac{1-\alpha}{\alpha}t}} + \frac{(1-\alpha)^2}{4\alpha} \left(\frac{1}{e^{\frac{1}{\alpha}t}} - \frac{1}{e^{\frac{1-\alpha}{\alpha}t}} \right) \\ &= \frac{\sqrt{(2-\alpha)^2 + 64\alpha^2} - 2(2-\alpha)}{2(2-\alpha)\sqrt{(2-\alpha)^2 + 64\alpha^2}} e^{\frac{1}{\alpha}t} - \frac{1}{2(2-\alpha)e^{\frac{1-\alpha}{\alpha}t}} \\ &\quad - \frac{2-\alpha}{((2-\alpha)^2 + 64\alpha^2)e^{\frac{1-\alpha}{\alpha}t}} + \frac{(1-\alpha)^2}{4\alpha} \left(\frac{1}{e^{\frac{1}{\alpha}t}} - \frac{1}{e^{\frac{1-\alpha}{\alpha}t}} \right) \\ &\rightarrow \infty \end{aligned}$$

as $t \rightarrow \infty$. Thus, condition (2.2) is also satisfied. Furthermore, numerical simulations confirm that the non-trivial solutions of Equation (2.9) exhibit oscillatory behavior. By introducing the transformation $w = e^{-\frac{1}{\alpha}t} D^\alpha u$, Equation (2.9) can be rewritten as the system

$$\begin{cases} D^\alpha w = e^{\frac{1}{\alpha}t} w, \\ D^\alpha w = -\left(\frac{1}{2} e^{\frac{1}{\alpha}t} + e^{\frac{1}{\alpha}t} \cos(8t) \right) w. \end{cases} \quad (2.10)$$

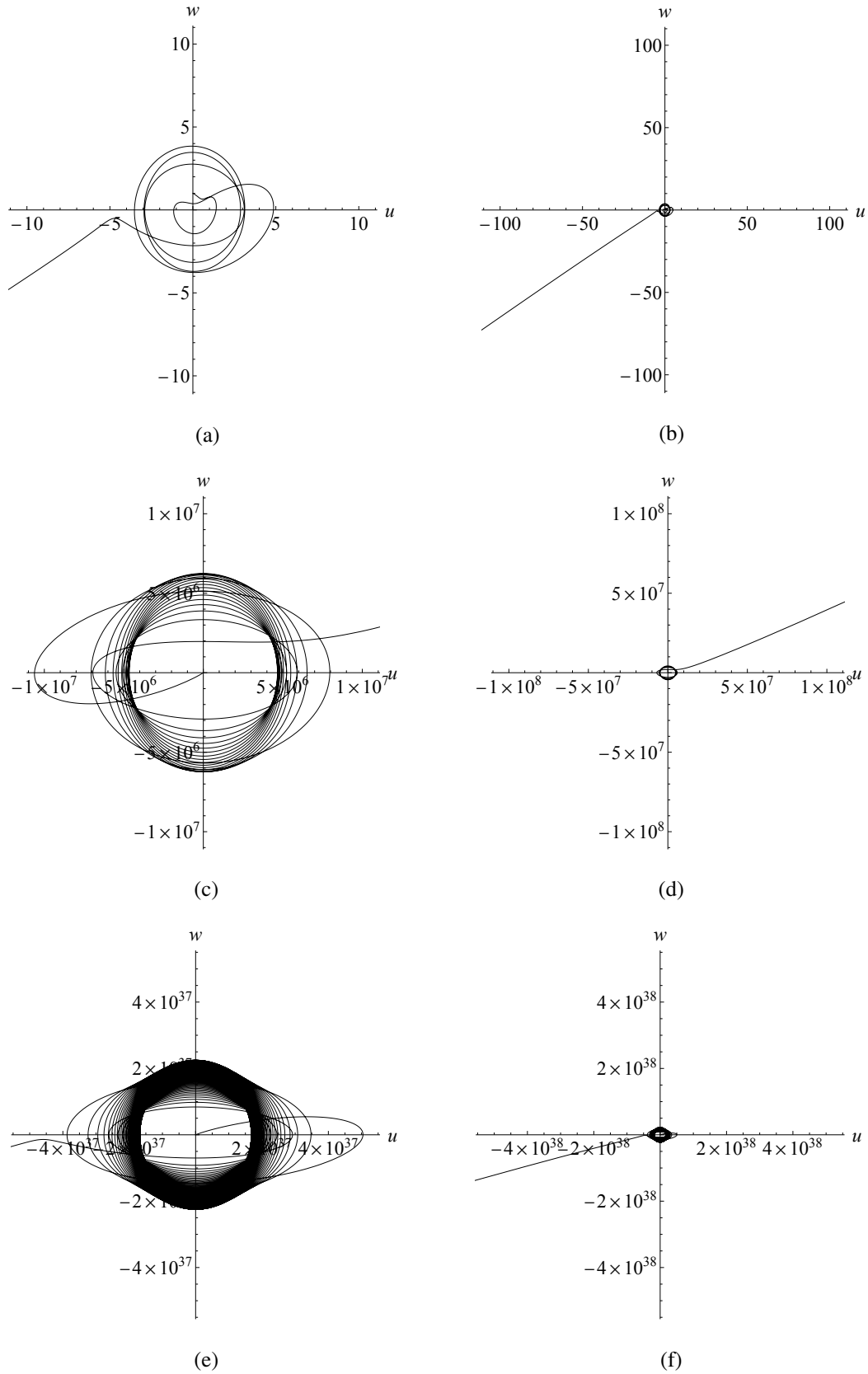


FIGURE 1 These six graphs represent the solution curve of (2.10) starting at the point $(0, 1)$ in the case that $\alpha = 0.5$.

Fig. 1 depicts the solution curve of System (2.10) for $\alpha = 0.5$, with the time interval set to $t \in [0, 4]$ and the initial condition $(u(0), w(0)) = (0, 1)$. A solution of System (2.10) is said to be oscillatory if its trajectory crosses the w -axis infinitely many times. Examining Figures 1-(a) through 1-(f) sequentially, we observe that the solution curve, starting from the given initial condition, first undergoes several rotations (see Fig. 1-(a)). It then remains in the fourth quadrant (Fig. 1-(b)). While staying in the fourth quadrant, the trajectory completes several more rotations (Fig. 1-(c)) before shifting to the first quadrant, where it remains (Fig. 1-(d)). Similarly, the trajectory undergoes additional rotations in the first quadrant (Fig. 1-(e)) before returning to the fourth quadrant. From Figures 1-(a) to 1-(f), we see that this pattern repeats indefinitely. The numerical simulation confirms that the solution of Equation (2.9) exhibits oscillatory behavior with an increasingly large amplitude.

Example 2. For $\alpha \in (0, 1)$, let $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$ and $\kappa_1(\alpha, t) \equiv 0$. For any real numbers $\varepsilon, \delta, \omega$, if $\varepsilon > |\delta|$, then all non-trivial solutions of

$$D^\alpha [t^\alpha D^\alpha u] + (\varepsilon + \delta \cos(\omega t))u = 0, \quad t \geq t_0 = 1 \quad (2.11)$$

are oscillatory. Comparing the coefficient functions of Equation (2.11) with those of Equation (1.5), we obtain

$$r(t) = t^\alpha \quad \text{and} \quad c(t) = \varepsilon + \delta \cos(\omega t).$$

Here, the coefficient function r is unbounded and diverges to infinity. We now verify the oscillation conditions for Equation (2.11). Since $e_0(t, 1) = e_0(t, s) = 1$, we obtain

$$\int_1^t \frac{e_0(t, 1)}{r(s)} d_\alpha s = \frac{1}{\alpha} \int_1^t \frac{1}{s} ds = \frac{1}{\alpha} \ln t \rightarrow \infty$$

as $t \rightarrow \infty$, which satisfies condition (2.1). On the other hand, considering $\varepsilon > |\delta|$, we have

$$\begin{aligned} \int_1^t \left(c(s) - \frac{r(s)\kappa_1^2(\alpha, s)}{4} \right) e_0(t, s) d_\alpha s &= \frac{1}{\alpha} \int_1^t (\varepsilon + \delta \cos(\omega s)) s^{\alpha-1} ds \\ &\geq \frac{\varepsilon - |\delta|}{\alpha^2} (t^\alpha - 1) \\ &\rightarrow \infty \end{aligned}$$

as $t \rightarrow \infty$, which satisfies condition (2.2). Moreover, Fig. 2 confirms that a non-trivial solution of Equation (2.11) exhibit oscillatory behavior.

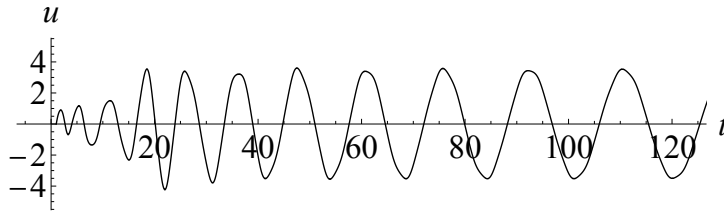


FIGURE 2 A non-trivial solution of Equation (2.11) for $\alpha = 0.5$, $\varepsilon = 3$, $\delta = 2$, $\omega = 2$.

3 | OSCILLATION AND NONOSCILLATION CRITERIA OF HILLE-KNESER TYPE

In this section, we consider the Euler-type equation with PD control:

$$D^\alpha [D^\alpha u] + \frac{\lambda}{t^{2\alpha}} u = 0, \quad t \geq t_0 = 1, \quad (3.1)$$

where $\alpha \in (0, 1)$, λ is a real number, $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, and $\kappa_1(\alpha, t)$ is a continuously differentiable function satisfying (1.1) and (1.2). If $\alpha \rightarrow 1$, then Equation (3.1) reduces to Equation (1.11).

Theorem 4. Let $\lambda_* = \alpha^4/4$. Then, the following results hold:

- (i) If $\lambda > \lambda_*$, then all non-trivial solutions of (3.1) are oscillatory;
- (ii) If $\lambda \leq \lambda_*$, then all non-trivial solutions of (3.1) are nonoscillatory.

Proof. For any $t \in [1, \infty)$, suppose that the functions a and b satisfy following relation:

$$b(t) - \frac{\kappa_0(\alpha, t)}{2} a'(t) - \frac{1}{4} a^2(t) = \frac{\lambda}{t^{2\alpha}}. \quad (3.2)$$

Then, Equation (3.1) can be transformed into the following differential equation with a damping coefficient:

$$D^\alpha[D^\alpha x] + a(t)D^\alpha x + b(t)x = 0, \quad t \geq t_0 = 1. \quad (3.3)$$

This transformation is an equivalence transformation. For details, see^{26, Theorem 3.1}. Here, a is a continuously differentiable function, and b is a continuous function. In other words, if the coefficients of Equation (3.1) can be expressed using appropriate functions a and b that satisfy the relation (3.2), then the oscillation (or nonoscillation) of all non-trivial solutions of Equation (3.1) is equivalent to the oscillation (or nonoscillation) of all non-trivial solutions of Equation (3.3). We define a and b , which satisfy the relation (3.2), as follows:

$$a(t) = -2\kappa_1(\alpha, t) \quad \text{and} \quad b(t) = \frac{\lambda}{t^{2\alpha}} + \kappa_1^2(\alpha, t) - \alpha t^{1-\alpha} \kappa_1'(\alpha, t). \quad (3.4)$$

In fact, noting that $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, we obtain

$$\begin{aligned} b(t) - \frac{\kappa_0(\alpha, t)}{2} a'(t) - \frac{1}{4} a^2(t) &= \frac{\lambda}{t^{2\alpha}} + \kappa_1^2(\alpha, t) - \alpha t^{1-\alpha} \kappa_1'(\alpha, t) + \alpha t^{1-\alpha} \kappa_1'(\alpha, t) - \kappa_1^2(\alpha, t) \\ &= \frac{\lambda}{t^{2\alpha}}. \end{aligned}$$

Thus, (3.4) satisfies the relation (3.2). From this point onward, we consider the oscillation problem for Equation (3.3) with (3.4). In fact, the general solution of Equation (3.3) with (3.4) can be explicitly obtained and expressed as follows. For $\alpha \in (0, 1)$ and arbitrary constants C_1 and C_2 we have:

- (a) If $\lambda > \lambda_*$, the solution of Equation (3.3) with (3.4) is given by

$$x(t) = C_1 \sqrt{t^\alpha} \cos\left(\frac{\sqrt{4\lambda - \alpha^4}}{2\alpha} \ln t\right) + C_2 \sqrt{t^\alpha} \sin\left(\frac{\sqrt{4\lambda - \alpha^4}}{2\alpha} \ln t\right).$$

- (b) If $\lambda = \lambda_*$, the solution of Equation (3.3) with (3.4) is given by

$$x(t) = C_1 \sqrt{t^\alpha} + C_2 \sqrt{t^\alpha} \ln t.$$

- (c) If $\lambda < \lambda_*$, the solution of Equation (3.3) with (3.4) is given by

$$x(t) = C_1 t^{\frac{\alpha^2 + \sqrt{\alpha^4 - 4\lambda}}{2\alpha}} + C_2 t^{\frac{\alpha^2 - \sqrt{\alpha^4 - 4\lambda}}{2\alpha}}.$$

We now proceed to prove this fact. By introducing the variable transformation

$$s = \ln t \quad \text{and} \quad y(s) = x(t),$$

Equation (3.3) with (3.4) is transformed into the second-order linear differential equation with constant coefficients:

$$\ddot{y} - \alpha \dot{y} + \frac{\lambda}{\alpha^2} y = 0, \quad s \geq 0, \quad (3.5)$$

where $\cdot = d/ds$. Indeed, from

$$\begin{aligned} D^\alpha x(t) &= D^\alpha y(s) \\ &= \kappa_1(\alpha, t) y(s) + \alpha t^{1-\alpha} \dot{y}(s) \frac{ds}{dt} \\ &= \kappa_1(\alpha, t) y(s) + \alpha t^{-\alpha} \dot{y}(s) \end{aligned}$$

and

$$\begin{aligned}
D^\alpha[D^\alpha x(t)] &= D^\alpha[D^\alpha y(s)] \\
&= D^\alpha [\kappa_1(\alpha, t)y(s) + \alpha t^{-\alpha} \dot{y}(s)] \\
&= D^\alpha[\kappa_1(\alpha, t)y(s)] + \alpha D^\alpha[t^{-\alpha} \dot{y}(s)] \\
&= (D^\alpha \kappa_1(\alpha, t))y(s) + \kappa_1(\alpha, t)(D^\alpha y(s)) - \kappa_1^2(\alpha, t)y(s) \\
&\quad + \alpha((D^\alpha[t^{-\alpha}])\dot{y}(s) + t^{-\alpha}(D^\alpha \dot{y}(s)) - \kappa_1(\alpha, t)t^{-\alpha} \dot{y}(s)) \\
&= \alpha^2 t^{-2\alpha} \ddot{y}(s) + (2\alpha t^{-\alpha} \kappa_1(\alpha, t) - \alpha^3 t^{-2\alpha}) \dot{y}(s) \\
&\quad + (\alpha t^{1-\alpha} \kappa_1'(\alpha, t) + \kappa_1^2(\alpha, t)) y(s),
\end{aligned}$$

we see that

$$D^\alpha[D^\alpha x(t)] + a(t)D^\alpha x(t) + b(t)x(t) = \alpha^2 t^{-2\alpha} \ddot{y}(s) - \alpha^3 t^{-2\alpha} \dot{y}(s) + \lambda t^{-2\alpha} y(s) = 0,$$

which confirms that Equation (3.5) is obtained. The characteristic equation of Equation (3.5) is given by $\sigma^2 - \alpha\sigma + \lambda\alpha^{-2} = 0$. Solving for σ , we obtain $\sigma = (\alpha^2 \pm \sqrt{\alpha^4 - 4\lambda})/2\alpha$. Thus, the general solution of Equation (3.5) is as follows. For $\alpha \in (0, 1)$:

(d) If $\lambda > \lambda_*$, the solution of Equation (3.5) is

$$y(s) = C_1 e^{\frac{\alpha}{2}s} \cos\left(\frac{\sqrt{4\lambda - \alpha^4}}{2\alpha}s\right) + C_2 e^{\frac{\alpha}{2}s} \sin\left(\frac{\sqrt{4\lambda - \alpha^4}}{2\alpha}s\right).$$

(e) If $\lambda = \lambda_*$, the solution of Equation (3.5) is

$$y(s) = C_1 e^{\frac{\alpha}{2}s} + C_2 s e^{\frac{\alpha}{2}s}.$$

(f) If $\lambda < \lambda_*$, the solution of Equation (3.5) is

$$y(s) = C_1 e^{\frac{\alpha^2 + \sqrt{\alpha^4 - 4\lambda}}{2\alpha}s} + C_2 e^{\frac{\alpha^2 - \sqrt{\alpha^4 - 4\lambda}}{2\alpha}s}.$$

Since $s = \ln t$, we obtain (a) from (d), (b) from (e), and (c) from (f). Thus, since all non-trivial solutions of Equation (3.5) oscillate if $\lambda > \lambda_*$, and all non-trivial solutions of Equation (3.5) do not oscillate if $\lambda \leq \lambda_*$, it follows that the oscillation constant of Equation (3.3) with (3.4) is λ_* . Consequently, Equation (3.1) also has the same oscillation constant λ_* . $\square$

Remark 4. From Theorem 4, it follows that the oscillation constant of Equation (1.12) is λ_* . This result clarifies Ishibashi's conjecture²⁵, which was posed as an open problem in Remark 2.

By utilizing the result of Theorem 4 and Equation (3.1), we obtain the following Hille–Kneser-type theorem.

Theorem 5. Let $r(t) \equiv 1$, $\kappa_1(\alpha, t)$ is a continuously differentiable function satisfying (1.1) and (1.2), $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$ and $\lambda_* = \alpha^4/4$. Suppose that

$$\omega_* = \liminf_{t \rightarrow \infty} t^{2\alpha} c(t), \quad \omega^* = \limsup_{t \rightarrow \infty} t^{2\alpha} c(t). \quad (3.6)$$

Then, all non-trivial solutions of (1.5) are oscillatory if $\omega_* > \lambda_*$ and all non-trivial solutions of (1.5) are nonoscillatory if $\omega^* < \lambda_*$.

Proof. If $\omega_* > \lambda_*$, there exists a number $\lambda > \lambda_*$ and a positive number $t_0 > 0$ such that $c(t) - \lambda t^{-2\alpha} > 0$ for $t \geq t_0$. Since all non-trivial solutions of (3.1) are oscillatory for $\lambda > \lambda_*$, the Sturm comparison theorem^{4, Theorem 7.2} shows that all non-trivial solutions of (1.5) with the conditions of Theorem 5 are oscillatory. On the other hand, if $\omega^* < \lambda_*$, there exists a number $\lambda < \lambda_*$ and a number $t_0 > 0$ such that $c(t) - \lambda t^{-2\alpha} < 0$ for $t \geq t_0$. If a solution of (1.5) with the conditions of Theorem 5 is oscillatory, then all non-trivial solutions of (3.1) with $\lambda < \lambda_*$ are oscillatory by the Sturm comparison theorem. However, by using Theorem 4 (ii), this contradiction shows that all non-trivial solutions of (1.5) with the conditions of Theorem 5 are nonoscillatory. $\square$

Remark 5. If $\alpha \rightarrow 1$, Theorem 5 coincides with Theorem 2.

Example 3. Consider the extended Equation of (3.1):

$$D^\alpha[D^\alpha u] + \lambda t^\mu u = 0, \quad t \geq t_0 = 1, \quad (3.7)$$

where $\alpha \in (0, 1)$, λ and μ are real numbers, $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, and $\kappa_1(\alpha, t)$ is a continuously differentiable function satisfying (1.1) and (1.2). The oscillation and nonoscillation conditions for Equation (3.7) are given as follows.

- (i) If $\mu > -2\alpha$ and $\lambda > 0$, then all non-trivial solutions of Equation (3.7) are oscillatory. If $\mu > -2\alpha$ and $\lambda \leq 0$, then all non-trivial solutions of Equation (3.7) are nonoscillatory.
- (ii) If $\mu = -2\alpha$ and $\lambda \in \mathbb{R}$, then Equation (3.7) reduces to Equation (3.1), and the result of Theorem 4 applies.
- (iii) If $\mu < -2\alpha$ and $\lambda \in \mathbb{R}$, then all non-trivial solutions of Equation (3.7) are nonoscillatory.

Indeed, by applying Theorem 5, we obtain the following as $t \rightarrow \infty$:

$$\lambda t^{2\alpha+\mu} \rightarrow \begin{cases} \infty & \text{if } \mu > -2\alpha \text{ and } \lambda > 0, \\ 0 & \text{if } \mu > -2\alpha \text{ and } \lambda = 0, \\ -\infty & \text{if } \mu > -2\alpha \text{ and } \lambda < 0, \\ \lambda & \text{if } \mu = -2\alpha \text{ and } \lambda \in \mathbb{R}, \\ 0 & \text{if } \mu < -2\alpha \text{ and } \lambda \in \mathbb{R}. \end{cases}$$

Thus, (i)–(iii) stated above are verified.

From Theorem 5, the following result is also obtained.

Corollary 1. Let $r(t) \equiv 1$, $\kappa_1(\alpha, t)$ be a continuously differentiable function satisfying (1.1) and (1.2), and $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$.

- (i) If there exists a constant $k > 0$ such that $c(t) \geq k$ for $t \geq t_0$, then all non-trivial solutions of Equation (1.5) are oscillatory.
- (ii) If $c(t) \leq 0$ for $t \geq t_0$, then all non-trivial solutions of Equation (1.5) are nonoscillatory.

Proof. Proof of (i). Since $t^{2\alpha}c(t) \geq t^{2\alpha}k \rightarrow \infty$ as $t \rightarrow \infty$, it follows from Theorem 5 that the result of Corollary 1 (i) holds.

Proof of (ii). Since $t^{2\alpha}c(t) \leq 0 < \alpha^4/4$ for any $t \geq t_0$, it follows from Theorem 5 that the result of Corollary 1 (ii) holds. $\square$

4 | CONCLUSION

The differential equation (1.5) considered in this paper can be transformed to study the oscillation problem of solutions for a linear differential equation using the definition (1.4) given by Khalil et al. As the first motivation of this study, we derived a Leighton–Wintner-type oscillation theorem applicable to Equation (1.5) (see Theorem 3). Moreover, Theorem 3 allows the coefficient function c to change sign and does not require the coefficient function r to be bounded.

Additionally, as the second motivation, we clarified the oscillation constant of Equation (1.12) and established Hille–Kneser-type oscillation and nonoscillation theorems. The significance of this result lies in demonstrating that the oscillation constant of Equation (3.1) is $\alpha^4/4$. Furthermore, by utilizing Theorem 5, we also established oscillation criteria for the more generalized Equation (3.7).

5 | APPENDIX: BASIC PROPERTIES ON CALCULUS FOR D^α

In this section, the background on calculus for (1.3) of Definition 1 is summarized³.

Theorem 6 (³). Let $\alpha \in (0, 1)$, points $s, t \in \mathbb{R}$ with $s \leq t$, and function $\phi : [s, t] \rightarrow \mathbb{R}$ be continuous. Furthermore, let $\kappa_0, \kappa_1 : (0, 1) \times \mathbb{R} \rightarrow [0, \infty)$ be continuous and satisfy (1.1) and (1.2), with ϕ/κ_0 and κ_1/κ_0 being Riemann integrable on $[s, t]$. Next, the exponential function with respect to D^α in (1.3) is defined as follows:

$$e_\phi(t, s) := e^{\int_s^t \frac{\phi(\tau) - \kappa_1(\alpha, \tau)}{\kappa_0(\alpha, \tau)} d\tau}, \quad e_0(t, s) = e^{-\int_s^t \frac{\kappa_1(\alpha, \tau)}{\kappa_0(\alpha, \tau)} d\tau}, \quad (5.1)$$

and

$$D^\alpha e_\phi(t, s) = \phi(t)e_\phi(t, s), \quad D^\alpha e_0(t, s) = 0.$$

Definition 2 ⁽³⁾. Let $\alpha \in (0, 1)$ and $t_0 \in \mathbb{R}$. The antiderivative is defined as follows:

$$\int D^\alpha f(t) d_\alpha t = f(t) + c e_0(t, t_0), \quad c \in \mathbb{R}.$$

Similarly, the integral of f over $[a, b]$ is defined as follows:

$$\int_a^t f(s) e_0(t, s) d_\alpha s := \int_a^t \frac{f(s) e_0(t, s)}{\kappa_0(\alpha, s)} ds, \quad d_\alpha s := \frac{1}{\kappa_0(\alpha, s)} ds.$$

Theorem 7 ⁽³⁾. Let differential operator D^α be expressed as (1.3), with $\alpha \in (0, 1)$. Let function $\phi : [s, t] \rightarrow \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : (0, 1) \times \mathbb{R} \rightarrow [0, \infty)$ be continuous and satisfy (1.1) and (1.2). We assume that functions f and g are differentiable as needed. Then,

- (i) $D^\alpha[kf(t) + lg(t)] = k(D^\alpha f(t)) + l(D^\alpha g(t))$ for all $k, l \in \mathbb{R}$;
- (ii) $D^\alpha k = k\kappa_1(\alpha, t)$ for all $k \in \mathbb{R}$;
- (iii) $D^\alpha[f(t)g(t)] = g(t)(D^\alpha f(t)) + f(t)(D^\alpha g(t)) - f(t)g(t)\kappa_1(\alpha, t)$;
- (iv) $D^\alpha[f(t)/g(t)] = \frac{g(t)(D^\alpha f(t)) - f(t)(D^\alpha g(t))}{g^2(t)} + \frac{f(t)}{g(t)}\kappa_1(\alpha, t)$;
- (v) for $\alpha \in (0, 1)$ and exponential function e_0 given in (5.1), we have

$$D^\alpha \left[\int_a^t f(s) e_0(t, s) d_\alpha s \right] = f(t);$$

- (vi) an integration by parts formula is given by

$$\begin{aligned} \int_a^t f(s)(D^\alpha g(s))e_0(t, s)d_\alpha s &= f(s)g(s)e_0(t, s)\Big|_{s=a}^t \\ &\quad - \int_a^t g(s)(D^\alpha f(s) - \kappa_1(\alpha, s)f(s))e_0(t, s)d_\alpha s. \end{aligned}$$

By (vi) of Theorem 7, we obtain

$$\int_a^t (D^\alpha g(s))e_0(t, s)d_\alpha s = g(s)e_0(t, s)\Big|_{s=a}^t = g(t) - g(a)e_0(t, a).$$

ACKNOWLEDGMENTS

The first author was supported by JSPS KAKENHI Grant number JP22K13933 and Grant for Basic Science Research Projects from The Sumitomo Foundation No. 2402156. The second author was supported by JSPS KAKENHI Grant number JP25K07092. We would like to thank Editage (www.editage.jp) for the English language editing.

CONFLICT OF INTEREST

The authors declare that there is no competing interest.

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How to cite this article: Ishibashi K., Onitsuka M. (2025), Oscillation criteria of the Leighton–Wintner-type and the Hille–Kneser-type for linear differential equations with a proportional derivative controller, *Math. Meth. Appl. Sci.*,