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A Locally Convex Topology of Simplicial Complexes

Dedicated to Professor Osamu Takenouchi on his 60th birthday

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In simplicial complexes, the Whitehead topology and the metric topology are well known. In the class of full simplicial complexes, the metric topology is locally convex, while the Whitehead topology is not necessarily locally convex. In this paper, we consider the strongest topology in the class of full simplicial complexes which is locally convex and is contained in the Whitehead topology, and study some interesting topological properties; for example, M_1-ness and ANR (M_1).

§1. Introduction

Since all complexes studied in this paper are simplicial, we shall usually drop the word simplicial. By obvious reasons, we shall require the topology of a complex K to satisfy the following two conditions:

(C1) Every subcomplex of K is a closed subset of K.

(C2) Every finite subcomplex of K, considered as a subspace of K, has the Euclidean topology.

As the topologies on a complex K satisfying (C1) and (C2), the Whitehead topology and the metric topology are well known. We denote the space K equipped with these topologies by $|K|_W$ and $|K|_M$, respectively. If the complex K is not locally finite, then $|K|_W$ does not coincide with $|K|_M$.

As easily seen, for a full complex K, $|K|_M$ is locally convex, but $|K|_W$ is not necessarily locally convex (cf. [5] pp. 416, 4.3). Therefore, in this paper, we consider the strongest topology of a full complex K which is locally convex and satisfies the conditions (C1) and (C2), and which is contained in the Whitehead topology — we call it the locally convex topology and denote the space K equipped with that topology by $|K|_C$. In section 2, we define the locally convex topology and consider some topological properties of $|K|_C$. In section 3, we shall prove that, for a full complex K, $|K|_C$ is M_1 and AR(M_1). In section 4, we consider the subcomplexes of $|K|_C$.

Throughout this paper, N denotes the set of all natural numbers and $M=\{1/n: n\in N\}$. For M_1-spaces, see [3]. For AR and ANR, see [6]. Every terminology is referred to [5] or [6], unless otherwise stated.

§ 2. Definitions and some properties

Let K be a full complex with the triangulation $\mathscr{K}$. Then we embed K in a suitable vector space with finite topology ([5] pp. 416) so that its vertices are at the unit points of the vector space. For $P \in \mathscr{K}$, d_P denotes the metric function of P inherited from the above embedding. For $x, y \in K$, let

$$I(x, y) = \{tx + (1-t)y \colon 0 \leqq t \leqq 1\}.$$

Then $U \subset K$ is said to be *convex* if $I(x, y) \subset U$ for any $x, y \in U$.

DEFINITION 2.1. Let K be a full complex. By the *locally convex topology* of K we mean the strongest topology $\mathscr{T}$ satisfying the following:

(1) If $U \in \mathscr{T}$, U is open in $|K|_W$.

(2) For each $U \in \mathscr{T}$ and each $x \in U$, there is a convex set $V \in \mathscr{T}$ with $x \in V \subset U$.

We denote the space K equipped with the locally convex topology by $|K|_C$. Let H be a subcomplex of K. Then if H is the subspace of $|K|_C$, we say that the space H has the locally convex topology (we denote it by $|H|_C$).

It is easily verified that the locally convex topology satisfies the conditions (C1) and (C2). Furthermore among the topologies satisfying (1) and (2) above there is the strongest topology. Indeed the family of all convex open subsets of $|K|_W$ is the base of the strongest topology.

The following proposition is obvious by [4] Lemma 4.4.

PROPOSITION 2.2. *Let K be a full simplicial complex with countable vertices. Then $|K|_C$ coincides with $|K|_W$.*

For k-leader, see [1]. For a space X, $k(X)$ denotes the k-leader of X.

PROPOSITION 2.3. *Let K be a full simplicial complex with uncountable vertices. Then $k(|K|_C)$ coincides with $|K|_W$.*

PROOF. To prove this proposition, it suffices to show that, for each compact set A of $|K|_C$, A is contained in some simplex. Suppose that A is not contained in any simplex of K. Then there are countable simplexes $\{P_n \colon n \in N\}$ satisfying the following:

$$P_1 \subsetneqq P_2 \subsetneqq \cdots \subsetneqq P_n \subsetneqq \cdots, \quad A \cap (P_n - P_{n-1}) \neq \emptyset, \quad n \in N.$$

Now choose a point $x_n \in A \cap (P_n - P_{n-1})$ and let $B = \{x_n \colon n \in N\}$. Since the locally convex topology and the Whitehead topology coincide in a countable full complex by Proposition 2.2, B is closed in $|K|_C$. Furthermore, it is easily seen (cf. the proof of Theorem 3.2 in this paper) that there are neighborhoods $N(x_n)$ of x_n for $n \in N$ such that $N(x_n) \cap N(x_m) = \emptyset$ if $n \neq m$. Since the open cover $\{N(x_n) \colon n \in N\} \cup \{|K|_C - B\}$ of A does not have a finite subcover of A, this contradicts the compactness of A.

§3. M_1-ness and AR(M_1) of $|K|_C$

In this section, K denotes a full complex with the triangulation $\mathscr{K}$. Then we use the following notations.

NOTATION 3.1. For $S\in\mathscr{K}$, let $\Delta(S)=\{T\in\mathscr{K}: T\subset S, T\neq S\}$. Define $\mathscr{K}_0=\{S\in\mathscr{K}: \Delta(S)=\emptyset\}$ (the set of all vertices) and, assuming $\mathscr{K}_m$ has been defined for $0\leqq m<n$, we define

$$\mathscr{K}_n=\{S\in\mathscr{K}: \Delta(S)\subset \bigcup_{i=0}^{n-1}\mathscr{K}_i\}-\bigcup_{i=0}^{n-1}\mathscr{K}_i.$$

Then $\cup\{\mathscr{K}_i: i\in N\cup\{0\}\}=\mathscr{K}$. For $S\in\mathscr{K}$, let $\partial S=\cup(\Delta(S))$. $S^\circ=S-\partial S$, and $\mathscr{A}_S=\{T\in\mathscr{K}: S\subset T, S\neq T\}$. For $n\in N$ and $S\in\mathscr{K}_{n-1}$, $\mathscr{A}_S^n=\mathscr{A}_S\cap\mathscr{K}_n$. Then obviously $\cup\{S^\circ: S\in\mathscr{K}\}=K$.

For $S\in\mathscr{K}_n$, $T\in\mathscr{K}_{n+1}$ such that $S\subset T$, let for each $k\in N$,

$$S_T(k)=\{x\in T: d_T(x, S)=1/k\}.$$

For $x\in T$ and a positive real number r, let

$$B(x, r; d_T)=\{y\in T: d_T(x, y)<r\}.$$

Since $S_T(k)$ is compact, there exists a finite subset $S_T(k\text{-finite})$ of $S_T(k)$ such that

$$\cup\{S_T(k)\cap B(x, 1/k; d_T): x\in S_T(k\text{-finite})\}=S_T(k).$$

Let $S_T=\cup\{S_T(k\text{-finite}): k\in N\}$ and $S(n+1)=\cup\{S_T: T\in\mathscr{A}_S^{n+1}\}$ for $S\in\mathscr{K}_n$.

Now we shall prove the following main theorems.

THEOREM 3.2. *For each full simplicial complex K, $|K|_C$ is an M_1-space.*

PROOF. We shall construct a σ-closure preserving base consisting of convex open subsets.

First, fix $P\in\mathscr{K}_0$. Let $g: \mathscr{A}_P^1\to M$. Then we define a candidate P_g for our local base at P as follows: For any $S\in\mathscr{K}_1\cap\mathscr{A}_P$ $(=\mathscr{A}_P^1)$, we define

$$P_g^S=\{x\in S: d_S(P, x)<g(S)\}.$$

Now assume that we have defined P_g^S for all $S\in\mathscr{K}_m\cap\mathscr{A}_P$ with $1\leqq m<n$. Then for any $T\in\mathscr{K}_n\cap\mathscr{A}_P$ we put

$$\partial P_g^T=\cup\{P_g^S: S\in\Delta(T)\cap\mathscr{A}_P\}$$

and

$$P_g^T=\cup\{I(x, y): x, y\in\partial P_g^T\}.$$

Finally we put

$$P_g = \cup \{P_g^T : T \in \mathscr{A}_P\}.$$

Next, we observe that for each $n \in N$, $P \in \mathscr{K}_n$, there exists a countable family $\mathscr{B}(P) = \{P_m : m \in N\}$ of open convex subsets of P° forming a base for points in P° so that $\bar{P}_m \subset P^\circ$ for all $m \in N$. Fix $n \in N$, $P \in \mathscr{K}_n$ and $B \in \mathscr{B}(P)$. Then for each map $g: \mathscr{A}_P^{n+1} \to P(n+1)$, we define a candidate B_g for our base as follows: Now for any $T \in \mathscr{A}_P^{n+1}$, we define $\partial B_g^T = B$. Let

$$W_T = \cup \{B(x, r_x; d_T) : x \in \partial B_g^T\}$$

where $r_x = d_P(x, P - \partial B_g^T)$. If g satisfies $g(T) \in W_T \cap P_T$ for any $T \in \mathscr{A}_P^{n+1}$, g is said to be *definable* for $B \in \mathscr{B}(P)$. If g is definable for $B \in \mathscr{B}(P)$, then we define

$$B_g^T = \cup \{I(x, g(T)) : x \in \partial B_g^T\} - \{g(T)\}.$$

Now assume that B_g^S has been defined for all $S \in \mathscr{K}_{n+k} \cap \mathscr{A}_P$ with $1 \leqq k < m$. Then for any $T \in \mathscr{K}_{n+m} \cap \mathscr{A}_P$ we put

$$\partial B_g^T = \cup \{B_g^S : S \in \Delta(T) \cap \mathscr{A}_P\},$$

and

$$B_g^T = \cup \{I(x, y) : x, y \in \partial B_g^T\}.$$

Finally we put

$$B_g = \cup \{B_g^T : T \in \mathscr{A}_P\}.$$

For $m, n \in N$, $P \in \mathscr{K}_n$ and $R \in \mathscr{K}_0$, we put

$$\mathscr{V}_P^m = \{(P_m)_g : g \in \mathrm{Map}(\mathscr{A}_P^{n+1}, P(n+1)) \text{ is definable for } P_m \in \mathscr{B}(P)\},$$

$$\mathscr{U}_n^m = \cup \{\mathscr{V}_P^m : P \in \mathscr{K}_n\},$$

$$\mathscr{V}_R = \{R_g : g \in \mathrm{Map}(\mathscr{A}_R^1, M)\}$$

and

$$\mathscr{U}_0 = \cup \{\mathscr{V}_R : R \in \mathscr{K}_0\}.$$

Then we shall prove the followings:

(a) $(P_m)_g$ and R_g are convex and open in $|K|_C$.
(b) $\cup \{\mathscr{V}_P^m : m \in N\}$ is a base for points in P°.
(c) $\mathscr{V}_R$ is a local base at the vertex R.
(d) $\mathscr{V}_P^m$ and $\mathscr{V}_R$ are closure preserving.
(e) $\mathscr{U}_n^m$ and $\mathscr{U}_0$ are closure preserving.

If these are proved, then $\mathscr{U}_0 \cup (\cup \{\mathscr{U}_n^m : m, n \in N\})$ is the desired σ-closure preserving base of $|K|_C$. Since it is obvious that $|K|_C$ is regular, the proof will be completed.

Proof of (a): For any $x, y \in (P_m)_g$, by the construction of $(P_m)_g$ there is a $T \in \mathscr{A}_P$ such that $x, y \in (P_m)_g^T$. Since $(P_m)_g^T$ is convex, $I(x, y) \subset (P_m)_g$. Therefore $(P_m)_g$ is convex. The convexity of R_g is much the same. Furthermore it is clear that $(P_m)_g$ and R_g are open in $|K|_C$.

Proof of (b): Let U be an open set of $|K|_C$ such that $x \in U \cap P^\circ$. Since U is open in $|K|_C$, there is a convex open set V of $|K|_C$ with $x \in V \subset U$. Since $x \in U \cap P^\circ$, there is an $m \in N$ such that $x \in P_m \subset U \cap P^\circ$. Then it is easily seen that there is a definable map g for $P_m \in \mathscr{B}(P)$ such that $x \in (P_m)_g \subset U$. Then $(P_m)_g$ is just an element of $\mathscr{V}_P^m$.

Proof of (c): Let U be a neighborhood of the vertex R. Then there is a convex neighborhood V of R such that $R \in V \subset U$. For each $S \in \mathscr{A}_R^1$, there is an $n_S \in N$ such that

$$\{x \in S: d_S(R, x) < 1/n_S\} \subset S \cap V.$$

Then we can define a map $g: \mathscr{A}_R^1 \to M$ by $g(S) = 1/n_S$. Since V is convex, it is clear that $R \in R_g \subset V \subset U$.

Proof of (d): Let $x \notin \overline{(P_m)_g}$ for each $(P_m)_g \in \mathscr{V}_P^m$. Then there is a simplex S with $x \in S$. In case $S \cap P_m = \emptyset$, there is a neighborhood W of x such that $W = (S_n)_h \in \mathscr{V}_S^n$ if $x \in S_n \subset S^\circ$ or $W = R_h \in \mathscr{V}_R$ if $x = R \in \mathscr{K}_0$, and $W \cap (P_m)_g = \emptyset$ for each $(P_m)_g \in \mathscr{V}_P^m$. In the other case $S \cap P_m \neq \emptyset$, then $S \in \mathscr{A}_P$. Since $\{(P_m)_g \cap S: (P_m)_g \in \mathscr{V}_P^m\}$ is closure preserving in S, there is a neighborhood $(S_n)_h \in \mathscr{V}_S^n$ of x such that $(S_n)_h \cap (P_m)_g = \emptyset$ for each $(P_m)_g \in \mathscr{V}_P^m$. Thus $\mathscr{V}_P^m$ is closure preserving. The closure preservingness of $\mathscr{V}_R$ is much the same.

Proof of (e): Let $x \notin \overline{U}$ for each $U \in \mathscr{U}_n^m$. Then there is a simplex $S \in \mathscr{K}_k$ with $x \in S$ for some k. In case $k \leq n$, it is easily verified that there is a neighborhood W of x such that $W = (S_j)_g \in \mathscr{V}_S^j$ if $x \notin \mathscr{K}_0$ or $W = R_g \in \mathscr{V}_R$ if $x = R \in \mathscr{K}_0$, and $W \cap U = \emptyset$ for each $U \in \mathscr{U}_n^m$. In the other case $k > n$ and $x \in S^\circ$, since $\{U \cap S: U \in \mathscr{U}_n^m\}$ is closure preserving in S, there is a neighborhood $(S_j)_g \in \mathscr{V}_S^j$ of x such that $(S_j)_g \cap U = \emptyset$ for each $U \in \mathscr{U}_n^m$. Thus $\mathscr{U}_n^m$ is closure preserving. The closure preservingness of $\mathscr{U}_0$ is much the same.

Thus the proof of Theorem 3.2 is completed.

THEOREM 3.3. *For each full simplicial complex K, $|K|_C$ is* AR (M_1).

PROOF. Since $|K|_C$ is locally convex, by the same method of [2] Theorem 4.3, $|K|_C$ is AE (stratifiable). Since $|K|_C$ is M_1 by Theorem 3.2, $|K|_C$ is AR (stratifiable) therefore AR (M_1).

§4. Subcomplexes

For subcomplexes with the locally convex topology, we shall prove the following theorems.

THEOREM 4.1. *Every subcomplex H of a simplicial complex K with the locally convex topology is a neighborhood retract of $|K|_C$.*

PROOF. The proof of this theorem is analogous to the case of $|K|_W$ (for example, see Hu [6] pp. 101, Lemma 10.1) except the following: As $|K|_C$ is not a k-space in general, note the proof of the fact that $\phi: K' \to I$ is continuous on K' ([6] pp. 102).

Any simplicial complex K can be embeded in a full simplicial complex $F(K)$ with the same vertices. Therefore the following is a direct consequence of Theorem 4.1 and Corollary 3.3.

COROLLARY 4.2. *Every simiplicial complex with the locally convex topology is* ANR (M_1).

Note that $|K|_C$ is M_1. Indeed, this fact is obvious if we consider the following facts: K is a subcomplex of $F(K)$, the topology of $F(K)$ (cf. the proof of Theorem 3.2) and the definition of subcomplex with the locally convex topology.

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