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BERGER SPHERES
— THEIR REDEFINITION AND RELATED EXAMPLES —

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Abstract. This expository paper consists of two parts. In the first half, using geometric properties of geodesic spheres with sufficiently big radii in a complex projective space, we redefine Berger spheres (cf. [18]). We study such Berger spheres from the viewpoints of contact geometry, submanifold geometry and length spectral geometry (see Theorems 1 and 2). In the latter half (cf. [17]), considering geodesic spheres of sufficiently small radii in a complex hyperbolic space, we present examples of non-Berger spheres related to the redefinition of Berger spheres (see Theorem 3). We finally characterize such non-Berger spheres considered as real hypersurfaces isometrically immersed into a complex hyperbolic space (see Theorem 4).

1. Introduction

In order to state the background of our study in [18] and [17], we first recall the fact on lengths of closed geodesics on a compact manifold to Klingenberg ([12]): On an even dimensional compact simply connected Riemannian manifold M whose sectional curvatures lie in the interval $(0, L]$ with a constant L, the length of every closed geodesic on M is not shorter than $2\pi/\sqrt{L}$. For odd dimensional manifolds, Berger ([6, 7]) gave examples of metrics on a 3-sphere S^3 whose sectional curvatures lie in $(0, L]$ and that has a closed geodesic of length shorter than this constant $2\pi/\sqrt{L}$. This 3-sphere is called a *Berger sphere* with a Riemannian metric from a one-parameter family, which can be obtained from the standard metric by shrinking along fibers of a Hopf fibration. In his paper [11], Chavel constructed similar metrics on higher odd-dimensional spheres. Three years after this paper, Weinstein ([25]) showed that these Berger and Chavel examples can be regarded as geodesic spheres of radius r $(0 < r < \pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in a complex projective

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space $\mathbb{C}P^n(c)$ of complex dimension n (≥ 2) and of constant holomorphic sectional curvature c (> 0).

In this context, it is natural to investigate geometric properties of these geodesic spheres of sufficiently big radii in $\mathbb{C}P^n(c)$ (see Proposition 1). Motivated by this study on geodesic spheres, we first give a redefinition of Berger spheres. Following this redefinition of Berger spheres, Tanabe and the first author showed in [18] that every complete simply connected Sasakian space form $N(k) := N^{2n-1}(k)$ of constant ϕ-sectional curvature k with $k > 9$ is an example of a Berger sphere and that in the case of $k = 8n + 5$ this space $N(k)$ can be regarded as a homogeneous submanifold with nonzero parallel mean curvature vector with respect to the normal connection in some Euclidean sphere (see Theorem 1). Moreover, they clarify the length spectrum of this Berger sphere $N(k)$ ($k > 9$) (see Theorem 2).

In [17], the authors found an example of non-Berger spheres as a geodesic sphere $G(r)$ of sufficiently small radius r with $2 < \coth(\sqrt{|c|}\, r/2) < 3/\sqrt{2}$ in a complex hyperbolic $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c(< 0)$ (see Theorem 3). We finally give a characterization of this non-Berger sphere in the class of real hypersurfaces isometrically immersed into the ambient space $\mathbb{C}H^n(c)$ (see Theorem 4).

2. Fundamental notions in contact metric structures

We denote by N an odd dimensional Riemannian manifold equipped with Riemannian metric g. A quartet (ϕ, ξ, η, g) of a $(1,1)$-tensor ϕ, a vector field ξ, a 1-form η, and the Riemannian metric g on N is called an *almost contact metric structure* if

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

hold for all tangent vectors $X, Y \in TN$. We see easily that these equalities yield both of $\phi\xi = 0$ and $\eta(\phi X) = 0$ for each $X \in TN$. An odd dimensional Riemannian manifold N is said to be an *almost contact metric manifold* if it admits an almost contact metric structure. Here, ϕ, ξ and η are called the structure tensor, the characteristic vector and the contact form on N, respectively. Following Theorem 6.3 in [8], we say an almost contact metric manifold N to be a *Sasakian manifold* if the structure tensor ϕ of N satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.1}$$

for all $X, Y \in TN$ with Riemannian connection ∇ on N associated with g. A Sasakian manifold (endowed with Riemannian curvature tensor R) is called a *Sasakian space form* of constant ϕ-sectional curvature k if the sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ of the ϕ-section of u satisfies $K(u, \phi u) = k$ for each unit vector u orthogonal to ξ. The following is a Sasakian analogue of Schur's Theorem.

Proposition A ([21]). *If ϕ-sectional curvatures at each point of a Sasakian manifold N of dimension ≥ 5 does not depend on the choice of ϕ-section at that point, then it is constant on N.*

In this paper, we denote by $N^{2n-1}(k)$ a $(2n-1)$-dimensional Sasakian space form of constant ϕ-sectional curvature k. For the standard construction of Sasakian space forms, see pp. 114–115 in [8]. The following shows the uniqueness of Sasakian space forms.

Proposition B ([24]). *For any two simply connected complete Sasakian manifolds of the same constant ϕ-sectional curvature k, there exists an isomorphism between their almost contact metric structures.*

The curvature tensor R of a Sasakian space form $N^{2n-1}(k)$ is given in [21].

3. Fundamental theory of real hypersurfaces in a nonflat complex space form

Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ isometrically immersed into a complex $n(\geq 2)$-dimensional nonflat complex space form $\widetilde{M}_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$, namely the space $\widetilde{M}_n(c)$ is globally congruent to either $\mathbb{C}P^n(c)$ $(c > 0)$ furnished with Fubini-Study metric or $\mathbb{C}H^n(c)$ $(c < 0)$ endowed with Bergman metric. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten with a unit normal local vector field $\mathcal{N}$:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX \tag{3.1}$$

for all vector fields X and Y tangent to M, where g is the Riemannian metric of M induced from the above canonical metric (, say) $\widetilde{g}$ of the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$ associated with $\mathcal{N}$.

On M an almost contact metric structure (ϕ, ξ, η, g) associated with $\mathcal{N}$ is canonically induced from the Kähler structure J of the ambient space $\mathbb{C}P^n(c)$. They are defined by

$$g(\phi X, Y) = \widetilde{g}(JX, Y), \ \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = \widetilde{g}(JX, \mathcal{N}),$$

where J is the Kähler structure of $\widetilde{M}_n(c)$. By the formulas of Gauss and Weingarten and by the property $\widetilde{\nabla} J = 0$, we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad \text{and} \quad \nabla_X \xi = \phi AX. \tag{3.2}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} &g(R(X,Y)Z,W) \\ &\quad = (c/4)\big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\phi Y,Z)g(\phi X,W) \\ &\qquad\qquad - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\big\} \\ &\qquad + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) \end{aligned} \tag{3.3}$$

for all vectors X, Y, Z and W on M. Hence, the sectional curvature $K(X,Y)$ of the real plane spanned by a pair of orthonormal vectors X, Y is given by

$$K(X,Y) = (c/4)\big(1 + 3g(\phi X,Y)^2\big) + g(AX,X)g(AY,Y) - g(AX,Y)^2. \tag{3.4}$$

An eigenvector X of the shape operator A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$ and an eigenvalue λ of A is called a *principal curvature* of M in this ambient space. We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. For a Hopf hypersurface $M^{2n-1}\,(n \geq 2)$ in $\widetilde{M}_n(c)$, the principal curvature ν corresponding to the characteristic vector field ξ is locally constant on M. Furthermore, every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface (cf. [10]). This fact shows that the set of all Hopf hypersurfaces of $\widetilde{M}_n(c)$ is an abundant class.

We here clarify the meaning of the condition that a real hypersurface M of $\widetilde{M}_n(c)$ through an isometric immersion is Sasakian with respect to the almost contact metric structure. On an orientable connected real hypersurface M in $\widetilde{M}_n(c)$, we have two almost contact metric structures (ϕ, ξ, η, g) and $(\phi, -\xi, -\eta, g)$ which are associated with a unit normal vector $\mathcal{N}$ and $-\mathcal{N}$ of M, respectively. We call a real hypersurface M *Sasakian* if M satisfies either (2.1) or

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for all X, $Y \in TM$.

4. Redefinition of Berger spheres

It is known that geodesic spheres in $\mathbb{C}P^n(c)$ are the simplest examples of Hopf hypersurfaces in this ambient space. The shape operator A of a geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is expressed as

$$A\xi = \sqrt{c}\,\cot(\sqrt{c}\,r)\xi \quad \text{and} \quad AX = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)X$$

for each tangent vector X perpendicular to ξ. In order to estimate sectional curvatures of $G(r)$, it suffices to compute $K(\sin\theta\cdot X+\cos\theta\cdot\xi, Y)$ for a pair of orthonormal vectors X and Y that are orthogonal to ξ. It follows from (3.4) that

$$K(\sin\theta \cdot X + \cos\theta \cdot \xi, Y) = (c/4)\{\sin^2\theta\big(1 + 3g(\phi X, Y)^2\big) + \cot^2(\sqrt{c}\,r/2)\}.$$

This gives the following sharp inequalities on sectional curvatures:

$$(c/4)\cot^2(\sqrt{c}\,r/2) \leq K \leq c + (c/4)\cot^2(\sqrt{c}\,r/2). \tag{4.1}$$

Here, for the minimum and the maximum values of sectional curvatures, we have

$$K_{\min} = (c/4)\cot^2(\sqrt{c}\,r/2) \qquad = K(X, \xi),$$
$$K_{\max} = c + (c/4)\cot^2(\sqrt{c}\,r/2) = K(X, \phi X)$$

for each unit vector X orthogonal to ξ. Solving the inequality $K_{\min}/K_{\max} < 1/9$ on radii, we have $\tan^2(\sqrt{c}\,r/2) > 2$ and vice versa.

Next, we take an integral curve $\gamma_\xi = \gamma_\xi(s)$ of the characteristic vector field ξ on $G(r)\,(0 < r < \pi/\sqrt{c}\,)$. Since the curve γ_ξ lies on a holomorphic line $\mathbb{C}P^1(c)(= S^2(c))$ as a circle of positive curvature $k = \sqrt{c}\,|\cot(\sqrt{c}\,r)|$, it is closed with length

$$\ell = \frac{2\pi}{\sqrt{k^2 + c}} = \frac{2\pi}{\sqrt{c\cot^2(\sqrt{c}\,r) + c}} = \frac{2\pi}{\sqrt{c}}\sin(\sqrt{c}\,r)$$

(c.f. [1, 4]). Moreover, from the second equality in (3.2), we know that $\nabla_\xi \xi = \phi A\xi = 0$, so that the curve $\gamma_\xi = \gamma_\xi(s)$ is a geodesic on $G(r)$, where ∇ is the Riemannian connection on $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ induced from the Riemannian connection $\widetilde{\nabla}$ in the ambient space $\mathbb{C}P^n(c)$. We set the following inequality:

$$\frac{2\pi}{\sqrt{K_{\max}}} = \frac{2\pi}{\sqrt{c + \frac{c}{4}\cot^2\left(\frac{\sqrt{c}\,r}{2}\right)}} > \frac{2\pi}{\sqrt{c}}\sin(\sqrt{c}\,r).$$

Then, solving this inequality, we get $\tan^2(\sqrt{c}\,r/2) > 2$ and vice versa. Hence, we obtain the following which is a key in this paper.

Proposition 1 ([18])**.** *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geq 2$. Then the following three conditions are mutually equivalent*:

(1) The radius r satisfies an inequality $\tan^2(\sqrt{c}\,r/2) > 2$;
(2) The sectional curvature K of $G(r)$ satisfies sharp inequalities $\delta L \leq K \leq L$ for some constant $\delta \in (0, 1/9)$ at its each point;
(3) The length of every integral curve of the characteristic vector field ξ on $G(r)$ is shorter than $2\pi/\sqrt{L}$, where L is the maximal sectional curvature of $G(r)$.

Needless to say, each of geodesic spheres $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is a Riemannian homogeneous manifold.

Inspired by Proposition 1, Tanabe and the first author redefined Berger spheres in [18].

Definition. An odd dimensional Riemannian homogeneous manifold M is called a *Berger sphere* if M satisfies the following three conditions:

(1) M is diffeomorphic to a Euclidean sphere;
(2) The sectional curvature K of M satisfies sharp inequalities $0 < \delta L \leq K \leq L$ on M for some constant $\delta \in (0, 1/9)$;
(3) M has a closed geodesic whose length is shorter than $2\pi/\sqrt{L}$, where L is a positive constant given by (2).

Characterizations of Berger spheres from the viewpoint of submanifold geometry are given in [13].

5. Length spectrum of geodesic spheres $G(r)$ in $\widetilde{M}_n(c)$

When we study the length spectrum of geodesics on a Riemannian manifold M, in order to avoid the influence of the action of the full isometry group $\mathrm{I}(M)$ of M, we consider the moduli space of geodesics under the action of isometries. We say two smooth curves σ_1, σ_2 are congruent to each other if there exist an isometry $\varphi \in \mathrm{I}(M)$ and some s_0 satisfying $\sigma_2(s) = (\varphi \circ \sigma_1)(s + s_0)$ for every s. The moduli space $\mathrm{Geod}(M)$ of geodesics of unit speed on M is the quotient space of the set of all geodesics on M under the congruency relation. We call a smooth curve σ parameterized by its arclength *closed* if there is a nonzero s_c with $\sigma(s + s_c) = \sigma(s)$ for all s. The minimum positive such s_c is said to be its length and is denoted by $\mathrm{length}(\sigma)$. We call a smooth curve σ *open* if it is not closed. For convenience we set length(σ) $= \infty$ for an open curve σ. We define the *length spectrum* $\mathcal{L}_M : \mathrm{Geod}(M) \to \mathbb{R} \cup \{\infty\}$ of M by $\mathcal{L}_M([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ denotes the congruency class containing a geodesic γ. We also call the image $\mathrm{LSpec}(M) = \mathcal{L}_M(\mathrm{Geod}(M)) \cap \mathbb{R}$ the length spectrum of M. For example, the length spectrum of $\mathbb{C}P^n(c)$ is $\mathrm{LSpec}(\mathbb{C}P^n(c)) = \{2\pi/\sqrt{c}\,\}$.

We first review the congruence theorem for geodesics on either a geodesic sphere $G(r)$ of radius $r\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ or a geodesic sphere $G(r)$ of radius $r\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$. For a geodesic γ on $G(r)$, we set $\rho_\gamma(s) = g(\dot\gamma(s), \xi_{\gamma(s)})$ and call it its *structure torsion.* Note that $\rho_\gamma = \rho_\gamma(s)$ is constant along the curve γ. Indeed, due to the commutativity $\phi A = A\phi$ on $G(r)$ and the second equality in (3.2) we see

$$\begin{aligned}\dot\gamma\big(g(\dot\gamma,\xi)\big) &= g(\dot\gamma, \nabla_{\dot\gamma}\xi) = g(\dot\gamma, \phi A\dot\gamma) = g(\dot\gamma, A\phi\dot\gamma)\\ &= g(A\dot\gamma, \phi\dot\gamma) = -g(\phi A\dot\gamma, \dot\gamma),\end{aligned}$$

which shows that $\dot\gamma\big(g(\dot\gamma,\xi)\big) = 0$, hence ρ_γ is a constant with $-1 \le \rho_\gamma \le 1$. Using the notion of structure torsions, the congruence theorem for geodesics on $G(r)$ can be stated as follows:

Lemma A ([5])**.** *Two geodesics $\gamma_1 = \gamma_2(s)$ and $\gamma_2 = \gamma_2(s)$ of unit speed on $G(r)$ in $\widetilde{M}_n(c)$ are congruent to each other if and only if their structure torsions satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.*

We remark that geodesics γ on $G(r)$ with structure torsions $\rho_\gamma = 0, \pm 1$ must be closed. But other geodesics are *not* necessarily closed in general. In the following, we pay particular attention to the length spectrum $\mathrm{LSpec}(G(r))$ of a geodesic sphere $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, that is, it is expressed as

$$\begin{aligned}&\mathrm{LSpec}(G(r)) = \big\{(2\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r)\big\} \cup \big\{(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)\big\}\\ &\bigcup\left\{\frac{4\pi}{\sqrt{c}}\sqrt{p^2\sin^2(\sqrt{c}\,r/2) + q^2\cos^2(\sqrt{c}\,r/2)}\ \middle|\ \begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is even and } q < p\tan^2(\sqrt{c}\,r/2)\end{array}\right\}\\ &\bigcup\left\{\frac{2\pi}{\sqrt{c}}\sqrt{p^2\sin^2(\sqrt{c}\,r/2) + q^2\cos^2(\sqrt{c}\,r/2)}\ \middle|\ \begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is odd and } q < p\tan^2(\sqrt{c}\,r/2)\end{array}\right\}\end{aligned}$$

(cf. [5]). If a pair (p,q) of integers satisfies $p^2\sin^2\theta + q^2\cos^2\theta \le L^2$ with a given real θ, it corresponds to a lattice point which is contained or is on the ellipse on the pq-plane. Therefore we obtain the following:

Theorem A ([5])**.** *On a geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, there exist countably infinite congruency classes of closed geodesics. Moreover the length spectrum* $\mathrm{LSpec}(G(r))$ *of $G(r)$ is a discrete unbounded subset in the real line* $\mathbb{R}$.

Remark 1. Theorem A also holds for a geodesic sphere $G(r)\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$.

For a length spectrum $\lambda \in \mathrm{LSpec}(M)$ we call the cardinality $m_M(\lambda)$ of the set $\mathcal{L}_M^{-1}(\lambda)$ the *multiplicity* of λ. When the multiplicity of a length spectrum is 1 we say it is *simple*. Clearly for a geodesic sphere $G(r)$ in a complex projective space, we see by the expression of LSpec$(G(r))$ that $m_{G(r)}(\lambda) < \infty$ at each λ. It follows from the above expression of LSpec$(G(r))$ that

Corollary 1 ([5]). *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. Except geodesics with structure torsion ± 1, the length of every geodesic γ on $G(r)$ is longer than $2\pi/\sqrt{K_{\max}}$.*

The following is worth mentioning.

Theorem B ([5]). *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$.*

(1) If $\tan^2(\sqrt{c}\,r/2)$ is irrational, then every length spectrum of $G(r)$ is simple.
(2) If $\tan^2(\sqrt{c}\,r/2)$ is rational, then the multiplicity of each length spectrum of $G(r)$ is finite. But it is not uniformly bounded; $\limsup_{\lambda\to\infty} m_{G(r)}(\lambda) = \infty$*. In this case, the growth order of $m_{G(r)}$ is not so rapid. It satisfies $\lim_{\lambda\to\infty} \lambda^{-\delta} m_{G(r)}(\lambda) = 0$ for arbitrary positive δ.*

This theorem guarantees that on a geodesic sphere of radius r with irrational $\tan^2(\sqrt{c}\,r/2)$ in $\mathbb{C}P^n(c)$, two closed geodesics are congruent to each other if and only if they have the same length. On the other hand, if $\tan^2(\sqrt{c}\,r/2)$ is rational, this theorem shows that we can not classify congruency classes of closed geodesics only by their length.

For a Riemannian manifold M, we denote by $n_M(\lambda)$ the cardinality of the set $\{[\gamma] \in \mathrm{Geod}(M) \mid \mathcal{L}_M([\gamma]) \leq \lambda\}$. We have

Theorem C ([5]). *For every geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ we have*

$$\lim_{\lambda\to\infty} \frac{n_{G(r)}(\lambda)}{\lambda^2} = \frac{3c\sqrt{c}\,r}{8\pi^4 \sin(\sqrt{c}\,r)}.$$

Since the function $\theta \mapsto \theta/\sin\theta$ on the interval $(0, \pi)$ is monotone increasing, we have

Corollary 2. *A geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is a Berger sphere if and only if the asymptotic behavior of the number of length spectrum satisfies*

$$\lim_{\lambda\to\infty} \frac{n_{G(r)}(\lambda)}{\lambda^2} > \frac{9\sqrt{2}\,c}{16\pi^4} \tan^{-1}\sqrt{2}.$$

6. Sasakian space forms of ϕ-sectional curvature greater than 9

We bridge between contact geometry and the redefinition of Berger spheres (for details, see [18]):

Theorem 1. *Every complete simply connected Sasakian space form $N(k) := N^{2n-1}(k)$, $n \geq 2$ of constant ϕ-sectional curvature k which is greater than 9 is a Berger sphere. In particular, when $k = 8n + 5$, the space $N(k)$ can be realized as a homogeneous submanifold with nonzero parallel mean curvature vector with respect* *to the normal connection in an d-dimensional sphere $S^d(\tilde{c})$ of constant sectional curvature $\tilde{c}$, where $d = n(n+2) - 1$ and $\tilde{c} = 2(n+1)(2n+1)/n$.*

Sketch of Proof. We first prove that every complete simply connected Sasakian space form $N(k)$ with $k > 1$ is represented as a real hypersurface M^{2n-1} isometrically immersed into $\mathbb{C}P^n(c), n \geq 2$, where k satisfies $k = c + 1$. To do this, for a real hypersurface M^{2n-1} in $\mathbb{C}P^n(c)$, $n \geq 2$, we verify that the following four conditions are mutually equivalent (cf. [2]):

(1) M is a Sasakian manifold with respect to the almost contact metric structure either (ϕ, ξ, η, g) or $(\phi, -\xi, -\eta, g)$ induced from the Kähler structure J of the ambient space $\mathbb{C}P^n(c)$;
(2) M is a Sasakian space form of constant ϕ-sectional curvature k with respect to the almost contact metric structure given by (1), where k satisfies automatically $k = c + 1$;
(3) The shape operator A of M in $\mathbb{C}P^n(c)$ satisfies either $AX = -X + (c/4)\eta(X)\xi$ for all $X \in TM$ or $AX = X - (c/4)\eta(X)\xi$ for all $X \in TM$;
(4) M is locally congruent to a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ with $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$.

These, together with Proposition B, yield the first half of the statement of our Theorem: Every complete simply connected Sasakian space form $N(k) := N^{2n-1}(k)$, $n \geq 2$ of constant ϕ-sectional curvature k which is greater than 9 is a Berger sphere.

We next explain an idea to prove the second half of the statement of our Theorem (cf. [19]). We shall use standard notations and terminologies without explanation in submanifold geometry. We denote by $(G(r), \iota_{G(r)})$ a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ through a natural embedding $\iota_{G(r)} : G(r) \to \mathbb{C}P^n(c)$. In the following, we regard geodesic spheres $G(r)$ in $\mathbb{C}P^n(c)$ as submanifolds of a sphere $S^{n(n+2)-1}((n+1)c/(2n))$ of constant sectional curvature $(n+1)c/(2n)$ through $f_1 \circ \iota_{G(r)}$, where f_1 is the parallel equivariant minimal embedding of $\mathbb{C}P^n(c)$ into this Euclidean sphere. We here give the definition and fundamental geometric properties of f_1. The embedding f_1 is defined by eigenfunctions of the first eigenvalue of the Laplacian Δ on $\mathbb{C}P^n(c)$ (cf. [9, 23]).

In submanifold geometry, the embedding f_1 is the unique minimal parallel full isometric immersion of a complex projective space (endowed with Fubini-Study metric and Kähler structure J) into a Euclidean sphere (furnished with standard metric g). It is known that the inner product of the first normal space of f_1 is given by

$$
\begin{aligned}
g(\sigma_{f_1}(X,Y), \sigma_{f_1}(Z,W)) = &-(c/(2n))g(X,Y)g(Z,W) \\
&+ (c/4)\{g(X,W)g(Y,Z) + g(X,Z)g(Y,W) \\
&\qquad + g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W)\}
\end{aligned}
\tag{6.1}
$$

for all vectors X, Y, Z and W on $\mathbb{C}P^n(c)$, where σ_{f_1} denotes the second fundamental form of the embedding f_1 (for details, see Lemma 8 in [16]). By virtue of (6.1) we have the following geometric properties of f_1:

(i) f_1 is minimal;
(ii) $\sigma_{f_1}(JX, JY) = \sigma_{f_1}(X, Y)$ for all tangent vectors X and Y on $\mathbb{C}P^n(c)$ (namely, σ_{f_1} is J-invariant);
(iii) $\|\sigma_{f_1}(X, X)\| = \sqrt{(n-1)c/(2n)}$ for each unit tangent vector X on $\mathbb{C}P^n(c)$ (i.e., f_1 is $\sqrt{(n-1)c/(2n)}$-isotropic (see [22])).

Thus we obtain a family of submanifolds $\{(G(r), f_1 \circ \iota_{G(r)})\}$ in the sphere. We note that every geodesic sphere $G(r)$ is homogeneous in $\mathbb{C}P^n(c)$, i.e., it is an orbit of some subgroup of the full isometry group $\mathrm{U}(n+1)$ of the ambient space $\mathbb{C}P^n(c)$. This, together with a fact that $\mathbb{C}P^n(c)$ is an orbit of some subgroup of the full isometry group $\mathrm{SO}(n(n+2))$ of the ambient sphere $S^{n(n+2)-1}((n+1)c/(2n))$ through the embedding f_1, shows that every submanifold $(G(r), f_1 \circ \iota_{G(r)})$ is also a homogeneous submanifold of the sphere, so that it has constant mean curvature, i.e., the length of the mean curvature vector $\mathfrak{h}$ of the embedding $f_1 \circ \iota_{G(r)} : G(r) \to S^{n(n+2)-1}((n+1)c/(2n))$ is constant.

On the other hand, there exist no submanifolds with parallel second fundamental form in the class $\{(G(r), f_1 \circ \iota_{G(r)})\}$ because every $G(r)$ is a homogeneous manifold but not a locally symmetric space isometrically immersed into the ambient sphere $S^{n(n+2)-1}((n+1)c/(2n))$ which is a Riemannian symmetric space in a trivial sense.

Hence it is natural to consider a submanifold with parallel mean curvature vector $\mathfrak{h}$ with respect to the normal connection D in the family of submanifolds $\{(G(r), f_1 \circ \iota_{G(r)})\}$ in the sphere $S^{n(n+2)-1}((n+1)c/(2n))$. We shall compute the mean curvature vector $\mathfrak{h}$ of each of submanifolds $\{(G(r), f_1 \circ \iota_{G(r)})\}_{0<r<\pi/\sqrt{c}}$ in the sphere. To do this, we take a local field of orthonormal frames $\{e_1, \dots, e_{n-1}, \phi e_1 (= Je_1), \dots, \phi e_{n-1} (= Je_{n-1}), \xi\}$ on $G(r)$. Then we see that $\{e_1, \dots, e_{n-1}, Je_1, \dots, Je_{n-1}, \xi, J\xi (= \mathcal{N})\}$ is a local field of orthonormal frames on $\mathbb{C}P^n(c)$ along $G(r)$. By the definition of the mean curvature vector $\mathfrak{h}$, we have

$$\mathfrak{h} = \frac{1}{2n-1}\Big[(\mathrm{Trace}\, A)\mathcal{N} + \sum_{i=1}^{n-1}(\sigma_1(e_i, e_i) + \sigma_1(Je_i, Je_i)) + \sigma_1(\xi, \xi)\Big],$$

which, together with the fundamental properties (i) and (ii) of f_1, shows that

$$\mathfrak{h} = \frac{1}{2n-1}[(\mathrm{Trace}\, A)\mathcal{N} - \sigma_1(\xi, \xi)]. \tag{6.2}$$

Here, A is the shape operator of $G(r)$ in $\mathbb{C}P^n(c)$ through the natural embedding $\iota_{G(r)}$ and $\mathrm{Trace}\, A$ is written as:

$$\mathrm{Trace}\, A = (2n-1)(\sqrt{c}/2)\cot(\sqrt{c}\, r/2) - (\sqrt{c}/2)\tan(\sqrt{c}\, r/2),$$

which is a constant.

Using (6.1) and the fundamental properties (i), (ii), (iii) of the embedding f_1 repeatedly, we shall compute the derivative $D\mathfrak{h}$ of the mean curvature vector $\mathfrak{h}$, where D is the normal connection of the embedding $f_1 \circ \iota_{G(r)}$. We first have

$$D_\xi \mathfrak{h} = 0. \tag{6.3}$$

Next, we find

$$D_{e_j}\mathfrak{h} = \frac{1}{2n-1}\sigma_{f_1}\big((\mathrm{Trace}\,A)e_j + 2Ae_j, J\xi\big) \tag{6.4}$$

for each $j\,(1 \le j \le n-1)$. Similarly, we get the following equality corresponding to (6.4):

$$D_{\phi e_j}\mathfrak{h} = \frac{1}{2n-1}\sigma_{f_1}\big((\mathrm{Trace}\,A)e_j - 2\phi A\phi e_j, \xi\big) \tag{6.5}$$

for each $j\,(1 \le j \le n-1)$. Then from (6.4) and (6.5), we can see that the mean curvature vector $\mathfrak{h}$ of the isometric embedding $f_1 \circ \iota_{G(r)}$ is parallel with respect to the normal connection D if and only if the following two equalities hold:

$$\sigma_{f_1}\big((\mathrm{Trace}\,A)e_j + 2Ae_j, J\xi\big) = 0 \quad \text{for } j\ (1 \le j \le n-1); \tag{6.6}$$

$$\sigma_{f_1}\big((\mathrm{Trace}\,A)e_j - 2\phi A\phi e_j, J\xi\big) = 0 \quad \text{for } j\ (1 \le j \le n-1). \tag{6.7}$$

On the other hand, it follows from (6.1) that

$$\|\sigma_{f_1}(X,\xi)\| = \|\sigma_{f_1}(X,J\xi)\| = \frac{\sqrt{c}}{2}\|X\| \quad \text{for all } X \in TG(r) \text{ with } X \perp \xi. \tag{6.8}$$

We here note that both of vectors $(\mathrm{Trace}\,A)e_j + 2Ae_j$ and $(\mathrm{Trace}\,A)e_j - 2\phi A\phi e_j$ on $G(r)$ are perpendicular to ξ for each $j \in \{1,2,\dots,n-1\}$. Hence in view of (6.6), (6.7) and (6.8) we obtain

$$Ae_j = -\frac{1}{2}(\mathrm{Trace}\,A)e_j \quad \text{for } j\ (1 \le j \le n-1) \tag{6.9}$$

and

$$\phi A\phi e_j = \frac{1}{2}(\mathrm{Trace}\,A)\phi e_j \quad \text{for } j\ (1 \le j \le n-1).$$

This, combined with the equality $\phi^2 X = -X + \eta(X)\xi$ and the fact that $G(r)$ is a Hopf hypersurface, yields that

$$A\phi e_j = -\frac{1}{2}(\mathrm{Trace}\,A)e_j \quad \text{for } j\ (1 \le j \le n-1). \tag{6.10}$$

In consideration of (6.9) and (6.10) we know that the mean curvature vector $\mathfrak{h}$ of the embedding $f_1 \circ \iota_{G(r)}$ is parallel with respect to the normal connection D if and only if the radius r of $G(r)$ satisfies the following equation:

$$\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r\Big) = -\frac{1}{2}\Big[(2n-1)\frac{\sqrt{c}}{2}\cot\Big(\frac{\sqrt{c}}{2}r\Big) - \frac{\sqrt{c}}{2}\tan\Big(\frac{\sqrt{c}}{2}r\Big)\Big].$$

Solving this equation, we obtain $\tan^2(\sqrt{c}\,r/2) = 2n+1$. Note that this equality is equivalent to $k = 8n+5$ because of $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ and $k = c+1$. Thus we have proved the second half of the statement of our theorem. □

Remark 2. In [19], Udagawa and the first author showed a fact that for every real hypersurface M^{2n-1} of $\mathbb{C}P^n(c)\,(n \ge 2)$ through an isometric immersion ι_M the isometric immersion $f_1 \circ \iota_M : M^{2n-1} \to S^{n(n+2)-1}((n+1)c/(2n))$ has parallel mean curvature vector with respect to the normal connection in this ambient sphere if and only if M is locally congruent to a geodesic sphere $G(r)$ of radius r with $\tan^2(\sqrt{c}\,r/2) = 2n+1$ up to the full isometry group $\mathrm{U}(n+1)$.

The following gives detailed information on the length spectrum $\mathrm{LSpec}(N(k))$ of each complete simply connected Sasakian space form $N(k)$ with $k > 1$.

Theorem 2 ([3])**.** *On the space $N(k)$ with $k > 1$, there exist countably infinite congruency classes of closed geodesics and the length spectrum* $\mathrm{LSpec}(N(k))$ *of $N(k)$ is a discrete unbounded subset in the real line $\mathbb{R}$. Moreover, the following hold.*

(1) *When k is irrational, two closed geodesics on $N(k)$ are congruent to each other if and only if they have the common length.*
(2) *If k is rational, then the multiplicity of each length spectrum of $N(k)$ is finite. But it is not uniformly bounded*; $\limsup_{\lambda\to\infty} m_{N(k)}(\lambda) = \infty$. *In this case, the growth order of $m_{N(k)}$ is not so rapid. It satisfies* $\lim_{\lambda\to\infty} \lambda^{-\delta} m_{N(k)}(\lambda) = 0$ *for arbitrary positive δ.*
(3) *For every $k(> 1)$*

$$\lim_{\lambda\to\infty} \frac{n_{N(k)}(\lambda)}{\lambda^2} = \frac{3(k+3)\sqrt{k-1}}{16\pi^4} \tan^{-1}(\sqrt{k-1}/2).$$

Sketch of proof. By virtue of Theorem A and the discussion in the proof of our Theorem we can see that on the space $N(k)$ with $k > 1$ there exist countably infinite congruency classes of closed geodesics and the length spectrum $\mathrm{LSpec}(N(k))$ of $N(k)$ is a discrete unbounded subset in the real line $\mathbb{R}$. Furthermore, $\mathrm{LSpec}(N(k))$ can be expressed as follows (for details, see [3]):

$$\begin{aligned}\mathrm{LSpec}(N(k)) = &\left\{\frac{8\pi}{k+3}, \frac{4\pi}{\sqrt{k+3}}\right\} \\ &\bigcup\left\{4\pi\sqrt{\frac{(k-1)p^2+4q^2}{(k-1)(k+3)}}\ \middle|\ \begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is even and } 4p < (k-1)q\end{array}\right\} \\ &\bigcup\left\{2\pi\sqrt{\frac{(k-1)p^2+4q^2}{(k-1)(k+3)}}\ \middle|\ \begin{array}{c}p \text{ and } q \text{ are relatively prime}\\ \text{positive integers which satisfy}\\ pq \text{ is odd and } 4p < (k-1)q\end{array}\right\}.\end{aligned}$$

Next, under equalities $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ and $k = c+1$ we see that $\tan^2(\sqrt{c}\,r/2)$ is irrational (resp. rational) if and only if k is irrational (resp. rational), so that we obtain (1) and (2). We finally check (3). It follows from $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ and $k = c+1$ that

$$r = \frac{2}{\sqrt{k-1}}\tan^{-1}\frac{\sqrt{k-1}}{2}, \quad \sin\frac{\sqrt{c}\,r}{2} = \sqrt{\frac{k-1}{k+3}}, \quad \cos\frac{\sqrt{c}\,r}{2} = \sqrt{\frac{4}{k+3}}.$$

These, together with Theorem C, yield the equality in (3). □

7. Examples of non-Berger spheres

Motivated by the redefinition of Berger spheres, we pose the following problem:

Problem. Does there exist an odd dimensional Riemannian homogeneous manifold M satisfying the following three conditions (1), (2) and (3)'?

(1) M is diffeomorphic to a Euclidean sphere;
(2) The sectional curvature K of M satisfies sharp inequalities $0 < \delta L \leq K \leq L$ on M for some constant $\delta \in (0, 1/9)$;
(3)' The length of each closed geodesic of M is longer than $2\pi/\sqrt{L}$, where L is a positive constant given by (2).

Our aim here is to give an affirmative answer to this problem. We have

Theorem 3 ([17]). *If the radius r of a geodesic sphere $G(r)$ in a complex hyperbolic space $\mathbb{C}H^n(c)\,(n \geq 2)$ satisfies the inequalities $2 < \coth(\sqrt{|c|}\,r/2) < 3/\sqrt{2}$, then it has the following properties*:

1) *The sectional curvature K of $G(r)$ satisfies sharp inequalities $0 < \delta L \leq K \leq L$ on $G(r)$ for some constant $\delta \in (0, 1/9)$;*
2) *The length of each closed geodesic of $G(r)$ is longer than $2\pi/\sqrt{L}$, where L is a positive constant given by 1).*

Sketch of proof. We shall prove the statement 1). We take an arbitrary geodesic sphere $G(r)\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$. Its shape operator A satisfies $A\xi = \sqrt{|c|}\coth(\sqrt{|c|}\,r)\xi$ and $AX = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)X$ for each vector X perpendicular to the characteristic vector ξ on $G(r)$. Hence we can see that the sectional curvature K of our geodesic sphere $G(r)$ satisfies the following inequalities corresponding to (4.1):

$$c + (|c|/4)\coth^2(\sqrt{|c|}\,r/2) \leq K \leq (|c|/4)\coth^2(\sqrt{|c|}\,r/2). \tag{7.1}$$

Here, we have

$$\begin{aligned} K_{\min} &= c + (|c|/4)\coth^2(\sqrt{|c|}\,r/2) = K(X, \phi X), \\ K_{\max} &= (|c|/4)\coth^2(\sqrt{|c|}\,r/2) \qquad = K(X, \xi) \end{aligned}$$

for each unit vector X orthogonal to ξ. Solving an inequality $K_{\min} > 0$, we get $\coth(\sqrt{|c|}\,r/2) > 2$ and vice versa. Next, solving $0 < K_{\min}/K_{\max} < 1/9$, we have $2 < \coth(\sqrt{|c|}\,r/2) < 3/\sqrt{2}$ and vice versa. Thus we have proved our assertion 1).

We next prove the statement 2). As was shown in [5], a geodesic γ on $G(r)$ in $\mathbb{C}H^n(c)$ is closed if and only if its structure torsion satisfies one of the following conditions:

1) $\rho_\gamma = \pm 1$,
2) $\rho_\gamma = 0$,
3) ρ_γ satisfies $0 < |\rho_\gamma| < 1$ and is of the form

$$\rho_\gamma = \frac{\pm q}{\sinh\bigl(\sqrt{|c|}\,r/2\bigr)\sqrt{p^2\tanh^2\bigl(\sqrt{|c|}\,r/2\bigr) - q^2}}$$

with some pair (p, q) of relatively prime positive integers p, q satisfying $q < p\tanh^2\bigl(\sqrt{|c|}\,r/2\bigr)$.

Corresponding to these cases, lengths of closed geodesics are given as $\big(2\pi/\sqrt{|c|}\,\big)\sinh\big(\sqrt{|c|}\,r\big)$, $\big(4\pi/\sqrt{|c|}\,\big)\sinh\big(\sqrt{|c|}\,r/2\big)$ and

$$\begin{cases} 4\pi\sqrt{\big\{p^2\sinh^2\big(\sqrt{|c|}\,r/2\big)-q^2\cosh^2\big(\sqrt{|c|}\,r/2\big)\big\}/|c|}\,, & \text{when } pq \text{ is even},\\ 2\pi\sqrt{\big\{p^2\sinh^2\big(\sqrt{|c|}\,r/2\big)-q^2\cosh^2\big(\sqrt{|c|}\,r/2\big)\big\}/|c|}\,, & \text{when } pq \text{ is odd}.\end{cases}$$

We now compare these with $2\pi/\sqrt{K_{\max}} = \big(4\pi/\sqrt{|c|}\,\big)\tanh\big(\sqrt{|c|}\,r/2\big)$. It is clear that $\big(2\pi/\sqrt{|c|}\,\big)\sinh\big(\sqrt{|c|}\,r\big)$ and $\big(4\pi/\sqrt{|c|}\,\big)\sinh\big(\sqrt{|c|}\,r/2\big)$ are greater than this constant. We study lengths of other closed geodesics. By the condition $p > q\coth^2\big(\sqrt{|c|}\,r/2\big)$, we have $\sinh^2\big(\sqrt{|c|}\,r/2\big) > q/(p-q)$. We take a quadratic function $f_{p,q}$ defined by

$$f_{p,q}(x) = \begin{cases}(p^2-q^2)x^2+(p^2-2q^2-1)x-q^2, & \text{when } pq \text{ is even},\\ (p^2-q^2)x^2+(p^2-2q^2-4)x-q^2, & \text{when } pq \text{ is odd},\end{cases}$$

and consider its vertex. When pq is even, by direct computation we have $f_{p,q}\big(\sinh^2\big(\sqrt{|c|}\,r/2\big)\big) > 0$, so that

$$\left\{p^2\sinh^2\left(\frac{\sqrt{|c|}\,r}{2}\right)-q^2\cosh^2\left(\frac{\sqrt{|c|}\,r}{2}\right)\right\}\cosh^2\left(\frac{\sqrt{|c|}\,r}{2}\right) > \sinh^2\left(\frac{\sqrt{|c|}\,r}{2}\right).$$

When pq is odd, by direct calculation we obtain $f_{p,q}\big(\sinh^2\big(\sqrt{|c|}\,r/2\big)\big) > 0$, which shows

$$\left\{p^2\sinh^2\left(\frac{\sqrt{|c|}\,r}{2}\right)-q^2\cosh^2\left(\frac{\sqrt{|c|}\,r}{2}\right)\right\}\cosh^2\left(\frac{\sqrt{|c|}\,r}{2}\right) > 4\sinh^2\left(\frac{\sqrt{|c|}\,r}{2}\right).$$

In both cases, those inequalities show that lengths of closed geodesics are larger than $2\pi/\sqrt{K_{\max}}$. This completes the proof of the case 2). □

Remark 3. The inequalities $2 < \coth(\sqrt{|c|}\,r/2) < 3/\sqrt{2}$ in Theorem 3 are equivalent to the following inequalities:

$$\frac{1}{\sqrt{|c|}}\log\frac{11+6\sqrt{2}}{7} < r < \frac{\log 3}{\sqrt{|c|}}.$$

Remark 4. For a geodesic sphere $G(r)\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$ the first length spectrum λ_1 of $\mathrm{LSpec}(G(r))$ is given by $\lambda_1 = \big(4\pi/\sqrt{|c|}\,\big)\sinh\big(\sqrt{|c|}\,r/2\big)$ which is the length of a closed geodesic of null structure torsion (see Proposition 3.9(1) in [5]).

We shall characterize geodesic spheres in $\mathbb{C}H^n(c)$ studied in Theorem 3 by observing geodesics on these real hypersurfaces. For this purpose, we review the notion of circles in Riemannian geometry.

A smooth curve parameterized by its arclength on a Riemannian manifold M is said to be a *circle* if it satisfies the following system of ordinary differential equations with some nonnegative constant k_γ and a field Y of unit vectors along the curve γ:

$$\nabla_{\dot\gamma}\dot\gamma = k_\gamma Y \quad\text{and}\quad \nabla_{\dot\gamma}Y = -k_\gamma\dot\gamma,$$

which is equivalent to the equation

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle\dot\gamma = 0.$$

Here, $\nabla_{\dot\gamma}$ is the covariant differentiation along the curve γ with respect to the Riemannian connection ∇ of M. We call k_γ the *curvature* of γ. We regard geodesics as circles of null curvature.

We here recall a fact that an n-dimensional Riemannian manifold M^n which is isometrically immersed into an $(n+1)$-dimensional Riemannian manifold $\widetilde{M}^{n+1}$ is totally umbilic and the Trace A of its shape operator A is locally constant on M if and only if every geodesic on M is mapped to a circle in the ambient space $\widetilde{M}^{n+1}$ (see [15]). However, the space $\mathbb{C}H^n(c)$ $(n \geq 2)$ admits no totally umbilic real hypersurfaces (see [20]), so that this space does not have a real hypersurface all of whose geodesics are mapped to circles in the ambient space $\mathbb{C}H^n(c)$. We hence weaken this condition. We shall say that for a real hypersurface M in $\mathbb{C}H^n(c)$ a unit tangent vector $v \in TM$ satisfies *the extrinsic k-circular geodesic condition* if the geodesic γ_v of initial vector $\dot\gamma_v(0) = v$ on M is mapped to a circle of curvature k in $\mathbb{C}H^n(c)$.

Theorem 4 ([17])**.** *Let M^{2n-1} be a connected real hypersurface of $\mathbb{C}H^n(c)$ $(n \geq 2)$ through an isometric immersion. It is locally congruent to a geodesic sphere $G(r)$ of radius r with $2 < \coth(\sqrt{|c|}\, r/2) < 3/\sqrt{2}$ with respect to the full isometry group $\mathrm{U}(1,n)$ of $\mathbb{C}H^n(c)$ if and only if at each point $p \in M$, there exists an orthonormal basis $\{v_1, v_2, \dots, v_{2n-2}, v_{2n-1} = \xi_p\}$ of T_pM and a positive constant $k(p)$ with $\sqrt{|c|} < k(p) < 3\sqrt{2|c|}\,/4$ such that v_i satisfies the extrinsic $k(p)$-circular geodesic condition for every i $(0 \leq i \leq 2n-2)$.*

In this case, the function $k = k(p)$ on M is automatically constant with $k = (\sqrt{|c|}\,/2)\coth(\sqrt{|c|}\, r/2)$.

In order to show this characterization of our geodesic spheres, we need to recall some results on homogeneous Hopf hypersurfaces in $\mathbb{C}H^n(c)$.

Lemma B ([14])**.** *For a Hopf hypersurface M in $\mathbb{C}H^n(c)$ $(n \geq 2)$, the following hold.*

1. *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \nu)A\phi v = (\nu\lambda + (c/2))\phi v$ holds with the principal curvature ν associated with ξ. In particular, when $2\lambda - \nu \neq 0$, we have $A\phi v = \big((\nu\lambda + (c/2))/(2\lambda - \nu)\big)\phi v$.*
2. *The principal curvature ν associated with ξ is locally constant.*
3. *When $2\lambda - \nu = 0$, we see $\lambda = \sqrt{|c|}/2$ and $\nu = \sqrt{|c|}$.*

Proof of Theorem 4. ($\Longrightarrow$) Suppose that we can regard M as a geodesic sphere $G(r)$ of radius r with $2 < \coth(\sqrt{|c|}\, r/2) < 3/\sqrt{2}$ in the ambient space $\mathbb{C}H^n(c)$. We take an arbitrary geodesic γ on M whose initial vector is perpendicular to the characteristic vector $\xi_{\gamma(0)}$. Then we have $\rho_\gamma = 0$. Recalling principal curvatures of $G(r)$, we see $A\dot\gamma(s) = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\, r/2)\dot\gamma(s)$ for every s. Thus, by use of the Gauss and Weingarten formulae (3.1), we find that the curve γ, considered as a curve in the ambient space $\mathbb{C}H^n(c)$, is a circle of the same positive curvature k with

$k = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\, r/2)$ which is independent of the choice of γ. Moreover, by the assumption that $2 < \coth(\sqrt{|c|}\, r/2) < 3/\sqrt{2}$ we get $\sqrt{|c|}\, < k < (3\sqrt{2|c|}\,)/4$. Thus we obtain the "only if" part.

($\Longleftarrow$) We take orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ at an arbitrary fixed point p of a real hypersurface M satisfying the condition. Then, the geodesic γ_i with initial vector v_i satisfies

$$\widetilde{\nabla}_{\dot\gamma_i}\widetilde{\nabla}_{\dot\gamma_i}\dot\gamma_i = -k(p)^2\dot\gamma_i$$

with positive $k(p)$. On the other hand, from (3.1) we have

$$\widetilde{\nabla}_{\dot\gamma_i}\widetilde{\nabla}_{\dot\gamma_i}\dot\gamma_i = \langle(\nabla_{\dot\gamma_i}A)\dot\gamma_i, \dot\gamma_i\rangle\mathcal{N} - \langle A\dot\gamma_i, \dot\gamma_i\rangle A\dot\gamma_i.$$

Comparing the tangential components of the right-hand sides of these equalities, we see that

$$\langle A\dot\gamma_i, \dot\gamma_i\rangle A\dot\gamma_i = k(p)^2\dot\gamma_i,$$

so that at $s = 0$ we get

$$\langle Av_i, v_i\rangle Av_i = k(p)^2 v_i \quad \text{for} \quad 1 \le i \le 2n-2.$$

Since $k(p) \ne 0$, we obtain

$$Av_i = k(p)v_i \quad \text{or} \quad Av_i = -k(p)v_i \quad \text{for} \quad 1 \le i \le 2n-2. \tag{7.2}$$

This implies that ξ is a principal curvature vector, because $\langle A\xi, v_i\rangle = \langle \xi, Av_i\rangle = 0$ for $1 \le i \le 2n-2$. Therefore our real hypersurface M is a Hopf hypersurface with at most three distinct principal curvatures $k(p), -k(p)$ and $\nu = \langle A\xi, \xi\rangle$ at each point $p \in M$. Since principal curvatures of M vary continuously, we may suppose that $k : M \to \mathbb{R}$ is continuous.

Next, we shall show that $k(p)$ does not depend on the choice of a point p, and determine the real hypersurface M. Our discussion is divided into the following three cases (i), (ii$_a$) and (ii$_b$).

(i) Suppose that $2k(p) - \nu(p) \ne 0$ at each point $p \in M$. Then, by Lemma B(1) and (7.2) we have

$$\frac{\nu k + (c/2)}{2k - \nu} = k \quad \text{or} \quad \frac{\nu k + (c/2)}{2k - \nu} = -k. \tag{7.3}$$

This, together with the local constancy of ν, implies that the function $k : M \to \mathbb{R}$ is locally constant on M. Hence we find that M is a Hopf hypersurface with at most three distinct constant principal curvatures $k, -k$ and ν. However, there does *not* exist a Hopf hypersurface M with just three distinct constant principal curvatures $k, -k$ and ν. So we can see that the shape operator A of M satisfies $Av = kv$ for every vector v orthogonal to ξ with some positive constant k. This, combined with the same discussion in the "only if" part, shows that every geodesic $\gamma = \gamma(s)$ with initial vector perpendicular to the characteristic vector on M is mapped to a circle of the same positive curvature k. Moreover, by hypothesis this constant k satisfies the inequalities $\sqrt{|c|}\, < k < (3\sqrt{2|c|}\,)/4$. Thus we obtain that M is our geodesic sphere given in Theorem 4.

(ii$_a$) Suppose that $2k - \nu = 0$ holds in some open subset U of M. Then, it follows from Lemma B(3) that on this open set U the shape operator A of M

satisfies $Av = (\sqrt{|c|}/2)v$ for every vector v perpendicular to ξ. Hence, by the above discussion our real hypersurface M does not satisfy the assumption on M. So, the case (ii$_a$) does not occur.

(ii$_b$) Suppose that there exists a point $p \in M$ satisfying that $(2k - \nu)(p) = 0$, $\lim_{n\to\infty} p_n = p$ and $(2k-\nu)(p_n) \neq 0$ for some point sequence $\{p_n\}$ $(n = 1, 2, \ldots)$ on M. Then by the discussion in the case (i) we see that the function $2k-\nu$ is locally a step function on M. This, together with the continuity of this function, implies that $(2k-\nu)(p_1) = (2k-\nu)(p_2) = \cdots = (2k-\nu)(p_n) = \cdots \neq 0$, so that $(2k-\nu)(p) \neq 0$, which is a contradiction. So, the case (ii$_b$) does not occur also. □

8. Appendix: Comments on metrics of Sasakian space forms of constant ϕ-sectional curvatures greater than 1

Rewriting Theorem 1, we obtain the following fact:

(1) A geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ $(n \geq 2)$ is a Sasakian space form $M^{2n-1}(k)$ of constant ϕ-sectional curvature k if and only if the radius r satisfies an equality $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$. Here, k satisfies automatically $k = c + 1(> 1)$.

(2) A geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$ $(n \geq 2)$ is both of a Berger sphere and a Sasakian space form $M^{2n-1}(k)$ of constant ϕ-sectional curvature k if and only if the radius r satisfies $\tan^2(\sqrt{c}\,r/2) > 2$ and $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$. Here, k satisfies automatically $k = c + 1$ and $k > 9$. Moreover, when $\tan^2(\sqrt{c}\,r/2) = 2n + 1$, this geodesic sphere $G(r)$ can be regarded as a homogeneous submanifold with nonzero parallel mean curvature vector with respect to the normal connection in a certain Euclidean sphere. In this case, k satisfies automatically $k = 8n + 5$.

We first clarify the relation between the metrics of $\mathbb{C}P^n(c)$ and $S^{2n+1}(c/4)$ of constant sectional curvature $c/4$ through the Hopf fibration $\pi : S^{2n+1}(c/4) \to \mathbb{C}P^n(c)$ (for details, see Section 5 in [2]). We denote by $S^m[R]$ an m-dimensional Euclidean sphere of radius R. So we can set

$$S^{2n+1}[R] = \{z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \,|\, |z_0|^2 + \cdots + |z_n|^2 = R^2\} = S^{2n+1}\Big(\frac{1}{R^2}\Big).$$

It is well-known that the standard inner product $\langle\ ,\ \rangle$ of $\mathbb{C}^{n+1}$ is defined by $\langle X, Y\rangle = \mathfrak{Re}(\sum_{i=0}^n X_i\overline{Y_i})$ for $X = (X_0. \ldots, X_n)$, $Y = (Y_0, \ldots, Y_n) \in \mathbb{C}^{n+1}$. Note that the horizontal part $\mathcal{H}_{\hat z}$ of $T_{\hat z}S^{2n+1}[R]$ with respect to $\pi : S^{2n+1}[R] \to \mathbb{C}P^n(4/R^2)$ at $\hat z \in S^{2n+1}[R]$ is expressed as:

$$\mathcal{H}_{\hat z} = \{(\hat z, \hat X) \in \{\hat z\} \times \mathbb{C}^{n+1} |\, \langle \hat z, \hat X\rangle = \langle i\hat z, \hat X\rangle = 0\}.$$

Then the metric g of $\mathbb{C}P^n(4/R^2)$ defines the metric $\hat g$ of $S^{2n+1}[R]$ which degenerates along the vertical vector $i\hat z$ as follows. For any $(\hat z, \hat X), (\hat z, \hat Y) \in T_{\hat z}S^{2n+1}[R] = \{\hat X \in \mathbb{C}^{n+1} |\, \langle \hat z, \hat X\rangle = 0\}$, we take two horizontal vectors

$$\hat X - \Big\langle \frac{i\hat z}{R}, \hat X\Big\rangle \frac{i\hat z}{R} = \hat X - \frac{1}{R^2}\langle i\hat z, \hat X\rangle i\hat z,\ \ \hat Y - \Big\langle \frac{i\hat z}{R}, \hat Y\Big\rangle \frac{i\hat z}{R} = \hat Y - \frac{1}{R^2}\langle i\hat z, \hat Y\rangle i\hat z \in \mathcal{H}_{\hat z}.$$

By direct computation we have

$$\hat{g}((\hat{z},\hat{X}),(\hat{z},\hat{Y})) = \langle \hat{X},\hat{Y}\rangle - \frac{1}{R^2}\langle i\hat{z},\hat{X}\rangle\langle i\hat{z},\hat{Y}\rangle \tag{8.1}$$

for each $(\hat{z},\hat{X}),\ (\hat{z},\hat{Y}) \in T_{\hat{z}}S^{2n+1}[R]$.

Next, we denote by $G(r) = G(r;4/R^2)$ a geodesic sphere of radius r $(0 < r < R\pi/2)$ in $\mathbb{C}P^n(4/R^2)$. The inverse image $\pi^{-1}(G(r))$ is expressed as:

$$\pi^{-1}(G(r)) = S^1[R_1] \times S^{2n-1}[R_2], \quad R_1^2 + R_2^2 = R^2.$$

We here note that $R_1 = R\cos(r/R)$ and $R_2 = R\sin(r/R)$. Indeed, we can set $R_1 = R\cos\theta$, $R_2 = R\sin\theta$. It is known that the hypersurface $\pi^{-1}(G(r))$ of $S^{2n+1}[R]$ has two constant principal curvatures $-R_2/(RR_1),\ R_1/(RR_2)$. Hence the principal curvatures λ_1, ν of the geodesic sphere $G(r)$ are given by $\lambda_1 = R_1/(RR_2),\ \nu = (R_1/(RR_2)) - (R_2/(RR_1))$. On the other hand, λ_1 can be expressed as $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = (1/R)\cot(r/R)$. So we find that $\theta = r/R$.

Next, in order to define a diffeomorphism $\varphi:\ S^{2n+1}(1)(= S^{2n-1}[1]) \to G(r)$, we consider a diffeomorphism $\tilde{\varphi}:\ S^{2n-1}(1) \to \tilde{\varphi}(S^{2n-1}(1))\,(\subset S^1[R_1] \times S^{2n-1}[R_2])$ given by $\tilde{\varphi}(z) = (R_1, R_2 z)$. Then we have a desirable mapping $\varphi:\ S^{2n-1}(1) \to G(r)$ as $\varphi = \pi \circ \tilde{\varphi}$. We here note that

$$(d\tilde{\varphi})_z(z,X) = (0, R_2X). \tag{8.2}$$

It follows from (8.1) and (8.2) that

$$\begin{aligned} g_\sharp((z,X),(z,Y)) &= \hat{g}(((R_1,R_2z),(0,R_2X)),((R_1,R_2z),(0,R_2Y))) \\ &= \langle (0,R_2X),(0,R_2Y)\rangle \\ &\quad - \frac{1}{R^2}\langle i(R_1,R_2z),(0,R_2X)\rangle\langle i(R_1,R_2z),(0,R_2Y)\rangle \\ &= R_2^2\langle X,Y\rangle - \frac{R_2^4}{R^2}\langle iz,X\rangle\langle iz,Y\rangle \\ &= R_2^2(\langle X,Y\rangle - \langle iz,X\rangle\langle iz,Y\rangle) + \frac{R_1^2R_2^2}{R^2}\langle iz,X\rangle\langle iz,Y\rangle. \end{aligned}$$

We here recall that $R_1 = RR_2$, $R_2/(RR_1) = c/4$ and $R_1^2 + R_2^2 = R^2$. Thus we see that $R = 2/\sqrt{c}$, $R_1 = 4/\sqrt{c(c+4)}$ and $R_2 = 2/\sqrt{c+4}$. We hence know that the metric g of the geodesic sphere $G(r)$ is realized as the following metric $g_\sharp$ which is nothing but the deformation of the standard metric g_0 of $S^{2n-1}(1)$:

$$g_\sharp = \frac{4}{c+4}(g_0 - \eta\otimes\eta) + \left(\frac{4}{c+4}\right)^2 \eta\otimes\eta.$$

Hence we can see that the metric g of a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ coincides with the well-known metric of a complete simply connected Sasakian space form $M^{2n-1}(c+1)$ of constant ϕ-sectional curvature $c+1$ for each $c > 0$ (cf. [8]).

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